PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 4 – 1994

A NOTE ON THE EXISTENCE OF TWO NONTRIVIAL SOLUTIONS OF A RESONANCE PROBLEM

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Abstract: We study the existence of two nontrivial solutions for an elliptic boundary value problem at resonance. In the variational setting, the associated action functional of the problem is bounded from below but not coercive.

1 – Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We study the existence of two nontrivial weak solutions of the nonlinear elliptic problem

$$(1.1) \qquad -\Delta u = \lambda_1\, u + g(x,u) \ \text{ in } \Omega\,, \qquad u = 0 \ \text{ in } \partial\Omega\ ,$$

where λ_1 is the first eigenvalue of the linear problem,

$$(1.2) \qquad -\Delta u = \lambda_1\, u \ \text{ in } \Omega\,, \qquad u = 0 \ \text{ in } \partial\Omega\ .$$

The function $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ will be taken as a Carathéodory function such that

$$(g_1) \qquad |g(x,s)| \le a\,|s|^p + b \quad \forall\, s\in\mathbb{R} \ \text{ a.e. in } \Omega$$

for some $a,b\ge 0$ with $0<p<(N+2)/(N-2)$ if $N\ge 3$ and $0<p<\infty$ if $N=1,2$. We also assume that there exists $k\in L^1(\Omega)$ such that

$$(g_2) \qquad |G(x,s)| \le k(x) \quad \forall\, s\in\mathbb{R} \ \text{ a.e. in } \Omega\ ,$$

where $G(x,s)=\int_0^s g(x,t)\,dt$.

Received: December 18, 1992; *Revised*: January 4, 1993.
Keywords and Phrases: Elliptic equation, resonance, multiplicity, minimax principle.
* Partially supported by CAPES/Brasil.

In order to find solutions for (1.1), we use variational methods. So let us consider the functional $F\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$F(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 u^2)\,dx - \int_\Omega G(x, u(x))\,dx\ . \tag{1.3}$$

From condition (g_1) it is easy to see that F is of class C^1 in $H_0^1(\Omega)$. Moreover, it is well known that, critical points of F are precisely the weak solutions of (1.1).

The condition (g_2) implies that the functional F is bounded from below but not coercive. In particular the Palais–Smale condition does not hold. Nevertheless, with a suitable hypothesis on the behaviour of G at the infinity, the $(PS)_c$ condition will be satisfied in some convenient interval (Theorem 2).

We assume that

$$g(x,0) = 0 \quad \text{a.e. in } \ \Omega\ . \tag{g_3}$$

Thus $u = 0$ is a trivial solution of (1.1) and so we are interested to find nonzero critical points of F. For this purpose we consider the following "twist condition" on the behaviour of g near zero.

There exist $m \in L^1(\Omega)$, $m \geq 0$, with strict inequality holding in some subset of positive measure, such that

$$\liminf_{s\to 0} \frac{G(x,s)}{s^2} = m(x) \quad \text{in the } L^1\text{-sense}\ . \tag{g_4}$$

In other words (cf. [FM]), there exist $\delta_n \downarrow 0$, $\varepsilon_n \in L^1(\Omega)$ with $\|\varepsilon_n\|_1 \to 0$, such that

$$\frac{G(x,s)}{s^2} \geq m(x) - \varepsilon_n(x) \quad \text{if } \ 0 < |s| \leq \delta_n\ .$$

In particular, if $\partial_u g(x,0) \equiv m > 0$, then (g_4) holds.

Now we state our main result.

Theorem 1. *Suppose that the conditions (g_1)–(g_4) hold and that*

$$\int_\Omega \limsup_{|s|\to\infty} G(x,s)\,dx \leq 0\ . \tag{g_5}$$

Then problem (1.1) has at least one nontrivial solution. If in addition,

$$G(x,s) \leq \frac{\lambda_2 - \lambda_1}{2}\, s^2 \quad \forall\, s \in \mathbb{R} \ \text{ a.e. in } \Omega\ , \tag{g_6}$$

then problem (1.1) has another nontrivial solution.

The proof of this theorem will be presented in Section 3. The idea is to show that F has two negative critical values, one corresponding to the minimum and another deduced from a contradiction argument using a deformation lemma and a minimax principle.

There is a large quantity of works about the existence of solutions for problem (1.1) when the associated functional F is coercive. In the case where F is bounded below and coerciveness is not assumed, existence results were studied in e.g. [BBF, CLS, RS, S, T]. Some of them also give multiplicity results, but with a symmetry condition. In a recent paper, Gonçalves and Miyagaki [GM] has obtained the existence of two nontrivial solutions for (1.1), without any symmetry, in the case of strong resonance, that is, when G is bounded and

$$g(x,s) \to 0 \quad \text{as} \;\; |s| \to \infty \;\; \text{a.e. in } \Omega \;.$$

They use a condition like (g_4), with $m \leq 0$, and the proof is based on the generalized saddle point theorem of Rabinowitz [R]. Of course, our result is also applicable in the case of strong resonance.

2 – Preliminaries

Let φ_1 be the λ_1-eigenfunction of the problem (1.2). We may choose $\varphi_1 > 0$ with $\|\varphi_1\| = 1$, where $\|u\| = \int_\Omega |\nabla u|^2\, dx$ is the usual norm of the Sobolev space $H_0^1(\Omega)$. The L^p-norm is denoted by $\|\cdot\|_p$.

Set $V = \operatorname{Span}\{\varphi_1\}$ and $W = V^\perp \cap H_0^1(\Omega)$ in the L^2-sense. Then we have the orthogonal decomposition $H_0^1(\Omega) = V \oplus W$ and for $u \in H_0^1(\Omega)$, we can write in a unique way $u = v + w$ such that $v \in V$ and $w \in W$.

Let us consider the equation (1.1) with a forcing term $h \in L^2(\Omega)$ satisfying

$$\int_\Omega h\, \varphi_1\, dx = 0 \;, \tag{2.1}$$

that is,

$$-\Delta u = \lambda_1\, u + g(x,u) + h \;\; \text{in } \Omega \,, \qquad u = 0 \;\; \text{in } \partial\Omega \;. \tag{2.2}$$

It follows that weak solutions of (2.2) are the critical points of

$$\widehat{F}(u) = \frac{1}{2} \int_\Omega (|\nabla w|^2 - \lambda_1\, w^2)\, dx - \int_\Omega G(x,u(x))\, dx - \int_\Omega h\, w\, dx \;.$$

Moreover given such $h \in L^2(\Omega)$, it is well known that the auxiliary problem

$$-\Delta w = \lambda_1\, w + h \;\; \text{in } \Omega \,, \qquad w = 0 \;\; \text{in } \partial\Omega \;,$$

is uniquely solvable in W and this solution

$$w_0 \in W$$

minimises in W the functional

$$I(w) = \frac{1}{2}\int_\Omega (|\nabla w|^2 - \lambda_1 w^2)\,dx - \int_\Omega h\,w\,dx\ .$$

Not that, for $u = v + w$, $\widehat{F}(u) = I(w) - \int G(x,u)\,dx$.

The following result is primarily concerned with the $(PS)_c$ condition. We recall that F satisfies the Palais–Smale condition at level c, $(PS)_c$ for short, if any sequence u_n such $F(u_n) \to c$ and $F'(u_n) \to 0$ has a convergent subsequence. If $(PS)_c$ holds for all $c \in \mathbb{R}$, then we say that F satisfies the Palais–Smale condition (PS).

Theorem 2. *Suppose that conditions (g_1), (g_2) and (2.1) hold. If*

$$(g_7) \qquad \int_\Omega \limsup_{|s|\to\infty} G(x,s) = \gamma$$

then $\widehat{F}$ satisfies the $(PS)_c$ condition for every $c < -\gamma + I(w_0)$. Moreover, if

$$\gamma < \int_\Omega G(x,w_0)\,dx\ ,$$

then (2.2) has at least one solution.

Proof: Let $u_n = v_n + w_n$ be a sequence such that $\widehat{F}(u_n) \to c$ and $\widehat{F}'(u_n) \to 0$ with $c < -\gamma + I(w_0)$. From the growth condition (g_1) it suffices to show that u_n has a bounded subsequence (see [R, appendix B]). Suppose by contradiction that $\|u_n\| \to \infty$. From the variational characterization of λ_1 and λ_2,

$$\widehat{F}(u_n) \ge \frac{1}{2}(1 - \lambda_1/\lambda_2)\,\|w_n\|^2 - \|k\|_1 - \|h\|_2\,\|w_n\|_2\ ,$$

which implies that $\|w_n\|$ is bounded and since $\|u_n\| \le \|v_n\| + \|w_n\|$, we have $\|v_n\| \to \infty$. On the other hand, for some subsequence, there exists $w \in W$ such that $w_n \to w$ a.e. in Ω and consequently $|u_n(x)| \to \infty$ as $n \to \infty$ a.e. in Ω. Then from (g_7) and the Fatou's Lemma,

$$c = \lim \widehat{F}(u_n) \ge I(w_0) - \limsup \int_\Omega G(x,u_n(x))\,dx \ge I(w_0) - \gamma\ .$$

This is impossible since $c < -\gamma + I(w_0)$.

As for the second part of the theorem, suppose that $\gamma < \int G(x,w_0)\,dx$. Since $\widehat{F}$ is bounded from below,

$$-\infty < l = \inf \widehat{F} \le \widehat{F}(w_0) = I(w_0) - \int_\Omega G(x,w_0)\,dx < I(w_0) - \gamma\ .$$

This shows that $(PS)_c$ condition holds at the level of infimum l and therefore it is attained in $H_0^1(\Omega)$. So (2.2) has a solution. ∎

Remarks 1.

a) The existence result in Theorem 2 was firstly reported in [RS], but we provide additional information about the validity of $(PS)_c$ condition.

b) If $h = 0$ then $(PS)_c$ holds for every $c < -\gamma$ since $w_0 = 0$ and $I(w_0) = 0$. In this case, if $\gamma < 0$, then problem (1.1) has at least one solution.

3 – The proof of Theorem 1

The proof of Theorem 1 makes use of a deformation lemma. We follow the version for C^1-functions stated in [RR], that is recalled here for completeness. See also [MP].

Let X be a real Banach space and $F\colon\ X \to \mathbb{R}$ a C^1-functional. We set

$$K_c = \Big\{u \in X;\ F'(u) = 0 \text{ and } F(u) = c\Big\}$$

and

$$F^c = \Big\{u \in X;\ F(u) \le c\Big\}\ .$$

Lemma 1 (Deformation)**.** *Suppose that F has no critical values in the interval (a,b) and that $F^{-1}(\{a\})$ contains at most a finite number of critical points of F. Then if $(PS)_c$ condition holds for every $c \in [a,b)$, there exists an F-decreasing homotopy of homeomorphisms $h\colon\ [0,1] \times F^b\backslash K_b \to X$ such that*

$$h(0,u) = u \quad \forall\, u \in F^b\backslash K_b\ ,$$

$$h(1, F^b\backslash K_b) \subset F^a \quad \textit{and} \quad h(t,u) = u\ \ \forall\, u \in F^a\ .$$

Proof of Theorem 1: From (g_4) we can fix n so large that

$$K = \int_\Omega m\,\varphi_1^2\,dx - \|\varepsilon_n\|_1\,\|\varphi_1\|_\infty^2 > 0\ .$$

Then for all $v \in V = \mathrm{Span}\{\varphi_1\}$, $\|v\| \le \delta_n/\|\varphi_1\|_\infty$,

$$F(v) \le -\int_\Omega m\,v^2\,dx + \int_\Omega \varepsilon_n\,v^2\,dx \le -K\,\|v\|^2\ . \tag{3.1}$$

Therefore $l = \inf F < 0$ and since by (g_5) $(PS)_c$ holds for $c < 0$ (Remark 1 b)), we conclude that the infimum l is attained at some point $u_0 \in H_0^1(\Omega)$, and this provides us with one nontrivial solution of problem (1.1).

Now we are going to find a second nonzero critical point of F. Suppose by contradiction that zero and u_0 are the only critical points of F. From (g_6) and (3.1) there exists $R > 0$ and $l < d < 0$ such that

$$d = \sup_{\partial B} F < \inf_W F = 0 \ , \tag{3.2}$$

where $B = \{v \in V;\ \|v\| \leq R\}$. Set

$$\alpha = \inf_{\gamma \in \Gamma} \sup_{u \in B} F(\gamma(u))$$

with $\Gamma = \{\gamma \in C(B, H_0^1(\Omega));\ \gamma(v) = v\ \forall\, v \in \partial B\}$. It is proved in [BBF] that ∂B and W link. So $\gamma(B) \cap W \neq \emptyset\ \forall\, \gamma \in \Gamma$ and since $\inf_W F = 0$ we have $\alpha \geq 0$.

Now since $(PS)_c$ holds for $c < 0$, and there are no critical values in $(l, 0)$, the above deformation lemma yields an homotopy $h\colon\ [0,1] \times F^0 \backslash \{0\} \to H_0^1(\Omega)$ such that

$$h(0,u) = u \qquad \forall\, u \in F^0 \backslash \{0\} \ ,$$

$$h(1,u) \in F^l = \{u_0\} \qquad \forall\, u \in F^0 \backslash \{0\} \ .$$

We define $\gamma_0\colon\ B \to H_0^1(\Omega)$ by

$$\gamma_0(v) = \begin{cases} u_0 & \text{if } \|v\| < R/2, \\ h\Big(\dfrac{2(R - \|v\|)}{R}, \dfrac{Rv}{\|v\|}\Big) & \text{if } \|v\| \geq R/2 \ . \end{cases}$$

It is easy to see that γ_0 is continuous and that $\gamma_0(v) = v$ if $v \in \partial B$. So $\gamma_0 \in \Gamma$. Moreover, as h is F-decreasing, that is, $F(h(s,u)) < F(h(t,u))$ if $s > t$, we have also $F(\gamma_0(v)) \leq d\ \forall\, v \in B$. But this implies (that $\max_{u \in B} F(\gamma_0(u)) \leq d$ and therefore) $0 \leq \alpha \leq d < 0$; a contradiction. This ends the proof. ∎

Remarks 2.

a) In some applications, condition (3.2) may not hold, as is the case if (g_6) is dropped. Consequently we cannot apply the linking argument in the above proof. However, local linking theorems may be used if (PS) condition is provided. In this direction, we refer the reader to [L] and [BN].

b) Of course, conditions (g_2) and (g_5) can be replaced by any set of conditions on g near infinity that guarantees lower boundedness for F and $(PS)_c$ for $c < 0$. For example, it is proved in [FG] that if G is bounded above and $G(x,s) \to -\infty$ as $|s| \to \infty$ in a subset of positive measure, then F is coercive. In particular F is bounded below and satisfies $(PS)_c$ for all c.

ACKNOWLEDGEMENTS – The author is grateful to Miguel Ramos for his suggestions and helpful conversations, and to the Referee for pointing out a mistake.

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