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BOOLEAN ALGEBRAS IN ALGEBRAIC TOPOLOGY

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Abstract. The appearance of the complete Heyting algebra in the realm of Algebraic Topology is the main topic of the paper.

1. Introduction

Actually, the title of this paper is misleading. We are really interested in the concrete complete Heyting algebra which appears in Algebraic Topology and Boolean algebras from the title are only subalgebras of this one. How is it that complete Heyting algebras enter Algebraic Topology? The answer is presented in this short paper. Briefly, in order to investigate *important* topological spaces, topologists introduce homology theories. To simplify matters, they look not at the category of spaces, but rather to the stable category of spectra and they even *localize* these new objects with respect to some numbers or even to some theories. It turns out that spectra and homology theories are tied together in the not such a great unification. By further identifying certain spectra one comes to the structure of distributive lattice which turns out to be a complete Heyting algebra.

Since it is a useless task to try to show the complete structure in the short or even very long paper, we have tried to present some basic ideas and notions so that the diligent reader can at least get some feeling for this extremely technical, but also interesting and important subject.

2. Spaces

Topologists study topological spaces. All of them? Well, yes, but not all topologists stydy *all* topological spaces at the same time. It depends on what area of Topology you are in. The most important classes of spaces algebraic topologist is interested in are: manifolds, CW-complexes and loop spaces. That is a little simplified view but it will do for our purpose.

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First of all, manifolds are important not only to topogists, but also to geometers, analysts, physicists... We have all kinds of manifolds — smooth manifolds, piecewise-linear, complex, symplectic... And they are all important.

CW-complexes may not be that well known among non-topologists. But, they are all around us. Every CW-complex is built in the following way. Start with a bunch of isolated points and add 1-cells (which are 1-dimensional discs) to them. So, we join some of them, maybe form a circle around some etc. We next proceed to the dimension 2 and glue 2-cells to the previous space. This means, that we *identify* points on the boundary of 2-discs with some points in the previous space and we get a new one. We can glue 1, 2, any finite number or, even, infinitely many 2-cells. Next we glue 3-cells etc. If we only glue cells up to the certain dimension, we get a finite dimensional CW-complex and if we glue only finitely many cells we get a finite CW complex which is necessarily compact. One gets many important spaces like this. All polytopes are CW-complexes. It is a theorem in Morse theory that any compact smooth manifold has a homotopy type of a CW-complex.

Loop spaces are also not too well known among other mathematicians. But they are important for Algebraic Topology (in what follows we use AT for short). First of all, in AT we very often work with *pointed* topological spaces, namely spaces in which one point is selected as a *base* point, and pointed maps. A loop on a space X is nothing but the map (by a map we mean a *continuous* map) from the circle S^1 to X which sends 1 to x_0, where x_0 is the base point of X (as you can see, for the base point on S^1 we choose 1). All loops on X form a space of loops ΩX.

One very important construction one can perform on a topological space is that of a *suspension*. We form a quotient space $\Sigma X := X \times I/\sim$, where we identify all points in $X \times \{0\}$ and all points in $X \times \{1\}$ (so if you imagine $X \times I$ as a cylinder with base X, then you get ΣX by squeezing both the top and the bottom of this cylinder). If our space X is a space with base point x_0, we get the *reduced* suspension by additional identification of all points in $\{x_0\} \times I$. The relation between suspension and loop space construction is explained by

$$[X, \Omega Y] \longleftrightarrow [\Sigma X, Y],$$

where $[A, B]$ stands for all homotopy classes of maps (with respect to the base point) between spaces A and B. So, we see that the functors Σ and Ω (these constructions are functorial, of course) are adjoint. By taking a suspension of a space we slightly simplify it in some way. For example, we have that

$$\Sigma(S^1 \times S^1) \cong \Sigma(S^1 \vee S^1 \vee S^2),$$

but the original spaces were not homotopy equivalent (if X and Y are pointed spaces, then $X \vee Y$ stands for a *pointed* union — we take the ordinary union and then identify base points; that is the coproduct in the category of pointed spaces).

3. Spectra

A CW-*spectrum* E consists of a sequence (E_n, ε_n) $(n \in \mathbb{N})$ of spaces E_n and celular maps $\varepsilon_n : \Sigma E_n \to E_{n+1}$. These maps can be turned into CW-embeddings so some authors include that in the definition of a spectrum. In the case where the adjoint maps $\varepsilon' : E_n \to \Omega E_{n+1}$ are weak homotopy equivalences (so they induce isomorphisms on homotopy groups) we say that E is an Ω-spectrum. Let us give a few examples.

Example 1. If X is a space (CW complex here), we can form the so-called *suspension* spectrum $\Sigma^\infty X$: $(\Sigma^\infty X)_n := \Sigma^n X$, $\varepsilon_n : \Sigma\Sigma^n X \approx \Sigma^{n+1} X$. If X is a sphere S^0, then we call the resulting spectrum the *sphere spectrum* and denote it by $\mathcal{S}$.

Example 2. The Eilenberg–Mac Lane spaces $K(\pi, n)$ are uniquely defined (up to homotopy) by the condition

$$\pi_k(K(\pi, n)) = \begin{cases} 0, & k \neq n \\ \pi, & k = n \end{cases}$$

They form an Ω-spectrum since $\Omega K(\pi, n+1) \simeq K(\pi, n)$.

Example 3. A complex n-dimensional vector bundle is a map $p : E \to B$ such that every fibre $E_b := p^{-1}(b)$ (for $b \in B$) is a vector space over $\mathbb{C}$ of dimension n. In addition to that, the map is locally trivial which means that we can find an open covering of B such that $p^{-1}(U) \approx U \times \mathbb{C}^n$ for every open set U in that covering (we skip some details in this definition). It turns out that there is the *universal* vector bundle $\gamma_n^{\mathbb{C}}$ such that one can get any vector bundle from this one using a standard pull-back construction. One can introduce a Hermitian metric on a vector bundle ξ and so one defines *disc* and *sphere* bundles: $D(\xi)$ and $S(\xi)$ (in the disc bundle we gather all vectors in all fibres whose norm is at most 1, while in the sphere bundle we put all of them with norm 1). The *Thom space* of ξ is defined as $T(\xi) := D(\xi)/S(\xi)$. One can extend standard operations on vector spaces to vector bundles and it turns out that $T(\xi \oplus \varepsilon^1) \simeq \Sigma^2 T(\xi)$ (by ε^1 we denote the trivial 1-dimensional bundle). The Thom space $T(\gamma_n^{\mathbb{C}})$ of the universal bundle is denoted by $MU(n)$. Using the previous results and the fact that $\gamma_n^{\mathbb{C}} \oplus \varepsilon^1$ is an $(n+1)$-bundle and so one can get it from $\gamma_{n+1}^{\mathbb{C}}$, we can construct the spectrum MU as follows: $MU_{2n} := MU(n)$, $MU_{2n+1} := \Sigma MU(n)$. The map ε_{2n} is the obvious homeomorphism, while the map ε_{2n+1} one gets from the previous remark concerning the bundle $\gamma_n^{\mathbb{C}} \oplus \varepsilon^1$. This spectrum is of central importance in modern AT.

The most important constructions one can perform on spectra are the *wedge* of two (or more spectra) $X \vee Y$, the *smash product* $X \wedge Y$ and the function spectrum $F(X, Y)$. While the wedge product is easy to define ($(X \vee Y)_n := X_n \vee Y_n$) and it corresponds to the direct sum in the category of modules, the smash product is much harder. It is much more complicated than, e. g. $(X \wedge Y)_{2n} := X_n \wedge Y_n$ or some similar attempt (let us recall that for *spaces* X and Y the smash $X \wedge Y$ is defined by $X \wedge Y := X \times Y / X \vee Y$). The function spectrum is given by the condition $[W \wedge X, Y] \cong [W, F(X, Y)]$ (recall the corresponding relation in the category of modules). The existence of this and many other spectra is established by the *Brown representability theorem*.

4. Homology theories

Homology theories are functors from the category of (pointed) topological spaces to the category of graded Abelian groups which satisfy the *Eilenberg – Steenrod axioms*. To be short, they are required to be homotopy invariant and to induce certain short exact sequences. Depending of whether we speak of reduced or unreduced theories these sequences are somewhat different. One also defines *cohomology theories* in the same way (they are contravariant functors of course). *The group of coefficients* of the given theory is the homology of the one-point space (or of the zero-dimensional sphere in case of reduced theories). Ordinary homology is characterized by the fact that the homology of the point is rather simple — it is $\mathbb{Z}$ (or some other Abelian group) in dimension 0 and 0 in all other dimensions. But in case of other theories these coefficients may become rather complicated, for example in case of complex cobordisms the *ring* of coefficients is given by $MU_* = \mathbb{Z}[x_1, x_2, \ldots]$, where the degree of x_i is $2i$. We say the ring of coefficients because this is an example of the multiplicative theory, so we also have multiplication (in addition to addition!).

How one goes about constructing homology theories? Well, some of them come from some geometric constructions (and they are only *later* recognized as homology theories) and others come from spectra — for a spectrum E we define a homology theory by $E_*(X) := \pi_*(E \wedge X)$. It turns out that Brown representability theorem shows that actually all homology theories come from spectra and, more, they come from Ω-spectra. In case we work with *ring spectra* we get multiplicative theories. Ring spectrum E comes with the additional structure: multiplication $\mu : E \wedge E \to E$ and the unit $\eta : \mathcal{S} \to E$, which have to satisfy some natural conditions.

5. Bousfield equivalence

One uses homology theories and therefore spectra in the attempt to solve topological and geometric problems algebraically. But what we get by looking at the class of all spectra from the point of view of homology theories? The notion of Bousfield equivalence is very useful here. We say that the spectrum X is *acyclic* with respect to certain theory E if $E \wedge X$ is contractible. Two spectra are *Bousfield equivalent* if they have the same acyclic spectra:

$$E \sim F \quad \textbf{iff} \quad \text{for all spectra } X : E \wedge X \simeq pt \Leftrightarrow F \wedge X \simeq pt.$$

We denote the Bousfield class of the spectrum E by $\langle E \rangle$. It is the result of Ohkawa that Bousfield classes form a set of cardinality at most $\beth_2$. One can define partial ordering on this set by

$$\langle E \rangle \geqslant \langle F \rangle \quad \textbf{iff} \quad \text{for all } X : E \wedge X \simeq pt \Rightarrow F \wedge X \simeq pt.$$

Bousfield classes form a complete lattice. The join is given by the wedge $\vee$ but the meet *is not* given by the smash $\wedge$. Actually, this poset has the smallest element $\langle pt \rangle$ and the largest element $\langle \mathcal{S} \rangle$. The meet $\curlywedge$ is given by the join of all lower bounds. But, this meet does not distribute over infinite joins and the smash does. So, it is interesting to consider a structure where the meet coincides with the smash. By DL we denote all elements $\langle E \rangle$ in the Bousfield lattice such that $\langle E \rangle \wedge \langle E \rangle = \langle E \rangle$

(of course, we take $\langle E\rangle \wedge \langle F\rangle := \langle E\wedge F\rangle$). In this way we get the distributive lattice and actually, this lattice turns out to be a *complete Heyting algebra*. Bousfield has considered this lattice even before it was known that Bousfield classes form a set. All ring spectra and all finite spectra are in DL but not all interesting spectra are there. For example the Brown–Comenetz dual of the p-local sphere I is not there. Namely, $I \wedge I \simeq pt$.

As we know, in the complete Heyting algebra we have three operations $\wedge$, $\vee$ and $\Rightarrow$ where $a \Rightarrow b$ is the greatest x such that $a \wedge x \leqslant b$. Not much is known about this lattice. Much more is known about the Boolean algebra which is contained in it and which consists of all $\langle E\rangle$ for which the pseudocomplement $\langle E\rangle \Rightarrow \langle pt\rangle$ is really the complement. If we denote this Boolean algebra by BA and by FBA we denote the subalgebra of all finite p-local spectra then it is known that FBA is isomorphic to the Boolean algebra of finite and co-finite subsets of $\mathbb{N}$.

We conclude this short paper with the following message to logicians. The structure of the DL is of great importance in modern AT. There are very few known facts and a lot of conjectures. Maybe the logic can show the way to investigate this complete Heyting algebra, and maybe it can also be of some interest to logic as an interesting and highly nontrivial example of this important structure.

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