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ON δ-SUNS

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ABSTRACT. We prove that an approximatively compact Chebyshev set in an M-space is a δ-sun and a δ-sun in a complete strong M-space (or externally convex M-space) is almost convex.

The most outstanding open problem of Approximation Theory is: Whether a Chebyshev set in a Hilbert space is convex? Many attempts have been made to solve this problem. Several partial answers are known (see e.g. survey articles by Vlasov-1973 [10], Narang-1977 [6], Deutsch-1993 [3] and Balaganskii and Vlasov-1996 [1]) but in full generality, the problem is still unsolved. In order to solve the problem, Vlasov [9] introduced the concepts of δ-suns and almost convex sets in Banach spaces and proved that an approximatively compact Chebyshev set in a Banach space is a δ-sun and each δ-sun in a Banach space is almost convex. We extend these results to M-spaces [5] which are more general than Banach spaces.

To start with, we recall a few definitions. A subset K of a metric space (X,d) is said to be a *δ-sun* [9] if for every $x \in X \setminus K$, there is a sequence $\langle x_n \rangle$ for which $x_n \neq x$, $x_n \to x$ and $\frac{d(x_n,K)-d(x,K)}{d(x_n,x)} \to 1$. A closed set A in a metric space (X,d) is said to be *almost convex* [9] if for any closed ball B which does not intersect A, there exists a closed ball $B' \supseteq B$ of arbitrary large radius and which does not intersect A. For a subset K of a metric space (X,d) and $x \in X$, an element $k_0 \in K$ is said to be a *best approximation* to x if $d(x,k_0) \leqslant d(x,k)$ for all $k \in K$ i.e., $d(x,k_0) = d(x,K) \equiv \inf_{k\in K} d(x,k)$. The set of all such $k_0 \in K$ is denoted by $P_K(x)$. The set K is said to be *proximinal* if $P_K(x) \neq \emptyset$ for each $x \in X$ and *Chebyshev* if $P_K(x)$ is exactly singleton for each $x \in X$. The mapping $p \equiv P_K$ from X into subsets of K is called the *metric projection*. For Chebyshev sets, p is single-valued. The set K is said to be *approximatively compact* if for every $x \in X$ and

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every sequence $\langle k_n \rangle$ in K with $\lim_{n\to\infty} d(x, k_n) = d(x, K)$ there is a subsequence $\langle k_{n_i} \rangle$ converging to an element of K.

For a metric space (X, d) and a closed interval $I = [0, 1]$, a mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x, y \in X$, $\lambda \in I$,

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda) d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* [8]. A convex metric space (X, d) is called an *M-space* [5] if for every two points x, y in X with $d(x, y) = \lambda$, and for every $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that

$$B[x, r] \cap B[y, \lambda - r] = \{z_r\},$$

where $B[x, r] = \{y \in X : d(x, y) \leqslant r\}$.

An M-space (X, d) is called a *strong M-space* [5] if for every two points x, y in X with $d(x, y) = \lambda$ and for every positive real number $r \neq \lambda$, there exists a unique z_r such that $S[x, r] \cap S[y, |\lambda - r|] = \{z_r\}$, where $S[x, r] = \{y \in X : d(x, y) = r\}$. A metric space (X, d) is called *externally convex* [5] if for all distinct points x, y such that $d(x, y) = \lambda$, and $r > \lambda$ there exists a unique z of X such that $d(x, y) + d(y, z) = d(x, z) = r$.

Every normed linear space is a strong M-space as well as an externally convex M-space but not conversely. If (X, d) is a convex metric space then for each two distinct points $x, y \in X$ and for every λ, $0 \leqslant \lambda \leqslant 1$, there exists at least one point $z \in X$ such that $d(x, z) = (1 - \lambda) d(x, y)$ and $d(z, y) = \lambda d(x, y)$. For M-spaces such a z is always unique. For distinct points x, y of strong M-space (X, d) with $d(x, y) = \lambda$ and for every $r \neq \lambda$, there exists a unique point z of X such that $d(x, y) + d(y, z) = d(x, z) = r$.

We denote by $G[x, y]$ the line segment joining x and y, i.e., $G[x, y] = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$; $G(x, y, -)$ denotes the largest line segment containing $G[x, y]$ for which x is an extreme point, i.e., the ray starting from x and passing through y; $G_1(x, y, -)$ denotes the set of all those points on the ray starting from x and passing through y which do not lie between x and y.

We intend to show that approximatively compact Chebyshev sets in M-spaces are δ-suns. To develop the proof, we prove some properties of Chebyshev sets.

LEMMA 1. *Given a Chebyshev set K in an M-space (X, d) and metric projection $x \to p(x)$, for every $x \in X \setminus K$ and $x_\lambda \in G_1(p(x), x, -)$,*

$$d(x_\lambda, K) \geqslant d(x, K) + d(x, x_\lambda) \left[1 - \frac{d(p(x), p(x_\lambda))}{d(x, p(x))} \right].$$

PROOF. Since $x_\lambda \in G_1(p(x), x, -)$, x is between $p(x)$ and x_λ so we can find some α, $0 < \alpha < 1$, such that

$$(*) \qquad d(p(x), x) = (1 - \alpha) d(p(x), x_\lambda), \quad d(x, x_\lambda) = \alpha d(p(x), x_\lambda)$$

i.e., $x = W(p(x), x_\lambda, \alpha)$. Consider

$$\begin{aligned} d(x, p(x)) &\leqslant d(x, p(x_\lambda)) = d(W(p(x), x_\lambda, \alpha), p(x_\lambda)) \\ &\leqslant \alpha d(p(x), p(x_\lambda)) + (1 - \alpha) d(x_\lambda, p(x_\lambda)). \end{aligned}$$

This implies

$$d(x_\lambda, p(x_\lambda)) \geqslant \frac{1}{1-\alpha} d(x, p(x)) - \frac{\alpha}{1-\alpha} d(p(x), p(x_\lambda)).$$

Therefore

$$\begin{aligned} d(x_\lambda, K) &= d(x_\lambda, p(x_\lambda)) \\ &\geqslant \frac{1}{1-\alpha} d(x, p(x)) - \frac{\alpha}{1-\alpha} d(p(x), p(x_\lambda)) \\ &= d(x_\lambda, p(x)) - \frac{d(x, x_\lambda)}{d(p(x), x)} d(p(x), p(x_\lambda)) \qquad \text{(using (*))} \\ &= d(p(x), x) + d(x, x_\lambda) - \frac{d(x, x_\lambda)}{d(x, p(x))} d(p(x), p(x_\lambda)) \quad \text{as } x \in [p(x), x_\lambda] \\ &= d(x, p(x)) + d(x, x_\lambda)\Big[1 - \frac{d(p(x), p(x_\lambda))}{d(x, p(x))}\Big]. \end{aligned}$$

□

LEMMA 2. *Given a Chebyshev set K in an M-space (X, d), if the metric projection $x \to p(x)$ is continuous on X, then*

$$\lim_{x_\lambda \to x} \frac{d(x_\lambda, p(x_\lambda)) - d(x, p(x))}{d(x_\lambda, x)} = 1$$

for every $x \in X \setminus K$ and $x_\lambda \in G_1(p(x), x, -)$ i.e., K is a δ-sun.

PROOF. We have

$$\begin{aligned} 1 = \frac{d(x_\lambda, x)}{d(x_\lambda, x)} &= \frac{d(x_\lambda, p(x)) - d(x, p(x))}{d(x_\lambda, x)} \\ &\geqslant \frac{d(x_\lambda, p(x_\lambda)) - d(x, p(x))}{d(x_\lambda, x)} \\ &\geqslant 1 - \frac{d(p(x), p(x_\lambda))}{d(x, p(x))}, \text{ by Lemma 1} \\ &\to 1 \text{ as by the continuity of } p,\ p(x_\lambda) \to p(x). \end{aligned}$$

The lemma is proved. □

THEOREM 1. *An approximatively compact Chebyshev set in an M-space is a δ-sun.*

PROOF. Let K be an approximatively compact Chebyshev set in an M-space (X, d) and $p : X \to K$ be the metric projection. Since the metric projection onto an approximatively compact Chebyshev set is continuous [7, p.390], p is continuous and so by Lemma 2, K is a δ-sun. □

REMARK 1. For Banach spaces, this result was proved by Vlasov [9] (see also [2, p.44]).

Almost convex sets (which are very close to convex sets) and δ-suns were introduced by Vlasov [9] to solve the problem of convexity of Chebyshev sets. We now show that in complete strong M-spaces (or externally convex M-spaces), δ-suns are almost convex. For this, we shall use the Primitive Ekeland form of the Bishop–Phelps Theorem (see [4, p.167]) stated below to derive a property for a Chebyshev set in a complete strong M-space (or externally convex M-space) when the metric projection is continuous.

PRIMITIVE EKELAND THEOREM. *Let (X, d) be a complete metric space and ψ be a proper but extended real lower semi-continuous function on X bounded below. Then given $\epsilon > 0$ and $x_1 \in X$ there exists an $x_0 \in X$ such that $\psi(x_0) + \epsilon d(x_0, x_1) \leqslant \psi(x_1)$ and $\psi(x) > \psi(x_0) - \epsilon d(x_0, x)$ for all $x \in X \smallsetminus x_0$.*

LEMMA 3. *Let (X, d) be a complete strong M-space (or externally convex M-space), $K \subseteq X$ be a Chebyshev set with continuous metric projection $x \to p(x)$. Given $x \in X \smallsetminus K$, $r > 0$ and $\sigma > 1$, there exists an $x_0 \in X$ such that*

(1) $d(x, K) + \frac{1}{\sigma} d(x, x_0) \leqslant d(x_0, K)$,
(2) $d(y, K) < d(x_0, K) + \frac{1}{\sigma} d(y, x_0)$ *for all* $y \neq x_0$ *and* $d(y, x) \leqslant r$,
(3) $d(x_0, x) = r$.

PROOF. Apply Primitive Ekeland Theorem to the complete metric space $B[x, r]$ and the continuous real mapping ψ on $B[x, r]$ defined by $\psi(y) = -d(y, K)$. For $\epsilon = \frac{1}{\sigma}$, there exists an $x_0 \in B[x, r]$ such that $\psi(x_0) + \frac{1}{\sigma} d(x_0, x) \leqslant \psi(x)$ and

$$\psi(y) > \psi(x_0) - \frac{1}{\sigma} d(x_0, y) \text{ for all } y \in B[x, r] \smallsetminus \{x_0\}.$$

So,

$$d(x, K) + \frac{1}{\sigma} d(x_0, x) \leqslant d(x, K),$$

which proves (1), and

$$d(y, K) < d(x_0, K) + \frac{1}{\sigma} d(x_0, y) \text{ for all } y \neq x_0 \text{ and } d(y, x) \leqslant r$$

which proves (2).

Now, we shall prove (3). From (1), $d(x, K) \leqslant d(x_0, K)$ so $x_0 \notin K$. Also $x_0 \in B[x, r]$ implies $d(x, x_0) \leqslant r$. Suppose $d(x, x_0) < r$. Take $x_{0_\lambda} \in G_1(p(x_0), x_0, -)$, $\lambda > 0$. Then

$$d(x_{0_\lambda}, x) \leqslant d(x_{0_\lambda}, x_0) + d(x_0, x) < d(x_{0_\lambda}, x_0) + r \to r \text{ as } x_{0_\lambda} \to x_0, \ \ x_{0_\lambda} \neq x_0.$$

Therefore $x_{0_\lambda} \in B[x, r]$ as $x_{0_\lambda} \to x_0$, $x_{0_\lambda} \neq x_0$ i.e., for λ sufficiently small. So, from (2) we have,

$$\frac{1}{\sigma} > \frac{d(x_{0_\lambda}, K) - d(x_0, K)}{d(x_0, x_{0_\lambda})}$$

for sufficiently small λ. Since $\sigma > 1$,

$$\lim_{x_{0_\lambda} \to x} \frac{d(x_{0_\lambda}, K) - d(x_0, K)}{d(x_0, x_{0_\lambda})} < 1,$$

contradicting Lemma 2. Therefore $d(x_0, x) = r$, which proves (3). □

THEOREM 2. *Each δ-sun K in a complete strong M-space (or externally convex M-space) (X, d) is almost convex.*

Proof. Let $x \in X \setminus K$ and $B[x, \alpha]$ be a ball with positive distance from K. Then $d(x, K) > \alpha$. Choose $\beta > d(x, K) > \alpha$ i.e., $\beta - d(x, K) < \beta - \alpha$. Choose $\sigma > 1$ and $r > 0$ such that $\sigma(\beta - d(x, K)) < r < \beta - \alpha$. By Lemma 3, there exists an $x_0 \in X$ such that $d(x, x_0) = r$ and $d(x, x_0) \leqslant \sigma(d(x_0, K) - d(x, K))$.

Now $d(x, x_0) = r < \beta - \alpha$. Also

$$\sigma(\beta - d(x, K)) < r = d(x, x_0) \leqslant \sigma(d(x_0, K) - d(x, K))$$

implies $\beta - d(x, K) < d(x_0, K) - d(x, K)$ i.e., $d(x_0, K) > \beta$. We claim that

(1) $B[x_0, \beta]$ does not intersect K.
(2) $B[x_0, \beta] \supseteq B[x, \alpha]$.

Suppose $B[x_0, \beta]$ intersects K then there exists $y \in B[x_0, \beta] \cap K$ i.e., $d(y, x_0) \leqslant \beta$ and so $d(x_0, K) \leqslant \beta$, a contradiction. This proves (1).

Now, suppose $y \in B[x, \alpha]$. Then $d(x, y) \leqslant \alpha$. Consider

$$d(y, x_0) \leqslant d(y, x) + d(x, x_0) \leqslant \alpha + r < \beta$$

i.e., $y \in B[x_0, \beta]$. This proves (2) and hence K is almost convex. □

Combining Lemma 2 and Theorem 2, we get

Theorem 3. *If K is a Chebyshev set in a complete strong M-space (or externally convex M-space) (X, d) and the metric projection is continuous then K is almost convex.*

Remark 2. For Banach spaces, Theorem 2 is given in [2, p.44] and Theorem 3 is given in [4, p.240].

Combining Theorems 1 and 2, we get:

Theorem 4. *An approximatively compact Chebyshev set in a complete strong M-space (or externally convex M-space) is almost convex.*

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