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ON SOME EXTENSIONS OF KARAMATA'S THEORY AND THEIR APPLICATIONS

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Dedicated to the memory of Tatjana Ostrogorski

Abstract. This is a survey of the authors' results on the properties and applications of some subclasses of (so-called) O-regularly varying (ORV) functions. In particular, factorization and uniform convergence theorems for Avakumović–Karamata functions with non-degenerate groups of regular points are presented together with the properties of various other extensions of regularly varying functions. A discussion of equivalent characterizations of such classes of functions is also included as well as that of their (asymptotic) inverse functions. Applications are given concerning the asymptotic behavior of solutions of certain stochastic differential equations.

Contents

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It was in winter 2000, when we heard about Tatjana Ostrogorski for the first time. Being all at Marburg in those days, we were completing the paper [**9**] and, since we needed an advice concerning the references, we asked E. Seneta about his opinion and possible related results in the literature. He kindly provided us with a preliminary list of references and suggested to ask Tatjana concerning further results on this topic. Unfortunately, we were not able to take advantage of a discussion of the topic with her. On the other hand, among other references in Prof. Seneta's list, we came across the origin of the topic due to Avakumović [**2**]. Vojislav G. Avakumović was a late professor at the University of Marburg, where some of his students and collaborators are still working. Due to this special coincidence of the topic and the place, where we were working on it, we decided to continue these investigations further. Thanks to the support given by Deutsche Forschungsgemeinschaft this became possible. In the survey below, we present some of our results in this field.

1. Introduction

In a stimulating paper in 1930, Karamata [**29**] introduced the notion of *regular variation* and proved some fundamental theorems for *regularly varying* (RV) functions such as the Representation Theorem, the Uniform Convergence Theorem, and the Characterization Theorem (see also [**30**]). These results (together with later extensions and generalizations) turned out to be fruitful for various fields of mathematics (cf. Seneta [**47**] and Bingham, Goldie and Teugels [**6**] for excellent surveys on this topic and for the history of the theory and its applications).

After the papers of Karamata, various generalizations of the notion of regular variation (for functions of a single argument) appeared in the literature. In this paper, we are mainly concerned with a generalization due to Avakumović [**2**] that has been further investigated in Karamata [**31**], Feller [**22**], and Aljančić and Arandjelović [**1**]. The functions studied by these and several other authors are known in the literature as *O-regularly varying* (ORV) functions (see Definition 2.1 below). Bari and Stechkin [**3**], for example, independently studied ORV functions and their applications in the theory of best function approximation.

We also mention extensions of the Karamata theory to the cases of multidimensional arguments (see, for example, Yakymiv [**51**], Ostrogorski [**41, 42**]) and of multidimensional functions (Meerschaert and Scheffler [**40**]).

The defining property of ORV functions f is that f is positive, measurable, and such that

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} < \infty \qquad \text{for all } c > 0.$$

The function f^* is called the "limit function" of f. Note that, in the theory of regular variation, f^* is assumed to exist as a positive and finite limit for all $c > 0$. This results in the well-known characterization of f^* as a power function with some characteristic index ρ. The current paper is organized as follows. In a first and introductory part (Sections 1 and 2), we give a *historical overview* on the topic and collect a series of related *definitions* and *preliminaries.*

The main aim of the second part (Section 3) is to study the classes of ORV functions with "*non-degenerate groups of regular points*", that is, those functions f for which $f^*(c)$ exists as a positive and finite limit not necessarily for all $c > 0$, but possibly for some $c > 0$ (the set of those c necessarily forms a subgroup of $\mathbf{R}_+$). This extends the investigation of regularly varying functions to a larger class of functions, and leads us to new *representation, characterization* and *uniform convergence theorems* which, in general, are different from their well–known counterparts in RV theory. In particular, the limit functions can typically be represented as a product of a power function and a positive periodic component which, in turn, results in corresponding representations for the ORV functions themselves.

We give some typical examples of ORV functions with "non-degenerate groups of regular points" and discuss some *invariance properties* of the transformation $f \mapsto f^*$. In addition, we present the *factorization representations* for limit functions of ORV (and of other generalized RV) functions, which as corollaries cover well–known characterization theorems for RV functions. We also consider the corresponding factorization representations for the class of ORV functions with non-degenerate groups of regular points. Finally, some *uniform convergence theorems* are given for ORV functions (and some of their variants), which complement their corresponding counterparts in the RV theory. Section 3 contains results from Buldygin et al. [**10, 12, 14**]. For the proofs we refer to Buldygin et al. [**14**].

The third part of this paper (Sections 4–6) deals with some properties of functions *preserving the equivalence of functions*, that is with functions f satisfying $f(u(t))/f(v(t)) \to 1$, whenever $u(t)/v(t) \to 1$ (as $t \to \infty$), and with *asymptotic quasi-inverse functions* (confer Buldygin et al. [**9**], [**15**]–[**17**]; proofs can be found in [**11**] and [**15**]–[**17**]). This part is organized as follows. In Section 4, we consider the *Integral Representation Theorems* for *pseudo-regularly varying* (PRV) functions (cf. Definition 2.4) and obtain some equivalent characterizations of functions with *positive order of variation* (POV, see Definition 2.6). Moreover, a theorem on *increasing versions* for POV functions and a variant of *Potter's theorem* for PRV functions are considered. The solutions of some application problems in Section 7 below are closely connected with the question of when differentiable functions satisfy PRV conditions or are even *pseudo-monotone of positive variation* (PMPV, see Definition 2.5). Therefore, in Section 4, the latter question will be discussed in some detail. In Section 5, we consider *asymptotic quasi-inverse* and *asymptotic inverse functions* and investigate the problem of the existence of such functions. We also discuss conditions under which quasi-inverse functions preserve the equivalence of functions. Main properties and *characterizations* of POV functions and their asymptotic quasi-inverses are studied in Section 6. Moreover, the limiting behavior of the ratio of asymptotic quasi-inverse functions is discussed.

To be more precise, let a real-valued function f be locally bounded on $[t_0, \infty)$ for some $t_0 \geqslant 0$, and let $f(t) \to \infty$ as $t \to \infty$. Then its *generalized inverse function* $f^{\leftarrow}(s) = \inf\{t \in [t_0, \infty) : f(t) \geqslant s\}$ is defined on $[f(t_0), \infty)$, is nondecreasing and tends to ∞ as $s \to \infty$.

One of the pioneering works on generalized inverses in probability theory is Vervaat [**50**] where functional central limit theorems and laws of the iterated logarithm are proved for nondecreasing and unbounded stochastic processes and their generalized inverses. Confer also the book by Resnick [**45**] as a standard reference for properties of generalized inverses, in particular in relation to regular variation.

Below are two determining properties for the *inverse function* f^{-1}:

(i) $f(f^{-1}(t)) = t, \quad t \in [f(t_0), \infty),$ and
(ii) $f^{-1}(f(t)) = t, \quad t \in [t_0, \infty).$

The inverse function f^{-1} exists if f is continuous and strictly increasing. But, if either f is discontinuous or f is not strictly increasing then its inverse function f^{-1} does not exist, in which case the generalized inverse function is a natural substitution for the inverse function in many situations. However, generalized inverse functions do not always satisfy the above determining properties of inverse functions.

Various other definitions of (so-called) "*quasi-inverse*" functions are known in the literature. Any of these definitions either lacks part of the above determining properties or weakens them in one way or another.

In Section 5 we consider *asymptotic quasi-inverse* functions $\tilde{f}^{(-1)}$ defined, for a given function f, by the property $f(\tilde{f}^{(-1)}(t)) \sim t$ as $t \to \infty$, and *asymptotic inverse functions*, satisfying, in addition, $\tilde{f}^{(-1)}(f(t)) \sim t$ as $t \to \infty$. That is, we keep condition (i) (and (ii)) in an asymptotic sense in the definition of quasi-inverse (inverse) functions. Note that an asymptotic quasi-inverse function is not unique, even when its original function is continuous and strictly increasing. On the other hand, not all functions f have an asymptotic quasi-inverse function, and one of the questions is how to describe an appropriate class of functions f possessing asymptotic quasi-inverses.

Below (Sections 5 and 6) we consider the following four problems for asymptotic quasi-inverse functions:

(A) for which functions are their generalized inverse functions also asymptotic quasi-inverse functions;
(B) for which functions are their asymptotic quasi-inverse functions also asymptotic inverse functions;
(C) for which functions are their asymptotic quasi-inverse functions asymptotically equivalent;
(D) for which functions can the asymptotic behavior of their asymptotic quasi-inverses be obtained from the asymptotic behavior of the original functions or vice versa.

All four problems above are closely related to each other; we shall present here solutions for the classes of PRV, PMPV and POV functions.

Our motivation for applications of the four problems above results from certain correspondences in the theory of probability, particularly between the strong law of large numbers for random walks and the renewal theorem for counting processes (see, e.g., Gut et al. [**25**]). Namely, given a sequence of random variables $\{Z_n, n \geqslant 0\}$, generalized renewal processes can be defined in "a natural way" as follows: consider either $R(t) = \sup\{n \geqslant 0\colon Z_n \leqslant t\}$ or $R(t) = \sup\{n \geqslant 0\colon \max(Z_0, Z_1, \dots, Z_n) \leqslant t\}$ or $R(t) = \sum_{n=1}^{\infty} I(Z_n \leqslant t)$. If the sequence $\{Z_n, n \geqslant 0\}$ is strictly increasing, then all three functions coincide. Otherwise they are different, and further "natural" definitions of renewal processes could be given. In a certain sense, the sequence $\{Z_n\}$ and the process $\{R(t)\}$ can be viewed as generalized inverses to each other.

Given a continuous, strictly increasing, and unbounded function f, it is proved in Klesov et al. [**33**] that, under some mild conditions, if $Z_n/f_n \to 1$ almost everywhere (a.e.) as $n \to \infty$, then $R(t)/f^{-1}(t) \to 1$ a.e. as $t \to \infty$, where $f_n = f(n)$ and f^{-1} is the inverse to f. The main assumption posed on the function f in [**33**] is that either f or f^{-1} or both of them (the choice depends on the desired result) satisfy the PRV property. The above results are of the following nature: given an asymptotic behavior for an original function, find the corresponding limit behavior for its inverse function. Thus, in Klesov et al. [**33**], problem (D) has been considered for continuous, strictly increasing and unbounded functions, and an application of this problem has been discussed.

PRV functions and their various applications have been studied by Korenblyum [**36**], Matuszewska [**38**], Matuszewska and Orlicz [**39**], Stadtmüller and Trautner [**48**], [**49**], Berman [**4, 5**], Yakymiv [**53, 54**], Cline [**19**], Klesov et al. [**33**], Djurčić and Torgašev [**21**], Buldygin et al. [**9**], [**11**]–[**13**], [**15**]–[**18**]. Note that PRV functions are called *regularly oscillating* in Berman [**4**], *weakly oscillating* in Yakymiv [**53**] and *intermediate regularly varying* in Cline [**19**].

Recall that one of the main properties of PRV functions is that PRV functions, and only they, preserve the equivalence of functions and sequences, cf. Theorem 2.1 below.

In a more general setting, problem (D) has been considered in Djurčić and Torgašev [**21**] and Buldygin et al. [**9, 11**]. In these papers, among other questions, PMPV functions are defined (see, [**21**, Definition 3] and [**11**, relation (6.2)]). One of the main properties of PMPV functions is that their quasi-inverse functions are PRV functions and preserve the equivalence of functions. Moreover, the POV property has been introduced in [**11**] as a generalization of RV functions with positive index. In particular, it is proved in Buldygin et al. [**11**] that strictly increasing, unbounded POV functions and their quasi-inverse functions simultaneously preserve the equivalence of functions and sequences. Moreover, only POV functions possess this property. Note that condition (2.3) defining PMPV functions has also been used by Yakymiv [**52**] in connection with Tauberian theorems.

The complete solution of problems (A)–(D) for PRV and POV functions is given in [**15**] and [**16**].

In the fourth part of this paper (Section 7) we present various applications of the general results from Sections 4–6. We investigate the almost sure asymptotic behavior (as $t \to \infty$) of the solution of the stochastic differential equation $dX(t) = g(X(t))dt + \sigma(X(t))dW(t)$, where g and σ are positive continuous functions and W is a standard Wiener process. Applying the general results of the theory of PRV and PMPV functions, we find conditions on g and σ, under which $X(t)$, as $t \to \infty$, may be approximated almost everywhere on $\{X(t) \to \infty\}$ by the solution of the underlying deterministic differential equation $d\mu(t) = g(\mu(t))dt$. Moreover, the asymptotic stability with respect to initial conditions of solutions of the above stochastic differential equation as well as the asymptotic behavior of generalized renewal processes connected with this equation are considered in this part. This section is based on Buldygin et al. [**16**]–[**18**] (for the proofs we refer to Buldygin et al. [**17**]).

ORV functions also find their applications in the theory of renewal functions and processes constructed from random walks with multidimensional time (see, Klesov and Steinebach [**34, 35**] and Indlekofer and Klesov [**27**]) and in the strong law of large numbers for random walks with restricted domain (see, [**28**]).

Further applications to the asymptotic behavior of generalized renewal sample functions of continuous functions and sequences were considered in Buldygin et al. [**11, 16**].

2. Definitions and preliminaries

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}_+$ be the set of positive numbers, $\mathbf{Q}$ be the set of rational numbers, $\mathbf{Z}$ be the set of integers, and $\mathbf{N}$ be the set of positive integers. Also let $\mathbb{F}$ be the space of real-valued functions $f = (f(t), t > 0)$, and $\mathbb{F}_+ = \bigcup_{A>0}\{f \in \mathbb{F} | f(t) > 0, t \in [A, \infty)\}$. Thus $f \in \mathbb{F}_+$ if and only if f is eventually positive.

Let $\mathbb{F}^{(\infty)}$ be the space of functions $f \in \mathbb{F}_+$ such that

(i) $\sup_{0\leqslant t\leqslant T} f(t) < \infty$, $\forall T > 0$; (ii) $\limsup_{t\to\infty} f(t) = \infty$.

Further let $\mathbb{F}^{\infty}$ and $\mathbb{F}^{\infty}_{\mathrm{ndec}}$ be the spaces of functions $f \in \mathbb{F}^{(\infty)}$ such that $f(t) \to \infty$, $t \to \infty$, and f be nondecreasing for large t, respectively.

We also use the subspaces $\mathbb{C}^{(\infty)}$, $\mathbb{C}^{\infty}$, and $\mathbb{C}^{\infty}_{\mathrm{ndec}}$ of continuous functions in $\mathbb{F}^{(\infty)}$, $\mathbb{F}^{\infty}$, and $\mathbb{F}^{\infty}_{\mathrm{ndec}}$, respectively. Finally, the space $\mathbb{C}^{\infty}_{\mathrm{inc}}$ contains all functions $f \in \mathbb{C}^{\infty}$, which are strictly increasing for large t. Throughout the paper "measurability" means "Lebesgue measurability" and "meas" denotes the Lebesgue measure.

For given $f \in \mathbb{F}_+$, introduce the *upper* and *lower limit functions*

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} \quad \text{and} \quad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad c > 0,$$

which take values in $[0, \infty]$.

RV and ORV functions. Recall that a measurable function $f \in \mathbb{F}_+$ is called *regularly varying* (RV) if $f_*(c) = f^*(c) = \varkappa(c) \in \mathbf{R}_+$ for all $c > 0$ (see Karamata [**29**]). In particular, if $\varkappa(c) = 1$ for all $c > 0$, then the function f is called *slowly varying* (SV). For any RV function f, $\varkappa(c) = c^{\alpha}$, $c > 0$, for some number $\alpha \in \mathbf{R}$ which is called the *index* of the function f. Moreover, $f(t) = t^{\alpha}\ell(t)$, $t > 0$, where ℓ is a slowly varying function.

A measurable function $f \in \mathbb{F}_+$ is called *O-regularly varying* (ORV) if $f^*(c) < \infty$ for all $c > 0$ (see Avakumović [**2**] and Karamata [**31**]). It is obvious that any RV function is an ORV function.

On some versions of ORV functions. There exist various versions of ORV functions. Three of them will be considered in our context. In what follows "weakly" always means that the corresponding functions are not assumed to be measurable.

DEFINITION 2.1. A function $f \in \mathbb{F}_+$ is called *O-weakly regularly varying* (OWRV) if

$$0 < f_*(c) \leqslant f^*(c) < \infty \qquad \text{for all } c > 0. \tag{2.1}$$

Moreover, a function $f \in \mathbb{F}_+$ is called *O-regularly varying* (ORV) if it is OWRV (i.e. condition (2.1) holds) and measurable.

DEFINITION 2.2. A function $f \in \mathbb{F}_+$ is called *O-weakly uniformly regularly varying* (OWURV) if there exists an interval $[a, b] \subset (0, \infty)$ with $a < b$ such that

$$0 < \inf_{c \in [a,b]} f_*(c) \leqslant \sup_{c \in [a,b]} f^*(c) < \infty.$$

Note that any OWURV function is an OWRV function.

REMARK 2.1. It is known (see, for example, [**1**, Theorem 1]; [**6**, Theorems 2.0.1 and 2.0.4]) that all ORV functions are OWURV. Moreover, if f is an ORV function, then

$$0 < \liminf_{t \to \infty} \inf_{c \in [a,b]} \frac{f(ct)}{f(t)} \leqslant \limsup_{t \to \infty} \sup_{c \in [a,b]} \frac{f(ct)}{f(t)} < \infty$$

for any interval $[a, b] \subset (0, \infty)$.

Certain *subclasses* of ORV functions have also been discussed in the literature. For example, Drasin and Seneta [**20**] studied the so-called OSV functions.

DEFINITION 2.3. A function $f \in \mathbb{F}_+$ is called *O-weakly slowly varying* (OWSV) if it is an OWRV function such that $\sup_{c>0} f^*(c) < \infty$. Moreover, a function $f \in \mathbb{F}_+$ is called *O-slowly varying* (OSV) if it is OWSV and measurable.

PRV functions. For any RV function f, we have $f^*(c) \to 1$ as $c \to 1$. In order to generalize this property to a wider class of functions, we introduce the following definition (see Buldygin et al. [**11**]).

DEFINITION 2.4. A function $f \in \mathbb{F}_+$ is called *weakly pseudo-regularly varying* (WPRV) if

$$\limsup_{c \to 1} f^*(c) = 1. \tag{2.2}$$

A function $f \in \mathbb{F}_+$ is called *pseudo-regularly varying* (PRV) if it is a measurable WPRV function (cf. Buldygin et al. [**11**]).

It is obvious that from (2.2) it follows that the function f is an ORV function. Thus every PRV function is an ORV function. Any quickly growing function, e.g. $f(t) = e^t$, $t \geqslant 0$, is not PRV.

REMARK 2.2. (Buldygin et al. [**11**]) Let $f \in \mathbb{F}_+$. Then,

1) condition (2.2) is equivalent to any of the following four conditions:

(i) $\liminf_{c\to 1} f_*(c) = 1$, (iii) $\lim_{c\downarrow 1} f^*(c) = \lim_{c\downarrow 1} f_*(c) = 1$,

(ii) $\lim_{c\to 1} \limsup_{t\to\infty} \left|\frac{f(ct)}{f(t)} - 1\right| = 0$, (iv) $\lim_{c\uparrow 1} f^*(c) = \lim_{c\uparrow 1} f_*(c) = 1$;

2) condition (2.2) holds if and only if the upper limit function f^* (or the lower limit function f_*) is continuous at the point $c = 1$, that is, $\lim_{c\to 1} f^*(c) = 1$ or $\lim_{c\to 1} f_*(c) = 1$;

3) if f is a function with a nondecreasing upper limit function f^*, then condition (2.2) holds if and only if $\lim_{c\downarrow 1} f^*(c) = 1$ or $\lim_{c\uparrow 1} f_*(c) = 1$; moreover, under these conditions, f^* is continuous at every point $c \in (0, \infty)$.

Example 2.1. Any PRV function is ORV, but not vice versa. For example, the function $f(t) = 2 + (-1)^{[t]}, t \geqslant 0$, is ORV, but not PRV. □

Example 2.2. Any RV function is PRV, but not vice versa. For example, let α be a fixed real number. Then, the function

$$f(t) = \begin{cases} 0, & \text{for } t = 0, \\ t^\alpha \exp\{\sin(\log t)\}, & \text{for } t > 0, \end{cases}$$

is PRV, but not RV. □

Example 2.3. Also, the function

$$f(t) = \begin{cases} 1, & \text{for } t \in [0, 1); \\ 2^k, & \text{for } t \in \left[2^{2k}, 2^{2k+1}\right),\ k = 0, 1, 2 \ldots; \\ t/2^{k+1}, & \text{for } t \in \left[2^{2k+1}, 2^{2k+2)}\right),\ k = 0, 1, 2 \ldots; \end{cases}$$

is PRV, but not RV. □

PMPV and POV functions. Next we define further classes of functions playing an important role in the context of this paper (see also Buldygin et al. [**11**]).

Definition 2.5. A function $f \in \mathbb{F}_+$ is called *weakly pseudo-monotone of positive variation* (WPMPV) if

$$f_*(c) > 1 \qquad \text{for all } c > 1, \tag{2.3}$$

or, equivalently, if $f^*(c) < 1$ for all $c \in (0, 1)$. A function $f \in \mathbb{F}_+$ is called *pseudo-monotone of positive variation* (PMPV) if f is a measurable WPMPV function.

Note that every slowly varying function f is *not* a PMPV function. On the other hand, any *RV function of positive index* as well as any *quickly increasing monotone function*, for example $f(t) = e^t, t \geqslant 0$, is PMPV.

Remark 2.3. Observe that any function f satisfying condition (2.3) belongs to $\mathbb{F}^{(\infty)}$.

Using condition (2.3) we introduce a subclass of PRV functions, which is similar to the class of RV functions with positive index (cf. Buldygin et al. [**11**]).

Definition 2.6. A WPRV (PRV) function f is said to have *positive order of variation* WPOV (POV) if it satisfies condition (2.3).

Any *slowly varying function* f as well as any *quickly growing function*, e.g. $f(t) = e^t, t \geqslant 0$, is *not* POV. On the other hand, any RV function of positive index is a POV function. Example 2.3 presents a PRV function, which is neither an RV function nor a POV function. Example 2.2, with $\alpha \geqslant 1$, gives a PRV function, which is not an RV function, but is a POV function.

Functions preserving asymptotic equivalence. In this subsection, the functions u and v are nonnegative and eventually positive.

Two functions u and v are called (*asymptotically*) *equivalent* if $u(t) \sim v(t)$ as $t \to \infty$, that is $\lim_{t\to\infty} u(t)/v(t) = 1$. The equivalence of functions is denoted by $u \sim v$.

DEFINITION 2.7. A function f *preserves the equivalence of functions* if $\frac{f(u(t))}{f(v(t))} \to 1$ as $t \to \infty$ for all nonnegative functions u and v such that $u \sim v$ and $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) = \infty$.

In a similar way, one can introduce the notion of functions f preserving the equivalence of sequences. Below, all sequences $\{u_n, n \geqslant 0\}$ and $\{v_n, n \geqslant 0\}$ are assumed to be nonnegative and eventually positive.

Two sequences $\{u_n, n \geqslant 0\}$ and $\{v_n, n \geqslant 0\}$ are called (*asymptotically*) *equivalent* if $\lim_{n\to\infty} u_n/v_n = 1$. Equivalent sequences $\{u_n, n \geqslant 0\}$ and $\{v_n, n \geqslant 0\}$ are denoted by $\{u_n\} \sim \{v_n\}$. A function f *preserves the equivalence of sequences* if $f(u_n)/f(v_n) \to 1$ as $n \to \infty$ for all sequences of positive numbers $\{u_n, n \geqslant 0\}$ and $\{v_n, n \geqslant 0\}$ such that $\{u_n\} \sim \{v_n\}$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$. One of the most important properties of WPRV functions is that they and only they preserve the equivalence of both functions and sequences.

THEOREM 2.1. (Buldygin et al. [**11**]) *Let $f \in \mathbb{F}_+$. The following conditions are equivalent:*

(a) *a function f preserves the equivalence of functions;*
(b) *a function f preserves the equivalence of continuous functions, which are strictly increasing to infinity;*
(c) *a function f preserves the equivalence of sequences;*
(d) *a function f is WPRV.*

Theorem 2.1 implies the following version of a Uniform Convergence Theorem (see also Yakymiv [**53**], Buldygin et al. [**11**]).

THEOREM 2.2. *Let f be a WPRV function. Then*

$$\lim_{a\downarrow 1} \limsup_{t\to\infty} \sup_{a^{-1}\leqslant c\leqslant a} \left| \frac{f(ct)}{f(t)} - 1 \right| = 0.$$

3. On Factorization representations for Avakumović–Karamata functions with non-degenerate groups of regular points

Regular points. Consider $f \in \mathbb{F}_+$. A number $\lambda > 0$ is called a *regular point* of the function f, denoted $\lambda \in \mathbb{G}_r(f)$, if

$$f_*(\lambda) = f^*(\lambda) \in (0, \infty), \tag{3.1}$$

that is, the limit $\kappa_f(\lambda) = \lim_{x\to\infty} f(\lambda x)/f(x)$ exists, and is positive and finite. The function $\kappa_f = (\kappa_f(\lambda), \lambda \in \mathbb{G}_r(f))$ is called the *limit function* of f. In the sequel, for the sake of brevity, all functions f^*, f_* and κ_f are sometimes just called limit functions.

The set $\mathbb{G}_r(f)$ of regular points of f is a multiplicative subgroup of $\mathbf{R}_+$ with $1 \in \mathbb{G}_r(f)$. If $\mathbb{G}_r(f) = \{1\}$, then $\mathbb{G}_r(f)$ is called *degenerate*, otherwise *non-degenerate.*

Given $f \in \mathbb{F}_+$, if f is measurable and $\mathbb{G}_r(f) = \mathbf{R}_+$, then f is regularly varying (RV) in Karamata's sense.

The next theorems are well-known (see [**30**], [**23**], [**26**], [**47**], [**6**]) and are fundamental in the theory of regular variation.

Characterization Theorem 1. *Let $f \in \mathbb{F}_+$, and let f be measurable. If* $\mathrm{meas}(\mathbb{G}_r(f)) > 0$*, then $\mathbb{G}_r(f) = \mathbf{R}_+$, that is, f is an RV function, and there exists a real number $\rho = \rho_f$ such that*

$$\kappa_f(\lambda) = \lambda^\rho, \quad \lambda > 0. \tag{3.2}$$

For any RV function f, one has

$$f(x) = x^\rho \ell(x), \quad x > 0, \tag{3.3}$$

where $(\ell(x), x > 0)$ is a slowly varying function. Moreover, (3.2) and (3.3) are equivalent.

There are various extensions of the notion of regularly varying functions. For example, weakly regularly varying functions and their characterizations have been studied (cf. [**37**], [**46**], [**47**], [**6**]).

Given $f \in \mathbb{F}_+$, if $\mathbb{G}_r(f) = \mathbf{R}_+$, then f is called *weakly regularly varying* (WRV). Here the function f is not assumed to be measurable.

Characterization Theorem 2. *Let $f \in \mathbb{F}_+$, and let both f and $1/f$ be bounded on all finite intervals far enough to the right. If there exists a measurable set $\Lambda \subset \mathbb{G}_r(f)$ such that* $\mathrm{meas}(\Lambda) > 0$*, then $\mathbb{G}_r(f) = \mathbf{R}_+$, that is, f is a WRV function, and there exists a real number $\rho = \rho_f$ such that* (3.2) *holds.*

Note that for any WRV function f, from the above theorem we have

$$f(x) = x^\rho w(x), \quad x > 0, \tag{3.4}$$

where $(w(x), x > 0)$ is a *weakly slowly varying function* (WSV), that is $\kappa_w(\lambda) = 1$, $\lambda > 0$. Moreover, (3.4) and (3.2) are equivalent.

The above characterization theorems show that, if the set of regular points $\mathbb{G}_r(f)$ is "sufficiently large", then, with some additional conditions on f, assertions (3.2), (3.3) and (3.4) hold, and the function f is regularly or weakly regularly varying.

Note for later use that $\mathbb{H}_r(f) = \log(\mathbb{G}_r(f))$ is an additive subgroup of numbers u in $\mathbf{R}$ such that $\exp(u) \in \mathbb{G}_r(f)$.

Some examples. Next we present some examples of functions f with nondegenerate, but "small" groups of regular points $\mathbb{G}_r(f)$. It will be seen later on, that the form of their limit functions is typical in some sense for the general situation.

EXAMPLE 3.1. Let $(r(x), x > 0)$ be a regularly varying function with index ρ, and put $f(x) = r(x)\exp\{\sin(\log x)\}, x > 0$. Then, for all $\lambda > 0$,

$$f^*(\lambda) = \lambda^\rho \exp\{2|\sin(\log\sqrt{\lambda})|\}, \quad f_*(\lambda) = \lambda^\rho \exp\{-2|\sin(\log\sqrt{\lambda})|\}. \tag{3.5}$$

By (3.5), $\mathbb{G}_r(f) = \{e^{2\pi n} : n \in \mathbf{Z}\}$, and this multiplicative group is non-degenerate. Moreover, $\kappa_f(\lambda) = \lambda^\rho, \lambda \in \mathbb{G}_r(f)$. Note that $(\exp\{2|\sin(u/2)|\}, u \in \mathbf{R})$ is a positive periodic function with set of periods $\mathbb{H}_r(f) = \{2\pi n : n \in \mathbf{Z}\}$. □

EXAMPLE 3.2. Let $(r(x), x > 0)$ be a regularly varying function with index ρ and put $f(x) = r(x)\exp\{\operatorname{sign}(\sin(\log x))\}, x > 0$, where $\operatorname{sign}(x) = 1$, if $x > 0$, $\operatorname{sign}(x) = -1$, if $x < 0$, and $\operatorname{sign}(0) = 0$. Then, for all λ such that $\log\lambda \neq 2\pi n$, $n \in \mathbf{Z}$,

$$f^*(\lambda) = \lambda^\rho \exp\{2\}, \qquad f_*(\lambda) = \lambda^\rho \exp\{-2\}, \tag{3.6}$$

whereas

$$f^*(\lambda) = f_*(\lambda) = \lambda^\rho, \tag{3.7}$$

for all λ such that $\log\lambda = 2\pi n, n \in \mathbf{Z}$. By relations (3.6), (3.7), $\mathbb{G}_r(f) = \{e^{2\pi n} : n \in \mathbf{Z}\}$, and this multiplicative group is non-degenerate. Moreover, $\kappa_f(\lambda) = \lambda^\rho$, $\lambda \in \mathbb{G}_r(f)$. Rewrite f^* in the form $f^*(\lambda) = \lambda^\rho \exp\{2(1 - I_{\mathbb{H}_r(f)}(\log\lambda))\}$, where $I_{\mathbb{H}_r(f)}$ is the indicator function of $\mathbb{H}_r(f)$, and note that $(\exp\{2(1-I_{\mathbb{H}_r(f)}(u)\}, u \in \mathbf{R})$ is a positive periodic function with set of periods $\mathbb{H}_r(f) = \{2\pi n : n \in \mathbf{Z}\}$. □

EXAMPLE 3.3. Let $(r(x), x > 0)$ be a regularly varying function with index ρ and let $(d(x), x > 0)$ be the Dirichlet function, i.e. $d(x) = 1$, if $x \in \mathbf{Q}$, and $d(x) = 0$ otherwise. Put $f(x) = r(x)\exp\{d(x)\}, x > 0$. Then, for all $\lambda > 0$,

$$f^*(\lambda) = \lambda^\rho \exp\{1 - d(\lambda)\}, \quad f_*(\lambda) = \lambda^\rho \exp\{d(\lambda) - 1\}. \tag{3.8}$$

By (3.8), $\mathbb{G}_r(f) = \mathbf{Q} \cap \mathbf{R}_+$, and this multiplicative group is non-degenerate. Moreover, $\kappa_f(\lambda) = \lambda^\rho, \quad \lambda \in \mathbb{G}_r(f)$. Note that the set $\mathbb{G}_r(f)$ is everywhere dense in $\mathbf{R}_+$, but $\operatorname{meas}(\mathbb{G}_r(f)) = 0$. Rewrite f^* from (3.8) in the form $f^*(\lambda) = \lambda^\rho \exp\{1 - d(e^{\log\lambda})\}$, and note that $(\exp\{1 - d(e^u)\}, u \in \mathbf{R})$ is a positive periodic function with set of periods $\mathbb{H}_r(f) = \{u \in \mathbf{R} : \exp(u) \in \mathbf{Q} \cap \mathbf{R}_+\}$. □

The next example shows that for every non-degenerate multiplicative subgroup of $\mathbf{R}_+$ there exists a function f such that $\mathbb{G}_r(f) = \mathbb{G}$.

EXAMPLE 3.4. Let $(r(x), x > 0)$ be a regularly varying function with index ρ and let $\mathbb{G}$ be a non-degenerate multiplicative subgroup of $\mathbf{R}_+$. Put

$$f(x) = r(x)\exp\{I_{\mathbb{G}}(x)\}, \quad x > 0,$$

where $I_{\mathbb{G}}$ is the indicator function of $\mathbb{G}$. Then, for all $\lambda > 0$,

$$f^*(\lambda) = \lambda^\rho \exp\{1 - I_{\mathbb{G}}(\lambda)\}, \quad f_*(\lambda) = \lambda^\rho \exp\{I_{\mathbb{G}}(\lambda) - 1\}. \tag{3.9}$$

By (3.9) $\mathbb{G}_r(f) = \mathbb{G}$, and this multiplicative group is non-degenerate. Moreover, $\kappa_f(\lambda) = \lambda^\rho, \lambda \in \mathbb{G}_r(f)$. Rewrite f^* from (3.9) in the form $f^*(\lambda) = \lambda^\rho \exp\{1 - I_{\mathbb{G}}(e^{\log\lambda})\}$, and note that $(\exp\{1 - I_{\mathbb{G}}(\exp\{u\})\}, u \in \mathbf{R})$ is a positive periodic function with set of periods $\mathbb{H}_r(f) = \{u \in \mathbf{R} : \exp(u) \in \mathbb{G}\}$. □

$*$-invariant limit functions. In a next step we consider some facts related to *invariants* of the transformations $f \mapsto f^*$ and $f \mapsto f_*$.

DEFINITION 3.1. A function $f \in \mathbb{F}_+$ is called *an upper $*$-invariant function*, if $f(\lambda) = f^*(\lambda)$ for all $\lambda > 0$, and it is called *a lower $*$-invariant function*, if $f(\lambda) = f_*(\lambda)$ for all $\lambda > 0$.

PROPOSITION 3.1. *Let f be an OWRV function with non-degenerate group of regular points $\mathbb{G}_r(f)$. Then its upper limit function f^* is upper $*$-invariant, i.e. $f^{**} = (f^*)^* = f^*$, and its lower limit function f_* is lower $*$-invariant, i.e., $f_{**} = (f_*)_* = f_*$. Moreover, $(f_*)^* = f^*$ and $(f^*)_* = f_*$.*

The following example contains some $*$-invariant functions.

EXAMPLE 3.5. The function $f = (x^a, x > 0)$, with $a \in \mathbf{R}$ fixed, is both upper $*$-invariant and lower $*$-invariant; the function $f = (\exp\{2|\sin(\log\sqrt{x})|\}, x > 0)$ is upper $*$-invariant and the function $f = (\exp\{-2|\sin(\log\sqrt{x})|\}, x > 0)$ is lower $*$-invariant.

COROLLARY 3.1. *Let φ be an OWRV function with non-degenerate group of regular points $\mathbb{G}_r(\varphi)$. If φ is upper $*$-invariant, then the function $g(\lambda) = 1/\varphi(1/\lambda)$, $\lambda > 0$, is lower $*$-invariant.*

A Factorization representation for the limit functions of OWRV functions with non-degenerate groups of regular points.

THEOREM 3.1. *Let f be an OWRV function with non-degenerate group of regular points $\mathbb{G}_r(f)$. Assume that $c \in \mathbb{G}_r(f)$ with $c \neq 1$. Then, for $\lambda > 0$,*

$$f^*(\lambda) = \lambda^\alpha P(\log\lambda) \quad \textit{and} \quad f_*(\lambda) = \frac{\lambda^\alpha}{P(-\log\lambda)},$$

where $\alpha = \log_c \kappa_f(c)$, $(P(u), u \in \mathbf{R})$ is a positive periodic function with $P(0) = 1$, for which its set of periods $S_{\rm per}(P)$ contains the set $\{nu_0, n \in \mathbf{Z}\}$ with $u_0 = \log c \neq 0$, and $S_{\rm per}(P) \subset \mathbb{H}_r(f)$. Moreover, the function $(P(\log\lambda), \lambda > 0)$ is upper $*$-invariant and the function $(1/P(-\log\lambda), \lambda > 0)$ is lower $*$-invariant, that is, for all $u \in \mathbf{R}$,

$$\limsup_{x\to\infty} \frac{P(u+x)}{P(x)} = P(u) \quad \textit{and} \quad \liminf_{x\to\infty} \frac{P(u+x)}{P(x)} = \frac{1}{P(-u)}.$$

COROLLARY 3.2. *Let f be an OWRV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and let $(P(u), u \in \mathbf{R})$ be as in Theorem 3.1. Then,*

(a) $P(u)P(-u) \geqslant 1$ *for all $u \in \mathbf{R}$;* (b) $\inf_{u\in\mathbf{R}} P(u) \sup_{u\in\mathbf{R}} P(u) \geqslant 1$.

A factorization representation for the limit functions of OWURV and ORV functions with non-degenerate groups of regular points. Theorem 3.1 demonstrates that, for any OWRV function f with non-degenerate group of regular points $\mathbb{G}_r(f)$, its limit functions f^* and f_* can be represented as a product of a power function and a positive periodic component with logarithmic argument. But, in general, such a representation need not necessarily be unique. The following theorem shows that, for any OWURV function with non-degenerate group of regular points, such a representation is indeed unique. Moreover, the form of the periodic component will be studied in more detail.

THEOREM 3.2. *Let f be an OWURV function with non-degenerate group of regular points $\mathbb{G}_r(f)$. Then,*

(i) *there exists a unique real number $\rho \in \mathbf{R}$, such that $\rho = \log_c \kappa_f(c)$, $c \in \mathbb{G}_r(f) \setminus \{1\}$;*

(ii) *if $1 \in \{\kappa_f(c), c \in \mathbb{G}_r(f) \setminus \{1\}\}$, then $\rho = 0$;*

(iii) *we have*

$$\kappa_f(\lambda) = \lambda^\rho, \quad \lambda \in \mathbb{G}_r(f); \tag{3.10}$$

(iv) *for $\lambda > 0$,*

$$f^*(\lambda) = \lambda^\rho \mathcal{P}(\log \lambda) \quad \textit{and} \quad f_*(\lambda) = \frac{\lambda^\rho}{\mathcal{P}(-\log \lambda)}, \tag{3.11}$$

where $(\mathcal{P}(u), u \in \mathbf{R})$ is a positive periodic function such that $\mathcal{P}(0) = 1$,

$$1 = \min_{-\infty<u<\infty} \mathcal{P}(u) \leqslant \mathcal{P}(u) \leqslant \sup_{-\infty<u<\infty} \mathcal{P}(u) < \infty, \quad u \in \mathbf{R},$$

and $S_{\mathrm{per}}(\mathcal{P}) = \mathbb{H}_r(f)$, with $S_{\mathrm{per}}(\mathcal{P})$ denoting the set of periods of $\mathcal{P}$;

(v) *$(\mathcal{P}(\log \lambda), \lambda > 0)$ is upper $*$-invariant and $(1/\mathcal{P}(-\log \lambda)$, $\lambda > 0)$ is lower $*$-invariant, that is, for all $u \in \mathbf{R}$,*

$$\limsup_{x\to\infty} \frac{\mathcal{P}(u+x)}{\mathcal{P}(x)} = \mathcal{P}(u) \quad \textit{and} \quad \liminf_{x\to\infty} \frac{\mathcal{P}(u+x)}{\mathcal{P}(x)} = \frac{1}{\mathcal{P}(-u)};$$

(vi) *$\mathfrak{p} = \log \mathcal{P}$ is subadditive, that is, $\mathcal{P}(u+x) \leqslant \mathcal{P}(u)\mathcal{P}(x)$, and $\mathfrak{p}(u+x) \leqslant \mathfrak{p}(u) + \mathfrak{p}(x)$, for all $u, x \in \mathbf{R}$;*

(vii) *for given f, the representations* (3.10), (3.11) *are unique.*

DEFINITION 3.2. The exponent ρ in (iii) of Theorem 3.2 is called the *index*, and the function $\mathcal{P}$ in (iv) of Theorem 3.2 is called the *periodic component* of the OWURV function f with non-degenerate group of regular points $\mathbb{G}_r(f)$.

By Theorem 3.2, for a given OWURV function f with non-degenerate group of regular points $\mathbb{G}_r(f)$, the function f^* is uniquely defined by its index ρ and its periodic component $\mathcal{P}$. The next result is immediate from (vi) of Theorem 3.2.

COROLLARY 3.3. *Let f be an OWURV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with periodic component $\mathcal{P}$. Then the following statements are equivalent:*

(a) *f^* is continuous on $\mathbf{R}_+$;*

(b) *$\mathcal{P}$ is uniformly continuous on $\mathbf{R}$;*

(c) *$\mathcal{P}$ is continuous at* 0*;*

(d) *f^* is continuous at* 1.

The next results follow from Theorem 3.2 again.

COROLLARY 3.4. *Let f be an OWURV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with periodic component $\mathcal{P}$. Then the following statements are equivalent:*

(a) *assertion* (3.2) *holds;*
(b) $\mathbb{G}_r(f) = \mathbf{R}_+$*;*
(c) $S_{\mathrm{per}}(\mathcal{P}) = \mathbf{R}$*;*
(d) $\mathcal{P}(u) = 1$ *for all* $u \in \mathbf{R}$*;*
(e) $\mathcal{P}(u)\mathcal{P}(-u) = 1$ *for all* $u \in \mathbf{R}$*;*
(f) $\mathbb{G}_r(f)$ *is dense in* $\mathbf{R}_+$ *and* f^* *is continuous on* $\mathbf{R}_+$*;*
(g) $\mathbb{G}_r(f)$ *is dense in* $\mathbf{R}_+$ *and* f^* *is continuous at one point* $\lambda \in \mathbf{R}_+$*;*
(h) $S_{\mathrm{per}}(\mathcal{P})$ *is dense in* $\mathbf{R}$ *and* $\mathcal{P}$ *is continuous on* $\mathbf{R}$*;*
(i) $S_{\mathrm{per}}(\mathcal{P})$ *is dense in* $\mathbf{R}$ *and* $\mathcal{P}$ *is continuous at one point* $u \in \mathbf{R}$*.*

Corollary 3.5. *Let* f *be an OWURV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$ *and index* ρ*. Then*

$$\lim_{\lambda\to 0+} \frac{\log f^*(\lambda)}{\log \lambda} = \lim_{\lambda\to\infty} \frac{\log f^*(\lambda)}{\log \lambda} = \rho,$$
$$\lim_{\lambda\to 0+} \frac{\log f_*(\lambda)}{\log \lambda} = \lim_{\lambda\to\infty} \frac{\log f_*(\lambda)}{\log \lambda} = \rho.$$

In view of Remark 2.1 we conclude:

Theorem 3.3. *Let* f *be an ORV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$*. Then all statements of Theorem 3.2 retain.*

Corollaries (Characterization theorems). Theorem 3.2 immediately implies the following series of characterization theorems:

Corollary 3.6. *Let* f *be an OWURV function, and let its group of regular points* $\mathbb{G}_r(f)$ *contain a set of positive Lebesgue measure. Then* $\mathbb{G}_r(f) = \mathbf{R}_+$*, that is,* f *is a WRV function, and there exists a real number* ρ *such that assertion* (3.2) *holds. Moreover,* $f(x) = x^\rho w(x), x > 0$*, where* $(w(x), x > 0)$ *is a weakly slowly varying function, that is* $\kappa_w(\lambda) = 1, \lambda > 0$*.*

Corollary 3.7. *The Characterization Theorems* 1 *and* 2 *hold true.*

Corollary 3.8. (Bingham et al. [**6**, Theorem 1.4.3]) *Let* $f \in \mathbb{F}_+$ *and*

$$\limsup_{\lambda\downarrow 1} f^*(\lambda) \leqslant 1 \quad \text{or} \quad \limsup_{\lambda\uparrow 1} f^*(\lambda) \leqslant 1. \tag{3.12}$$

Then the following statements are equivalent:

(i) *there exists a real number* ρ *such that* (3.2) *holds;*
(ii) f *is a WRV function;*
(iii) $\mathbb{G}_r(f)$ *contains a set of positive Lebesgue measure;*
(iv) $\mathbb{G}_r(f)$ *is dense in* $\mathbf{R}_+$*;*
(v) *there exist positive numbers* $\lambda_1, \lambda_2 \in \mathbb{G}_r(f) \setminus \{1\}$ *such that* $\log\lambda_1/\log\lambda_2$ *is irrational.*

Moreover, (i) $\Rightarrow$ (3.12)*.*

Corollary 3.9. *Let* $f \in \mathbb{F}_+$*, and assume there exists* $\lambda_0 > 0$ *such that*

$$\lim_{\lambda\to\lambda_0} f^*(\lambda) = 1. \tag{3.13}$$

Then the following statements are equivalent:

(i) *there exists a real number* ρ *such that* (3.2) *holds;*
(ii) f *is a WRV function;*
(iii) $\mathbb{G}_r(f)$ *contains a set of positive Lebesgue measure;*
(iv) $\mathbb{G}_r(f)$ *is dense in* $\mathbf{R}_+$*;*
(v) *there exist positive numbers* $\lambda_1, \lambda_2 \in \mathbb{G}_r(f) \setminus \{1\}$ *such that* $\log \lambda_1 / \log \lambda_2$ *is irrational.*

Moreover, if (iv) *and* (3.13) *hold with* $\lambda_0 \neq 1$*, then* $\rho = 0$*, that is,* f *is a WSV function.*

Note that (3.13) is also necessary for (i) of Corollary 3.9.

A factorization representation for ORV functions having nondegenerate groups of regular points. In the previous subsections, factorization representations for the *limit functions* of Avakumović–Karamata functions have been considered. Now, we present a factorization representation for the functions themselves.

Proposition 3.2. *Let* f *be an OWURV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$*, and with index* ρ *and periodic component* $\mathcal{P}$*. Then there exists an OWSV function* $(s(x), x > 0)$ *such that*

$$f(x) = x^{\rho} s(x), \quad x > 0, \tag{3.14}$$

where $(s(x), x > 0)$ *has the upper limit function*

$$s^*(\lambda) = \mathcal{P}(\log \lambda), \quad \lambda > 0. \tag{3.15}$$

Corollary 3.10. *Let* f *be an ORV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$*, and with index* ρ *and periodic component* $\mathcal{P}$*. Then there exists an OSV function* $(s(x), x > 0)$ *such that* (3.14) *and* (3.15) *hold.*

The following statement is due to Drasin and Seneta [**20**].

Proposition 3.3. *Let* $(\psi(x), x > 0) \in \mathbb{F}_+$*, and let* ψ *be measurable. Then* ψ *is an OSV function if and only if it can be written in the form* $\psi(x) = \ell(x)\theta(x)$*,* $x > 0$*, where* ℓ *is slowly varying and* θ *is measurable such that* θ *and* $1/\theta$ *are positive and bounded on* $(0, \infty)$*.*

Now, a factorization representation for ORV functions with non-degenerate group of regular points can be considered.

Theorem 3.4. *Let* $f \in \mathbb{F}_+$*, and let* f *be measurable. Then,* f *is an ORV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$ *if and only if* f *can be written in the form*

$$f(x) = r(x)\theta(x), \quad x > 0, \tag{3.16}$$

where

(A1) $(r(x), x > 0)$ *is an RV function;*
(A2) $(\theta(x), x > 0)$ *is a positive measurable function;*
(A3) θ *and* $1/\theta$ *are bounded on* $(0, \infty)$*;*
(A4) $\theta^* = (\mathcal{P}(\log \lambda), \lambda > 0)$*;*

(A5) $(\mathcal{P}(u), u \in \mathbf{R})$ *is a positive periodic function with* $\mathcal{P}(0) = 1$ *and*

$$0 < \inf_{-\infty<u<\infty} \mathcal{P}(u) \leqslant \sup_{-\infty<u<\infty} \mathcal{P}(u) < \infty.$$

Moreover, if (3.16) *and* (A1)–(A5) *hold, then*

(A6) *the index of* r *coincides with the index of* f*;*
(A7) *the set of periods of* $\mathcal{P}$ *coincides with* $\mathbb{H}_r(f)$*;*
(A8) $(\mathcal{P}(\log \lambda), \lambda > 0)$ *is upper* $*$*-invariant, that is, for all* $u \in \mathbf{R}$*,*

$$\limsup_{x\to\infty} \frac{\mathcal{P}(u+x)}{\mathcal{P}(x)} = \mathcal{P}(u) \quad \textit{and} \quad \liminf_{x\to\infty} \frac{\mathcal{P}(u+x)}{\mathcal{P}(x)} = \frac{1}{\mathcal{P}(-u)};$$

(A9) $\min_{-\infty<u<\infty} \mathcal{P}(u) = 1$*;*
(A10) $\log \mathcal{P}$ *is subadditive;*
(A11) $\mathbb{G}_r(f) = \mathbb{G}_r(\theta)$.

It is well known (see, for example, [**47**] and [**6**]), that

$$\lim_{x\to\infty} \frac{\log r(x)}{\log x} = \rho_r$$

for any RV function $(r(x), x > 0)$ with index ρ_r. The following statement extends this result to ORV functions with non-degenerate groups of regular points.

Corollary 3.11. *Let* f *be an ORV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$ *and index* ρ*. Then,* $\lim_{x\to\infty} \log f(x)/\log x = \rho$.

The representation (3.16) can be rewritten in the following form.

Theorem 3.5. *Let* $f \in \mathbb{F}_+$*, and let* f *be measurable. Then,* f *is an ORV function with non-degenerate group of regular points* $\mathbb{G}_r(f)$ *if and only if* f *can be written in the form*

$$f(x) = x^\rho \ell(x) \exp\{h(\log x)\}, \quad x > 0, \tag{3.17}$$

where

(B1) $\rho \in \mathbf{R}$*;*
(B2) $(\ell(x), x > 0)$ *is an SV function;*
(B3) $(h(u), u \in \mathbf{R})$ *is a measurable function such that* $\sup_{u\in\mathbf{R}} |h(u)| < \infty$*;*
(B4) *for all* $u \in \mathbf{R}$*,*

$$\limsup_{x\to\infty}[h(u+x) - h(x)] = \mathfrak{p}(u) \quad \textit{and} \quad \liminf_{x\to\infty}[h(u+x) - h(x)] = -\mathfrak{p}(-u);$$

(B5) $(\mathfrak{p}(u), u \in \mathbf{R})$ *is a periodic function such that* $\mathfrak{p}(0) = 0$*, and*

$$-\infty < \inf_{u\in\mathbf{R}} \mathfrak{p}(u) \leqslant \sup_{u\in\mathbf{R}} \mathfrak{p}(u) < \infty.$$

Moreover, if (3.17) *and* (B1)–(B5) *hold, then*

(B6) ρ *is the index of* f*;*
(B7) *the set of periods of* $\mathfrak{p}$ *coincides with* $\mathbb{H}_r(f)$*;*
(B8) *for all* $u \in \mathbf{R}$*,*

$$\limsup_{x\to\infty}[\mathfrak{p}(u+x) - \mathfrak{p}(x)] = \mathfrak{p}(u) \quad \textit{and} \quad \liminf_{x\to\infty}[\mathfrak{p}(u+x) - \mathfrak{p}(x)] = -\mathfrak{p}(-u);$$

(B9) $\mathfrak{p}$ *is nonnegative, and* $\min_{u\in\mathbf{R}}\mathfrak{p}(u)=0$;
(B10) $\mathfrak{p}$ *is subadditive, that is,* $\mathfrak{p}(u+x)\leqslant\mathfrak{p}(u)+\mathfrak{p}(x)$ *for all* $u,x\in\mathbf{R}$.

The next statement follows from Theorem 3.5 in combination with a well–known result about infinitely differentiable variants of SV functions (cf. [**8**] and [**6**, Theorem 1.3.3]).

COROLLARY 3.12. *Let f be an ORV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with index ρ and periodic component $\mathcal{P}$. Then $f\sim f_1$, that is, $f(x)/f_1(x)\to 1$ as $x\to\infty$, where $f_1(x)=cx^\rho\ell_1(x)\exp\{h(\log x)\}$, $x>0$, with c a positive number, h as in Theorem* 3.5, $\mathfrak{p}=\log\mathcal{P}$, *and ℓ_1 an infinitely differentiable SV function such that, for all $n\in\mathbf{N}$,*

$$\lim_{x\to\infty}\frac{d^n h_1}{dx^n}(x)=0,$$

where $h_1(u)=\log\ell_1(e^u)$, $u\in\mathbf{R}$.

The next result, for which we first introduce some additional notation, complements Theorem 3.5.

DEFINITION 3.3. The function $(g(u),u\in\mathbf{R})$ is called *uniformly continuous at infinity* if, for every $\varepsilon>0$, there exist positive numbers $x'=x'(\varepsilon)$ and $\delta=\delta(\varepsilon)$ such that $|g(x_1)-g(x_2)|<\varepsilon$ for all $x_1,x_2\geqslant x'$ with $|x_1-x_2|<\delta$.

It is clear, that if the function $(g(u),u\in\mathbf{R})$ is uniformly continuous on $[A,\infty)$, for some $A\in\mathbf{R}$, then it is uniformly continuous at infinity.

DEFINITION 3.4. The function $(g(u),u\in\mathbf{R})$ is called *almost periodic at infinity* if, for every $\varepsilon>0$ and for all $x_1,x_2\in\mathbf{R}$, there exists a sequence of positive numbers $u_n=u_n(\varepsilon,x_1,x_2)$, $n\geqslant 1$, such that $u_n\to\infty$, as $n\to\infty$, and

$$\limsup_{n\to\infty}|g(x_i+u_n)-g(x_i)|<\varepsilon,\quad i=1,2.$$

Obviously, if the function $(g(u),u\in\mathbf{R})$ is almost periodic (in Bohr's sense, [**7**]), then it is almost periodic at infinity.

PROPOSITION 3.4. *Let f be an ORV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with periodic component $\mathcal{P}$. Then*

(i) *if $\mathcal{P}$ $[f^*]$ is continuous at* 0 [1], *then $\mathcal{P}$ and $\mathfrak{p}=\log\mathcal{P}$ are uniformly continuous on $\mathbf{R}$, and f^* is continuous on $\mathbf{R}_+$;*
(ii) *if $h=(h(u),u\in\mathbf{R})$ is uniformly continuous at infinity, then $\mathcal{P}$ and $\mathfrak{p}$ are uniformly continuous on $\mathbf{R}$, where $(h(u),u\in\mathbf{R})$ is as in* (3.17).

Moreover, if the function h is almost periodic at infinity, then, with $\mathfrak{p}=\log\mathcal{P}$,

(iii) $h(u+x)\leqslant h(x)+\mathfrak{p}(u)$ *for all* $u,x\in\mathbf{R}$, *and*
(iv) $|h(u+x)-h(x)|\leqslant\max\{|\mathfrak{p}(u)|,|\mathfrak{p}(-u)|\}$ *for all* $u,x\in\mathbf{R}$;
(v) *if $\mathfrak{p}$ $[f^*]$ is continuous at* 0 [1], *then h is uniformly continuous on $\mathbf{R}$, and f is continuous on $\mathbf{R}_+$;*
(vi) *h is a periodic function, and* $S_{\rm per}(h)=S_{\rm per}(\mathfrak{p})$.

By (vi) of Proposition 3.4, in the class of functions h being almost periodic at infinity, only periodic functions are relevant in representation (3.17)

Note that various integral representations for ORV functions with non-degenerate groups of regular points can be obtained from Theorems 3.4, 3.5 and Proposition 3.4 in combination with well-known integral representations for RV functions (cf. [**47, 6**]), and with integral representations for PRV functions (see Section 4).

Uniform convergence theorems for OWURV and ORV functions with non-degenerate groups of regular points. In this subsection some uniform convergence theorems for OWURV and ORV functions with non-degenerate groups of regular points will be presented. These theorems complete the results above, and, in combination with the well-known Uniform Convergence Theorem for RV functions (see, for example, [**47**, Theorem 1.2], and [**6**, Theorem 1.2.1]), they complement its counterpart for general ORV functions [**1**, Theorem 1].

The next statement refines (v) of Theorem 3.2, and (A8) of Theorem 3.4.

PROPOSITION 3.5. *Let f be an OWURV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with periodic component $\mathcal{P}$. Then*

$$\limsup_{x\to\infty} \sup_{-\infty<u<\infty} \left(\frac{\mathcal{P}(u+x)}{\mathcal{P}(x)} - \mathcal{P}(u) \right) = 0,$$

$$\liminf_{x\to\infty} \inf_{-\infty<u<\infty} \left(\frac{\mathcal{P}(u+x)}{\mathcal{P}(x)} - \frac{1}{\mathcal{P}(-u)} \right) = 0.$$

The next result complements [**1**], Theorem 1.

THEOREM 3.6. *Let f be an ORV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with index and periodic component ρ and $\mathcal{P}$, respectively. If $(\theta(x), x>0)$ in the representation* (3.16) *is uniformly continuous at infinity* (*recall Definition* 3.3)*, then, for any $[a,b]\subset \mathbf{R}_+$,*

$$\limsup_{x\to\infty} \sup_{\lambda\in[a,b]} \left(\frac{f(\lambda x)}{f(x)} - \lambda^{\rho}\mathcal{P}(\log\lambda) \right) = 0,$$

$$\liminf_{x\to\infty} \inf_{\lambda\in[a,b]} \left(\frac{f(\lambda x)}{f(x)} - \frac{\lambda^{\rho}}{\mathcal{P}(-\log\lambda)} \right) = 0.$$

COROLLARY 3.13. *Let f be an ORV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with periodic component $\mathcal{P}$. If $h=(h(u), u\in\mathbf{R})$ in the representation* (3.17) *is uniformly continuous at infinity, then, for any $[a,b]\subset\mathbf{R}$,*

$$\limsup_{x\to\infty} \sup_{u\in[a,b]} [h(u+x)-h(x)-\mathfrak{p}(u)] = 0,$$

$$\liminf_{x\to\infty} \inf_{u\in[a,b]} [h(u+x)-h(x)+\mathfrak{p}(-u)] = 0,$$

where $\mathfrak{p}=\log\mathcal{P}$.

From Corollary 3.13 in combination with Proposition 3.4 we also have:

Corollary 3.14. *Let f be an ORV function with non-degenerate group of regular points $\mathbb{G}_r(f)$, and with periodic component $\mathcal{P}$. If $h = (h(u), u \in \mathbf{R})$ in the representation* (3.17) *is almost periodic at infinity* (*recall Definition* 3.4), *and if f^** [$\mathfrak{p} = \log \mathcal{P}$] *is continuous at* 1 [0], *then $\mathfrak{p}$ and h are uniformly continuous on $\mathbf{R}$, h is periodic, and*

$$\limsup_{x\to\infty} \sup_{-\infty<u<\infty} [h(u+x) - h(x) - \mathfrak{p}(u)] = 0,$$

$$\liminf_{x\to\infty} \inf_{-\infty<u<\infty} [h(u+x) - h(x) + \mathfrak{p}(-u)] = 0.$$

4. On some properties of PRV, PMPV and POV functions

A representation theorem for prv functions and some characterizations of POV functions. There is a basic result concerning RV functions, namely the Representation Theorem, for which several proofs have been given in the literature (see, e.g., Karamata [**29**] and Bingham et al. [**6**]). For ORV functions, the Representation Theorem has been proved in Karamata [**31**] and Aljančić and Arandelović [**1**]. Here we briefly recall a Representation Theorem for PRV functions in the manner of Karamata's representation for RV functions. Moreover, as an application, we will obtain some equivalent characterizations of POV functions.

Recall that a function f is RV if and only if

$$f(t) = \exp\left\{\alpha(t) + \int_{t_0}^{t} \beta(s)\frac{ds}{s}\right\} \tag{4.1}$$

for some $t_0 > 0$ and all $t \geqslant t_0$, where α and β are bounded measurable functions such that the limits $\lim_{t\to\infty} \alpha(t)$ and $\lim_{t\to\infty} \beta(t)$ exist. For SV functions, $\lim_{t\to\infty} \beta(t) = 0$. ORV functions have the same characterization representation (4.1), with α and β only being bounded measurable functions (see Aljančić and Arandelović [**1**]).

Note that all of these representations are not unique. For example, one can start from a discontinuous function β and obtain a similar representation with other functions $\tilde{\alpha}$ and $\tilde{\beta}$, where $\tilde{\beta}$ is continuous or even infinitely differentiable.

The proof of the Representation Theorem for PRV functions is based on that for ORV functions (see Aljančić and Arandelović [**1**]).

Theorem 4.1. (Yakymiv [**53**], Buldygin et al. [**11**]) *A function f is PRV if and only if it has a representation* (3.1), *where α and β are bounded measurable functions such that*

$$\lim_{c\to 1} \limsup_{t\to\infty} |\alpha(ct) - \alpha(t)| = 0. \tag{4.2}$$

Remark 4.1. Condition (4.2) characterizes the so-called *slowly oscillating* functions (see Bingham et al. [**6**]).

Another representation for PRV functions is based on that for SV functions and the fact that $(f \circ \log)$ is an SV function for any PRV function f.

THEOREM 4.2. (Buldygin et al. [**11**]) *A function f is PRV if and only if*

$$f(t)=\exp\left\{a(t)+\int_{t_0}^{t}b(s)\,ds\right\}$$

for some $t_0>0$ and all $t\geqslant t_0$, where a and b are measurable functions such that the limit $\lim_{t\to\infty}a(t)$ *exists,* $\lim_{t\to\infty}b(t)=0$*, and*

$$\lim_{c\to 1}\limsup_{t\to\infty}\int_{t_0}^{ct}b(s)\,ds=\lim_{c\to 1}\liminf_{t\to\infty}\int_{t_0}^{ct}b(s)\,ds=0.$$

Next, we discuss some characterizations for POV functions.

PROPOSITION 4.1. *Let f be a PRV function. Then condition* (2.3) *is equivalent to any of the following two conditions:*

1) *for any sequence of positive numbers $\{c_n\}$ such that* $\limsup_{n\to\infty}c_n>1$ *and for any sequence of positive numbers $\{t_n\}$ such that* $\lim_{n\to\infty}t_n=\infty$*, one has*

$$\limsup_{n\to\infty}\frac{f(c_nt_n)}{f(t_n)}>1; \tag{4.3}$$

2) *for any sequence of positive numbers $\{c_n\}$ such that* $\limsup_{n\to\infty}c_n\in(1,\infty)$ *and for any sequence of positive numbers $\{t_n\}$ such that* $\lim_{n\to\infty}t_n=\infty$*, condition* (4.3) *holds.*

We need the following auxiliary result.

LEMMA 4.1. *Let f be a POV function. Then, for any sequence of positive numbers $\{c_n\}$ and for any sequence of positive numbers $\{t_n\}$ such that* $\lim_{n\to\infty}c_n=\infty$ *and* $\lim_{n\to\infty}t_n=\infty$*, one has*

$$\lim_{n\to\infty}\frac{f(c_nt_n)}{f(t_n)}=\infty.$$

COROLLARY 4.1. *A function $f\in\mathbb{F}_+$ is a POV function if and only if f is a PRV function and, for any sequence of positive numbers $\{c_n\}$ such that* $\limsup_{n\to\infty}c_n>1$ *and for any sequence of positive numbers $\{t_n\}$ with* $\lim_{n\to\infty}t_n=\infty$*, condition* (4.3) *holds.*

REMARK 4.2. If $f\in\mathbb{F}_+$ is a POV function, then $f\in\mathbb{F}^\infty$.

Increasing versions of POV functions. Many problems related to POV functions become easier if the functions are monotone. Hence, we next consider the problem of existence of strictly increasing versions for POV functions.

THEOREM 4.3. *Assume that f is a POV function. Then there exists a strictly increasing and continuous POV function f_1 tending to ∞ such that $f\sim f_1$.*

Potter bounds for PRV functions. Due to the Representation Theorem and the Uniform Convergence Theorem, we can prove a variant of Potter's Theorem [**44**] (see, e.g., Bingham et al. [**6**, p.25]) for PRV functions. The following theorem improves Corollary 4 of Yakymiv [**53**].

Theorem 4.4. *Let f be a PRV function. Then there exists some $b > 0$ and, for any $A > 1$, there exist $\lambda_A > 1$, and $t_A > 0$ such that, for all $\lambda \in (1, \lambda_A]$,*

$$A^{-1}\lambda^{-b} f(s) \leqslant f(t) \leqslant A\lambda^{b} f(s) \tag{4.4}$$

for all $t \geqslant t_A$ and all $s \in [\lambda^{-1}t, \lambda t]$.

Remark 4.3. Let $f \in \mathbb{F}_+$. If condition (4.4) holds then f is a WPRV function.

The following result provides a characterization of PRV functions in terms of Potter bounds.

Corollary 4.2. *Let $f \in \mathbb{F}_+$ be measurable. Then f is a PRV function if and only if condition* (4.4) *holds.*

Conditions for differentiable functions to be PRV or PMPV. The solutions of the main application problems in this paper (see Section 7) are closely connected with the question of when differentiable functions satisfy PRV or PMPV conditions. In this subsection, the latter question will be discussed. In the sequel, the following five conditions on a function f and its derivative f' will play a role:

- (D) $f \in \mathbb{F}^{\infty}$ and there exists $t_0 = t_0(f) > 0$ such that f is positive and continuously differentiable for all $t \geqslant t_0$;
- (DM) condition (D) holds and $f'(t) \geqslant 0$ for all $t \geqslant t_0$;
- (DM+) condition (D) holds and $f'(t) > 0$ for all $t \geqslant t_0$;
- (DM1) condition (DM+) holds and f' is nonincreasing for all $t \geqslant t_0$;
- (DM2) condition (DM+) holds and f' is nondecreasing for all $t \geqslant t_0$.

For a function f satisfying condition (D), the following integral representation holds:

$$f(t) = f(t_0) \exp\left\{\int_{t_0}^{t} \frac{f'(u)}{f(u)} du\right\} \tag{4.5}$$

for any $t > t_0$.

Conditions for differentiable functions to be PRV. The following statement is immediate from Definition 2.4 in combination with (4.5).

Lemma 4.2. *Assume condition* (D). *Then f is a PRV function if and only if*

$$\lim_{c \to 1} \limsup_{t \to \infty} \int_{t}^{ct} \frac{f'(u)}{f(u)} du = 0.$$

On applying Lemma 4.2 in combination with Remark 2.2, we get the following result.

Lemma 4.3. *Assume condition* (DM). *Then f is a PRV function if and only if*

$$\lim_{c \downarrow 1} \limsup_{t \to \infty} \int_{t}^{ct} \frac{f'(u)}{f(u)} du = 0.$$

Let us consider some corollaries of the above lemmas.

Corollary 4.3. *Assume condition* (D).

1) *If* $\limsup_{t\to\infty} \dfrac{t|f'(t)|}{f(t)} < \infty$, *then* f *is a PRV function.*

2) *If* f *is a PRV function, then* $\liminf_{t\to\infty} \dfrac{tf'(t)}{f(t)} < \infty$.

3) *If condition* (DM) *holds and*

$$\limsup_{t\to\infty} \frac{tf'(t)}{f(t)} < \infty, \tag{4.6}$$

then f *is a PRV function.*

4) *If condition* (DM1) *holds, then* f *is a PRV function.*

REMARK 4.4. If condition (D) holds and $\limsup_{t\to\infty} t|f'(t)| < \infty$, then $f^*(c) = 1$ for all $c > 0$. This means that f is an SV function, and hence it is a PRV function. Thus we can confine ourselves to the case, when $\limsup_{t\to\infty} t|f'(t)| = \infty$.

COROLLARY 4.4. *Assume condition* (DM2). *Then* f *is a PRV function if and only if* (4.6) *holds true.*

The integral in the next statement means the Lebesgue integral.

COROLLARY 4.5. *Assume condition* (DM+). *If*

$$\int_{0+}^{1} (f')_*(c)dc > 0,$$

then f *is a PRV function.*

On applying Corollary 4.5, we get the following result.

COROLLARY 4.6. *Assume condition* (DM+). *If the set* $\{c\in(0,1]:(f')_*(c)>0\}$ *has positive Lebesgue measure, then* f *is a PRV function. In particular, this condition holds if* f' *is an ORV function.*

Conditions for differentiable functions to be PMPV. The following statement is also immediate from Definition 2.5 in combination with (4.5).

LEMMA 4.4. *Assume condition* (D). *Then* f *is a PMPV function if and only if*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{f'(u)}{f(u)} du > 0 \quad \textit{for all } c > 1.$$

Next, we consider some corollaries of Lemma 4.4.

COROLLARY 4.7. *Assume condition* (DM).

1) *If*

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} > 0, \tag{4.7}$$

then f *is a PMPV function.*

2) *If* f *is a PMPV function, then* $\limsup_{t\to\infty} \dfrac{tf'(t)}{f(t)} > 0$.

3) *If* f *is a PMPV function, then* $\limsup_{t\to\infty} tf'(t) = \infty$.

4) *If condition* (DM2) *holds, then f is a PMPV function.*

COROLLARY 4.8. *Assume condition* (DM1). *Then f is a PMPV function if and only if* (4.7) *holds true.*

The next result gives a condition in terms of the function $(f')_*$.

LEMMA 4.5. *Assume condition* (DM+). *If $c(f')_*(c) > 1$ for all $c > 1$, then f is a PMPV function.*

5. Asymptotic quasi-inverse and asymptotic inverse functions

In this section we study asymptotic quasi-inverse and asymptotic inverse functions and investigate the problem of their existence. Theorem 5.1 below shows that any PRV function $f \in \mathbb{F}^\infty$ has an asymptotic quasi-inverse function, and Theorem 5.2 proves that any POV function has an asymptotic inverse function.

Moreover, in this section we discuss conditions under which quasi-inverse functions preserve the equivalence of functions.

First, we recall the definition of a *quasi-inverse function* which will be useful for our considerations below (cf. Buldygin et al. [**11**]).

DEFINITION 5.1. Let $f \in \mathbb{F}^{(\infty)}$. A function $f^{(-1)} \in \mathbb{F}^\infty$ is called a *quasi-inverse function* for f if $f(f^{(-1)}(s))) = s$ for all large s.

For any $f \in \mathbb{C}^{(\infty)}$, a quasi-inverse function exists, but may not be unique. If $f \in \mathbb{C}^\infty_{\text{inc}}$, then its *inverse function* $f^{-1}(\cdot)$ exists, that is, $f(f^{-1}(s)) = s$ and $f^{-1}(f(t)) = t$ for all sufficiently large s and t.

EXAMPLE 5.1. Let $x \in \mathbb{C}^{(\infty)}$. Put

$$x_1^{(-1)}(s) = \inf\{t \geqslant 0 : x(t) = s\} = \inf\{t \geqslant 0 : x(t) \geqslant s\},$$

for $s \geqslant s_0 = x(0)$, and $x_1^{(-1)}(s) = 0$, for $0 \leqslant s < s_0$, if $s_0 > 0$. The function $x_1^{(-1)}$ is a quasi-inverse function for x. If $x \in \mathbb{C}^\infty_{\text{inc}}$, then $x_1^{(-1)} = x^{-1}$. □

EXAMPLE 5.2. Let $x \in \mathbb{C}^\infty$. Put

$$x_2^{(-1)}(s) = \sup\{t \geqslant 0 : x(t) = s\} = \sup\{t \geqslant 0 : x(t) \leqslant s\},$$

for $s \geqslant s_0 = x(0)$, and $x_2^{(-1)}(s) = 0$, for $0 \leqslant s < s_0$, if $s_0 > 0$. The function $x_2^{(-1)}$ is a quasi-inverse function for x. Observe that $x_1^{(-1)}(s) \leqslant x_2^{(-1)}(s)$, $s > 0$, and, in general, $x_1^{(-1)} \neq x_2^{(-1)}$. If $x \in \mathbb{C}^\infty_{\text{inc}}$, then $x_2^{(-1)} = x_1^{(-1)} = x^{-1}$. □

EXAMPLE 5.3. Let f be a POV function. Then, by Theorem 4.3, f has a strictly increasing and continuous version g, which, in turn, has an inverse function g^{-1}.

Next we introduce the notion of an *asymptotic quasi-inverse function.*

DEFINITION 5.2. Let $f \in \mathbb{F}^{(\infty)}$. A function $\tilde{f}^{(-1)}$ is called an *asymptotic quasi-inverse function* for f if (i) $\tilde{f}^{(-1)} \in \mathbb{F}^\infty$; (ii) $f(\tilde{f}^{(-1)}(s)) \sim s$ as $s \to \infty$.

Of course, every quasi-inverse function is an *asymptotic quasi-inverse* function. Also recall (cf. Bingham et al. [**6**]) that a function $\tilde{f}^{-1}$ is called an *asymptotic inverse function* for $f \in \mathbb{F}^{(\infty)}$ if $\tilde{f}^{-1}$ is an *asymptotic quasi-inverse function* and

(iii) $\tilde{f}^{-1}(f(s)) \sim s \quad \text{as} \quad s \to \infty.$

It is clear that any inverse function is an *asymptotic inverse function.* If $f \in \mathbb{F}^{\infty}$ and, if an asymptotic inverse function $\tilde{f}^{-1}$ exists, then f is also an asymptotic inverse function for $\tilde{f}^{-1}$.

Remark 5.1. Let $f \in \mathbb{F}^{(\infty)}$ and let $\tilde{f}^{(-1)}$ be an asymptotic quasi-inverse function for f.

(A) If a function g is asymptotically equivalent to f, then $\tilde{f}^{(-1)}$ is an asymptotic quasi-inverse function for g.
(B) If f is a WPRV function and g is asymptotically equivalent to $\tilde{f}^{(-1)}$, then, by Theorem 2.1, g is an asymptotic quasi-inverse function for f.

Example 5.4. Let f be an RV function with positive index α. Then (see, e.g., Bingham et al. [**6**, p.28]) there exists an asymptotic inverse function $\tilde{f}^{-1}$ which is an RV function with index $1/\alpha$, and one version of this function is

$$\tilde{f}^{-1}(s) = \inf\{t \geqslant 0 : f(t) > s\}.$$

In this case, the asymptotic inverse function $\tilde{f}^{-1}$ is uniquely determined up to asymptotic equivalence. □

Next we shall consider some important examples of asymptotic quasi-inverse functions for PRV functions.

Lemma 5.1. *Let f be a PRV function.*

1) *If $f \in \mathbb{F}^{(\infty)}$, then*

$$\varphi(s) = \inf\{t \geqslant 0 : f(t) > s\}, \quad \varphi_1(s) = \inf\{t \geqslant 0 : f(t) \geqslant s\}$$

are nondecreasing asymptotic quasi-inverse functions for f.

2) *If $f \in \mathbb{F}^{\infty}$, then φ, φ_1 and*

$$\psi(s) = \sup\{t \geqslant 0 : f(t) < s\}, \quad \psi_1(s) = \sup\{t \geqslant 0 : f(t) \leqslant s\}$$

are nondecreasing asymptotic quasi-inverse functions for f.

Lemma 5.2. *Let $f \in \mathbb{F}^{\infty}$. If the functions φ_1 and ψ_1 of Lemma* 5.1 *are asymptotically equivalent, then f is an asymptotic quasi-inverse function for both φ_1 and ψ_1.*

On combining the above lemmas we get the following result.

Theorem 5.1. *Let $f \in \mathbb{F}^{\infty}$ be a PRV function. Then, the functions $\varphi_1 \leqslant \varphi \leqslant \psi \leqslant \psi_1$ of Lemma* 5.1 *are nondecreasing asymptotic quasi-inverse functions for f. Moreover, if the functions φ_1 and ψ_1 are asymptotically equivalent, then all these four functions are asymptotically equivalent and are nondecreasing asymptotic inverse functions for f.*

Next, we discuss some conditions, given in terms of the function f itself, which guarantee the existence of an asymptotic inverse function.

THEOREM 5.2. *Let $f \in \mathbb{F}_+$ be a POV function. Then, the four functions φ_1, φ, ψ, ψ_1 of Lemma* 5.1 *are asymptotically equivalent and are nondecreasing asymptotic inverse functions for f.*

In general, the functions φ_1 and ψ_1 of Lemma 5.1 are either not asymptotically equivalent or are not quasi-inverse for the original function f.

EXAMPLE 5.5. Let $f(t) = [\log t], t \geqslant 1$, where $[x]$ denotes the integer part of a real number x. The function f is an SV function, and hence is not POV, but is PRV. By Lemma 5.1, the functions φ_1 and ψ_1 are asymptotic quasi-inverse functions for f. Nevertheless, φ_1 and ψ_1 are not asymptotically equivalent. Moreover, the function $(e^t, t \geqslant 0)$ is also an asymptotic quasi-inverse function for f. Note, that $[\log t] \sim \log t$ as $t \to \infty$, and that $(\log t, t \geqslant 1)$ is not a POV function. But, for the latter function, $(e^t, t \geqslant 0)$ is the inverse function and φ_1 and ψ_1 are equal to $(e^t, t \geqslant 0)$. □

EXAMPLE 5.6. The functions φ_1 and ψ_1 of Lemma 5.1 are not asymptotic quasi-inverse functions for $f(t) = e^{[t]}, t \geqslant 0$. Nevertheless, φ_1 and ψ_1 are asymptotically equivalent. Observe that f is not PRV. □

For non-PRV functions, asymptotic quasi-inverse functions may not exist at all.

EXAMPLE 5.7. For the non-PRV function $f(t) = e^{[t]}, t \geqslant 0$, there is no asymptotic quasi-inverse function.

Indeed, for any $\varphi \in \mathbb{F}_+$ and for $t_n = \exp\{n + \frac{1}{2}\}$, $n \geqslant 1$, we have that either $[\varphi(t_n)] \geqslant n + 1$ or $[\varphi(t_n)] \leqslant n$, which implies that either

$$\begin{aligned}\limsup_{t\to\infty} \frac{\exp\{[\varphi(t)]\}}{t} &\geqslant \limsup_{n\to\infty} \frac{\exp\{[\varphi(t_n)]\}}{t_n} \\ &= \exp\left\{\limsup_{n\to\infty}\left([\varphi(t_n)] - n - \frac{1}{2}\right)\right\} \geqslant \exp\left\{\frac{1}{2}\right\} > 1,\end{aligned}$$

or

$$\begin{aligned}\liminf_{t\to\infty} \frac{\exp\{[\varphi(t)]\}}{t} &\leqslant \liminf_{n\to\infty} \frac{\exp\{[\varphi(t_n)]\}}{t_n} \\ &= \exp\left\{\liminf_{n\to\infty}\left([\varphi(t_n)] - n - \frac{1}{2}\right)\right\} \leqslant \exp\left\{-\frac{1}{2}\right\} < 1. \quad \square\end{aligned}$$

EXAMPLE 5.8. By Example 5.7 and Remark 5.1, the function $f(t) = \frac{t}{[t]} \cdot e^{[t]}$, $t \geqslant 1$, does not have an asymptotic quasi-inverse function, since $f(t) \sim e^{[t]}$ as $t \to \infty$. Observe, that the function f is strictly increasing. □

EXAMPLE 5.9. Let $a \in \mathbb{F}_+$. By the method of Example 5.7, the function $(e^{[a(t)]}, t > 0)$ does not have an asymptotic quasi-inverse function. Hence, by Lemma 5.1, the function $(e^{[a(t)]}, t > 0)$ is not PRV for any $a \in \mathbb{F}^{(\infty)}$. □

The PRV property for asymptotic quasi-inverse functions. In this subsection, we discuss conditions under which quasi-inverse functions preserve the equivalence of functions, i.e., in view of Theorem 2.1, are WPRV (or PRV).

LEMMA 5.3. *Let $f \in \mathbb{F}^{(\infty)}$. Then, an asymptotic quasi-inverse function $\tilde{f}^{(-1)}$ for f is WPRV (and thus preserves the equivalence of functions) if and only if*

$$\limsup_{c\to 1} \limsup_{t\to\infty} \frac{\tilde{f}^{(-1)}(ct)}{\tilde{f}^{(-1)}(t)} = 1.$$

PROPOSITION 5.1. *Let $f \in \mathbb{F}^{\infty}_{\text{ndec}}$ and let $\tilde{f}^{(-1)}$ be an asymptotic quasi-inverse function for f. If* (2.3) *holds, then $\tilde{f}^{(-1)}$ is WPRV* (*and thus preserves the equivalence of functions*).

THEOREM 5.3. *Let $f \in \mathbb{F}^{\infty}_{\text{ndec}}$ and let $\tilde{f}^{-1}$ be an asymptotic inverse function for f. Then $\tilde{f}^{-1}$ is WPRV (and thus preserves the equivalence of functions) if and only if condition* (2.3) *holds.*

COROLLARY 5.1. *Let $f \in \mathbb{F}^{\infty}$ and let $\tilde{f}^{-1}$ be a nondecreasing asymptotic inverse function for f. Then f is WPRV (and preserves the equivalence of functions) if and only if the following condition holds:*

$$(\tilde{f}^{-1})_*(c) = \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(ct)}{\tilde{f}^{-1}(t)} > 1 \qquad \text{for all } c > 1. \tag{5.1}$$

COROLLARY 5.2. (Buldygin et al. [**11**]) *Let $f \in \mathbb{C}^{\infty}_{\text{inc}}$. Then its inverse function f^{-1} is PRV (and thus preserves the equivalence of functions) if and only if condition* (2.3) *holds. Moreover, f is a PRV function if and only if*

$$(f^{-1})_*(c) = \liminf_{t\to\infty} \frac{f^{-1}(ct)}{f^{-1}(t)} > 1 \qquad \text{for all } c > 1.$$

6. Properties and characterizations of POV functions and their asymptotic quasi-inverses. Related limit results

The main results of this section (Theorems 6.1–6.4) show that the class of POV function is similar to the class of RV functions with positive index.

PROPOSITION 6.1. *Let $f \in \mathbb{F}^{(\infty)}$ be a WPRV function and let $\tilde{f}^{(-1)}$ be an asymptotic quasi-inverse function for f. Then,*

1) *if there exists a nondecreasing function g such that $f \sim g$, then condition* (5.1) *holds* (*with $\tilde{f}^{(-1)}$ replacing $\tilde{f}^{-1}$*);
2) *if there exists a nondecreasing function g such that $\tilde{f}^{(-1)} \sim g$, then condition* (5.1) *holds* (*with $\tilde{f}^{(-1)}$ replacing $\tilde{f}^{-1}$*);
3) *if f is a POV function, then $\tilde{f}^{(-1)}$ is a WPRV function;*
4) *if f is a POV function, then $\tilde{f}^{(-1)}$ is a WPOV function.*

COROLLARY 6.1. *Let $f \in \mathbb{F}^{\infty}$ and let $\tilde{f}^{(-1)}$ be an asymptotic inverse function for f. Assume that $\tilde{f}^{(-1)}$ is a PRV function. Then,*

1) *if there exists a nondecreasing function g which is asymptotically equivalent either to f or to $\tilde{f}^{(-1)}$, then condition* (2.3) *holds;*
2) *if $\tilde{f}^{(-1)}$ is a POV function, then f is a WPRV function;*
3) *if $\tilde{f}^{(-1)}$ is a POV function, then f is a WPOV function.*

Applying Proposition 6.1, Corollary 6.1 and Theorem 2.1 we get the following result.

THEOREM 6.1. *Let both $f \in \mathbb{F}^{\infty}$ and its asymptotic inverse function $\tilde{f}^{-1}$ be measurable functions. Assume that there exists a nondecreasing function g which is asymptotically equivalent either to f or to $\tilde{f}^{(-1)}$. Then, the following four conditions are equivalent:*

(a) *f is POV;* (b) *$\tilde{f}^{-1}$ is POV;* (c) *both f and $\tilde{f}^{-1}$ are PRV;*
(d) *both f and $\tilde{f}^{-1}$ preserve the equivalence of functions and sequences.*

THEOREM 6.2. *Let $f \in \mathbb{F}_+$ be a POV function. Then, the four functions φ_1, φ, ψ, ψ_1 from Lemma* 5.1 *are asymptotically equivalent and are nondecreasing asymptotic inverse functions for f. Moreover, each of these functions possesses the POV property.*

Theorem 6.2 follows from Theorems 5.2 and 6.1, since every nondecreasing function is measurable.

THEOREM 6.3. *Let $f \in \mathbb{F}_+$ be a POV function and let $\tilde{f}^{-1}$ be an asymptotic inverse function for f. If q is an asymptotic quasi-inverse function for f, then*

1) *$q \sim \tilde{f}^{-1}$;* 2) *q is an asymptotic inverse function for f.*

This means that an asymptotic quasi-inverse function for f is also an asymptotic inverse function, and it is uniquely determined up to asymptotic equivalence.

The next result complements Theorem 6.3.

THEOREM 6.4. *Let $f \in \mathbb{F}_+$ be a POV function. Then*

1) *there exists a continuous POV function f_0, asymptotically equivalent to f and strictly increasing to ∞, for which the inverse function f_0^{-1} is a continuous POV function, strictly increasing to ∞;*
2) *f_0^{-1} is an asymptotic inverse function for f;*
3) *if $g \sim f$, then any asymptotic quasi-inverse function for g is asymptotically equivalent to any asymptotic quasi-inverse function for f.*

Limit behavior of ratios of asymptotic quasi-inverse functions. The following theorem extends Theorems 6.3 and 6.4 to the case of functions satisfying condition (2.3).

THEOREM 6.5. *Let f be a WPMPV function, that is, let f satisfy condition* (2.3). *If $f \sim f_0 \in \mathbb{F}^{\infty}_{\mathrm{ndec}}$, and $\tilde{f}_0^{-1}$ is an asymptotic inverse function for f_0, then*

1) *$\tilde{f}_0^{-1}$ is an asymptotic inverse function for f;*
2) *any asymptotic quasi-inverse function for f is equivalent to $\tilde{f}_0^{-1}$ and is a version of an asymptotic inverse function for f.*

This means that all asymptotic quasi-inverse functions for f are asymptotically equivalent, and that the asymptotic inverse function $\tilde{f}^{-1}$ is uniquely determined up to asymptotic equivalence.

The following results describe the relationship between the limiting behavior of the ratio of asymptotic quasi-inverse functions and that of their original functions.

COROLLARY 6.2. *Let f be a WPMPV function, that is f satisfies condition* (2.3). *Assume that $f \sim f_0 \in \mathbb{F}^\infty_{\text{ndec}}$, and $\tilde{f}_0^{-1}$ is an asymptotic inverse function for f_0. If, for some function $x \in \mathbb{F}^\infty$,*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \qquad \text{for some } a \in (0,\infty), \tag{6.1}$$

then, for any asymptotic quasi-inverse function $\tilde{x}^{(-1)}$ of x and for any asymptotic quasi-inverse function $\tilde{f}^{(-1)}$ of f, we have

$$\lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s/a)} = \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}_0^{-1}(s/a)} = 1. \tag{6.2}$$

COROLLARY 6.3. (Buldygin et al. [**11**]) *Let $f \in \mathbb{C}^\infty_{\text{inc}}$ and let f satisfy condition* (2.3)*. If, for some function $x \in \mathbb{F}^\infty$,*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \qquad \text{for some } a \in (0,\infty),$$

then, for any quasi-inverse function $x^{(-1)}$ of x, we have $\lim_{s\to\infty} \frac{x^{(-1)}(s)}{f^{-1}(s/a)} = 1.$

Corollary 6.2 and Examples 5.1 and 5.2 imply the following result.

COROLLARY 6.4. *Let $f \sim f_0 \in \mathbb{F}^\infty_{\text{ndec}}$ and let $\tilde{f}^{-1}$ be an asymptotic inverse function for f_0. Assume that condition* (2.3) *and relation* (6.1) *hold for some function $x \in \mathbb{F}^\infty$. Then,*

1) *if $x \in \mathbb{C}^{(\infty)}$, then* (6.2) *holds for $\tilde{x}_1^{(-1)}(s) = \inf\{t \geqslant 0 : x(t) \geqslant s\}$;*
2) *if $x \in \mathbb{C}^\infty$, then* (6.2) *holds for $\tilde{x}_2^{(-1)}(s) = \sup\{t \geqslant 0 : x(t) \leqslant s\}$;*
3) *if $x \in \mathbb{C}^\infty$, then*

$$\lim_{s\to\infty} \frac{z(s)}{\tilde{f}^{-1}(s/a)} = 1$$

for any function z satisfying $\tilde{x}_1^{(-1)}(s) \leqslant z(s) \leqslant \tilde{x}_2^{(-1)}(s)$ for all large s.

The case of POV functions. For POV functions we have a more complete result compared to that of Theorem 6.5.

THEOREM 6.6. *Let $f \in \mathbb{F}_+$ be a POV function and let $\tilde{f}^{(-1)}$ be an asymptotic quasi-inverse function of f. Assume that $x \in \mathbb{F}^\infty$ and $\tilde{x}^{(-1)}$ is an asymptotic quasi-inverse function of x. Then, we have*

1) *relations* (6.1) *and* (6.2) *are equivalent, that is*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \in (0,\infty) \quad \iff \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{(-1)}(s/a)} = 1;$$

2) *if* (6.1) *holds and* x *is measurable, then it is a POV function and, as an asymptotic quasi-inverse function* $\tilde{x}^{(-1)}$ *for* x, *we can take in* 1) *the function* $\tilde{x}_1^{(-1)}(s) = \inf\{t \geqslant 0 : x(t) \geqslant s\}$ *or* $\tilde{x}_2^{(-1)}(s) = \sup\{t \geqslant 0 : x(t) \leqslant s\}$*;*

3) *if* x *is measurable, then the relation* (6.1) *implies the relation*

$$\lim_{s\to\infty} \frac{z(s)}{\tilde{f}^{(-1)}(s/a)} = 1,$$

for any function z *satisfying* $\tilde{x}_1^{(-1)}(s) \leqslant z(s) \leqslant \tilde{x}_2^{(-1)}(s)$ *for all large* s.

The proof of Theorem 6.6 follows from Theorems 6.2, 6.3 and 6.4.

Zero and infinite limits of ratios. The following results discuss relationships between the limiting behavior of the ratio of asymptotic quasi-inverse functions in case the limit of the ratio of the original functions equals 0 or ∞. In this situation, Corollary 6.2 can be retained for zero and infinite limits, but with the additional condition that

$$\liminf_{s\to\infty} \frac{\tilde{f}^{-1}(c_0 s)}{\tilde{f}^{-1}(s)} > 1 \quad \text{for some } c_0 > 1. \tag{6.3}$$

Proposition 6.2. *Let* $f \in \mathbb{F}^{\infty}_{\text{ndec}}$, *and let* $\tilde{f}^{-1} \in \mathbb{F}^{\infty}_{\text{ndec}}$ *be an asymptotic inverse for* f. *Assume that* $x \in \mathbb{F}^{(\infty)}$, *and let* $\tilde{x}^{(-1)}$ *be an asymptotic quasi-inverse function for* x. *Then, under conditions* (6.3) *and* (2.3), *the following relations hold:*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = \infty \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = 0\,;$$
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = 0 \quad \Longrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = \infty.$$

For POV functions the latter result reads as follows.

Proposition 6.3. *Let* f *be a POV function, and let* $\tilde{f}^{-1}$ *be an asymptotic inverse for* f. *Assume that* $x \in \mathbb{F}^{(\infty)}$ *and* $\tilde{x}^{(-1)}$ *is an asymptotic quasi-inverse function for* x. *Then the following relations hold:*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = \infty \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = 0\,;$$
$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = 0 \quad \Longleftrightarrow \quad \lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = \infty.$$

Proposition 6.3 follows from Proposition 6.2, Theorem 6.3 and Theorem 6.4.

Remark 6.1. Proposition 6.3 extends Theorem 6.6 and allows for considering $a = 0$ or ∞ in the limiting relations (6.1) and (6.2).

Limit behavior of the ratio of asymptotic quasi-inverse functions for RV functions. For RV functions, Theorem 6.6 and Proposition 6.3 read as follows.

COROLLARY 6.5. *Let f be an RV function with positive index α, and let $\tilde{f}^{-1}$ be an asymptotic inverse function for f. Assume that $x \in \mathbb{F}^{(\infty)}$ and $\tilde{x}^{(-1)}$ is an asymptotic quasi-inverse function for x. Then, we have*

1) *the relation*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \in [0,\infty] \tag{6.4}$$

is equivalent to

$$\lim_{s\to\infty} \frac{\tilde{x}^{(-1)}(s)}{\tilde{f}^{-1}(s)} = \left(\frac{1}{a}\right)^{1/\alpha} \in [0,\infty]; \tag{6.5}$$

Here and in the sequel, it is assumed that $(1/\infty) = 0$ and $(1/0) = \infty$.

2) *if (6.4) holds and x is measurable, then, as a quasi-inverse function $\tilde{x}^{(-1)}$ for x, we can take in (6.5) the function $\tilde{x}_1^{(-1)}(s) = \inf\{t \geqslant 0 : x(t) \geqslant s\}$ or $\tilde{x}_2^{(-1)}(s) = \sup\{t \geqslant 0 : x(t) \leqslant s\}$;*

3) *moreover, if x is measurable then (6.4) implies the relation*

$$\lim_{s\to\infty} \frac{z(s)}{\tilde{f}^{-1}(s)} = \left(\frac{1}{a}\right)^{1/\alpha} \in [0,\infty],$$

for any function z such that $\tilde{x}_1^{(-1)}(s) \leqslant z(s) \leqslant \tilde{x}_2^{(-1)}(s)$ for all large s.

REMARK 6.2. In statement 3) of Corollaries 6.4–6.5 and Theorem 6.6, we can take z as any function satisfying $\tilde{x}_1^{(-1)}(s) - \delta_1(s) \leqslant z(s) \leqslant \tilde{x}_2^{(-1)}(s) + \delta_2(s)$ for all large s, where δ_1 and δ_2 are two nonnegative functions for which

$$\lim_{s\to\infty} \frac{\delta_1(s)}{\tilde{f}^{-1}(s/a)} = 0 \quad \text{and} \quad \lim_{s\to\infty} \frac{\delta_2(s)}{\tilde{f}^{-1}(s/a)} = 0.$$

7. PRV and PMPV properties of functions and the asymptotic behavior of solutions of stochastic differential equations

Introduction. Gihman and Skorohod [**24**, §17] and later Keller et al. [**32**], considered the asymptotic behavior, as $t \to \infty$, of a solution $X = (X(t), t \geqslant 0)$ of the stochastic differential equation (SDE)

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t),\ t \geqslant 0, \quad X(0) \equiv 1. \tag{7.1}$$

Here W is a standard Wiener process and X denotes the Itô-solution of (7.1). One of the basic assumptions in the above papers was that both $\sigma = (\sigma(x), -\infty < x < \infty)$ and $g = (g(x), -\infty < x < \infty)$ are positive functions, and the authors were only interested in situations, in which the event $\{\lim_{t\to\infty} X(t) = \infty\}$ occurs with positive probability and such that infinity will not be reached in finite time.

Gihman and Skorohod [**24**, §17] and Keller et al. [**32**] gave conditions under which the asymptotic of $X(t)$, as $t \to \infty$, is determined by a nonrandom function. In this section, we reconsider this problem under the same basic conditions.

Denote by $\mu = (\mu(t), t \geqslant 0)$ the solution of the deterministic differential equation corresponding to (7.1) for $\sigma \equiv 0$, i.e.

$$d\mu(t) = g(\mu(t))dt,\ t \geqslant 0, \quad \mu(0) = 1. \tag{7.2}$$

We assume that the function g is such that the solution μ exists, is unique, tends to ∞, as $t \to \infty$, and that infinity will not be reached in finite time. Then an interesting question is, under which conditions it holds that

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = 1 \quad \text{a.e. on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\}. \tag{7.3}$$

Here "a.e." stands for "almost everywhere", which means that the property holds except for a subset of probability 0. The methods used in Gihman and Skorohod [**24**, S17 Theorem 4] and in Keller et al. [**32**] are similar and consist of two main steps. First, they study the process $Y(t) = G(X(t)),\ t \geqslant 0$, where

$$G(t) = \int_1^t \frac{ds}{g(s)}\,,\ t \geqslant 0, \tag{7.4}$$

and prove that, under some conditions,

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = 1 \quad \text{a.e. on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\}. \tag{7.5}$$

Note that $G = (G(t), t \geqslant 1)$ is the inverse function of μ ($G = \mu^{-1}$) if g is positive and continuous. In the second step, relation (7.5) is used to prove (7.3).

For the second step, Gihman and Skorohod [**24**, §17, Theorem 4] assume that, for some $C > 0$,

$$\lim_{\varepsilon\to 0} \sup_{z>C} \sup_{|z/u-1|\leqslant\varepsilon} \left|\frac{\mu(z)}{\mu(u)} - 1\right| = 0. \tag{7.6}$$

By Theorem 2.1 and Remark 2.2, under condition (7.6), the function μ preserves the equivalence of functions, so that (7.6) implies in this case that

$$\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = \lim_{t\to\infty} \frac{\mu(G(X(t)))}{\mu(t)} = 1 \quad \text{a.e. on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\},$$

that is, relation (7.3) holds. Note that Gihman and Skorohod [**24**] use another reasoning; the general idea above, however, simplifies the proof considerably.

Condition (7.6) is formulated in terms of the function μ, that is in terms of the solution of equation (7.2). It is more natural, however, to give conditions in terms of the functions g and G. Our goal in this section is to find conditions for the implication (7.6) $\Longrightarrow$ (7.3) expressed in terms of the functions g and G. For doing so, we follow the general approach developed in Sections 4–6. This approach allows for solving the following general problem: Find conditions on a given function under which its inverse or quasi-inverse function preserves the equivalence of functions.

Further in this section, we study the asymptotic stability with respect to initial conditions of the solution of SDE (7.1), as well as the asymptotic behavior of generalized renewal processes connected with this SDE.

The main problems of this section are closely connected with the problem of finding out when differentiable functions satisfy PRV or PMPV conditions. In Section 4, these questions were discussed in a general setting.

Asymptotic behavior of the solution of a stochastic differential equation. General statements. Let us now consider the stochastic differential equation (7.1), where both functions g and σ are positive and such that (7.1) has a.e. a unique and continuous solution X as well as (7.2) has a unique and continuous solution μ. For example, g and σ may be positive and continuously differentiable. Our main goal is to find conditions on g and σ, under which relation (7.3) holds. To do so, we first consider the following general statement, which describes extra conditions for relation (7.6) to imply or being equivalent to (7.3).

THEOREM 7.1. *Let g and σ be positive and continuous functions such that problem* (7.1) *has a.e. a unique continuous solution as well as* (7.2) *has a unique continuous solution. Assume*

$$\lim_{t\to\infty} G(t) = \int_1^{\infty} \frac{du}{g(u)} = \infty, \tag{7.7}$$

and let G (*see* (7.4)) *be such that*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g(u)G(u)} > 0 \qquad \text{for all } c > 1. \tag{7.8}$$

Then,

1) *if* (7.6) *holds, then also* (7.3) *holds true;*
2) *if*

$$\lim_{c\downarrow 1} \limsup_{t\to\infty} \int_t^{ct} \frac{du}{g(u)G(u)} = 0, \tag{7.9}$$

then (7.6) *and* (7.3) *are equivalent.*

Recall that $G = (G(t), t \geqslant 1)$ is the inverse function of μ. By condition (7.7), $\mu(t) \to \infty$ as $t \to \infty$. Moreover, (7.7) excludes the possibility of explosions (that is, the solution does not reach infinity in finite time). Note that the function $g(u) = u, u > 0$, satisfies (7.7), but does not satisfy condition (7.8).

REMARK 7.1. In view of Definition 2.5 above, condition (7.8) means that the function G is a PMPV function. Observe that, by Corollary 5.2, under condition (7.7) the function μ preserves the equivalence of functions (see Definition 2.7) if and only if (7.8) holds.

Condition (7.9) means that the function G is a PRV function (see Definition 2.1) and, by Theorem 2.1, this condition is equivalent to the condition that G preserves the equivalence of functions. The set of conditions (7.7), (7.8) and (7.9) means that both G and μ preserve the equivalence of functions.

Next, we consider some sufficient conditions for (7.8) (Proposition 7.1) and (7.9) (Proposition 7.2), which can be expressed in terms of the function g, and thus are more suitable for practical use.

PROPOSITION 7.1. *Let g be a positive and continuous function such that* (7.7) *holds. Assume that at least one of the following conditions holds:*

(i) $\limsup_{t\to\infty} g(t)G(t)/t < \infty$*;*
(ii) *g is eventually nonincreasing;*
(iii) *there exists $\alpha < 1$ such that $0 < \inf_{s\geqslant 1} g(s)s^{-\alpha}$, $\sup_{s\geqslant 1} g(s)s^{-\alpha} < \infty$;*
(iv) *$g^*(c) < c$ for all $c > 1$, with $g^*(c) = \limsup_{t\to\infty} g(ct)/g(t)$;*
(v) *g is an RV function with index $\alpha < 1$.*

Then, g satisfies condition (7.8).

REMARK 7.2. Under (7.7), condition (i) of Proposition 7.1 is equivalent to (7.8), if the function g is eventually nondecreasing.

REMARK 7.3. Substituting $t \to G(t)$, we get from (7.3)

$$\lim_{t\to\infty} \frac{X(G(t))}{t} = 1 \quad \text{a.e. on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\}.$$

This means that, under the conditions of Theorem 7.1, if (7.6) holds, then

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \lim_{t\to\infty} \frac{X(G(t))}{t} = 1 \quad \text{a.e. on } \Big\{\lim_{t\to\infty} X(t) = \infty\Big\},$$

that is, G is an asymptotic inverse function for the process X a.e. on the set $\{\lim_{t\to\infty} X(t) = \infty\}$.

REMARK 7.4. Condition (i) of Proposition 7.1 does not hold for any regularly varying function g of index 1, that is, for functions $g(t) = t\ell(t)$, where ℓ is slowly varying. This is due to a result of Parameswaran [**43**], which proves that

$$\lim_{t\to\infty} \ell(t) \int_1^t \frac{ds}{s\ell(s)} = \infty.$$

PROPOSITION 7.2. *Let g be a positive and continuous function such that* (7.7) *holds. Assume that at least one of the following conditions holds:*

(i) $\liminf_{t\to\infty} g(t)G(t)/t > 0$*;*
(ii) *g is eventually nondecreasing;*
(iii) $\int_{0+}^1 dc/g^*(c) > 0$*, with $g^*(c) = \limsup_{t\to\infty} g(ct)/g(t)$;*
(iv) *the set $\{c \in (0,1] : g^*(c) < \infty\}$ has positive Lebesgue measure;*
(v) *at least one of conditions* (iii), (iv)*, or* (v) *of Proposition* 7.1 *holds.*

Then, g satisfies condition (7.9).

REMARK 7.5. Under (7.7), condition (i) of Proposition 7.2 is equivalent to (7.9), if the function g is eventually nonincreasing.

Asymptotic behavior of the solution of a stochastic differential equation. Specific statements. The next two theorems contain sufficient conditions for relation (7.3). The following one is a condition from Gihman and Skorohod [**24**, §17].

(GS) Let g and σ be positive and continuous functions such that (7.1) has a.e. a unique and continuous solution X with arbitrary initial condition and with $\lim_{t\to\infty} X(t) = \infty$ a.e., as well as (7.2) has a unique and continuous solution with arbitrary positive initial condition. Let σ/g be bounded and let $g'(x)$ exist for all $x > 0$ with $g'(x) \to 0$ as $x \to \infty$.

REMARK 7.6. Under (GS), relation (7.6) holds true a.e., that is

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = 1 \quad \text{a.e.}$$

(see Gihman and Skorohod [**24**, §17, Theorem 4 and Remark 1]).

THEOREM 7.2. *Assume condition (GS) and let g be such that* (7.7) *holds. If* (7.8) *or at least one of the conditions* (i)–(v) *of Proposition* 7.1 *holds, then relation* (7.3) *follows a.e., that is* $\lim_{t\to\infty} \frac{X(t)}{\mu(t)} = 1$ *a.e.*

Observe that Theorem 7.2 provides conditions in terms of the functions g and σ only, under which relation (7.3) holds, and, from this point of view, Theorem 7.2 complements Theorem 4 in Gihman and Skorohod [**24**, §17].

Asymptotic stability of SDE's with respect to initial conditions. We start with a discussion of the deterministic differential equation (7.2) with positive initial condition, that is, we consider the Cauchy problem

$$d\mu_b(t) = g(\mu_b(t))dt,\ t \geqslant 0, \quad \mu_b(0) = b > 0, \tag{7.10}$$

where $g(u), u > 0$, is a positive and continuous function such that (7.10) has a unique and continuous solution μ_b for all fixed $b > 0$. We say that the Cauchy problem (7.10) is *asymptotically stable with respect to the initial condition* if

$$\lim_{t\to\infty} \frac{\mu_{b_1}(t)}{\mu_{b_2}(t)} = 1, \tag{7.11}$$

for all positive b_1 and b_2. Note, for example, that the problem (7.10) is not asymptotically stable with respect to the initial condition, if $g(u) = u$, $u > 0$, while it is asymptotically stable for $g(u) = u^r$, $u > 0$, with $r < 1$. Observe also that a solution reaches infinity in finite time, if $g(u) = u^r$, $u > 0$, with $r > 1$, so that we do not discuss this case here.

Given $b > 0$, consider the function

$$G_b(s) = \int_b^s \frac{du}{g(u)}, \quad s \geqslant b,$$

and note that G_b is a strictly increasing and continuous function, and it is the inverse of μ_b. Conditions for the asymptotic stability (7.11) can be obtained in terms of the function $G = G_1$.

THEOREM 7.3. *Let g be a positive and continuous function such that* (7.10) *has a unique and continuous solution μ_b for all fixed $b > 0$, and assume* (7.7) *and* (7.8). *Then, the Cauchy problem* (7.10) *is asymptotically stable with respect to the initial condition.*

REMARK 7.7. In view of Definition 2.5 above, condition (7.8) means that the function G is a PMPV function. Observe that, by Corollary 5.2, condition (7.7) implies that the function μ_b preserves the equivalence of functions (see Definition 2.7) if and only if condition (7.8) holds. Observe also, that there are different conditions under which the Cauchy problem (7.10) is asymptotically stable with respect to the initial condition, but later on, in particular, condition (7.8) will serve our purposes.

Next, we consider the SDE (7.1) with a positive initial condition, i.e.

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t),\ t \geqslant 0, \quad X(0) \equiv b > 0. \tag{7.12}$$

Let problem (7.12) have a.e. a unique continuous solution X_b and problem (7.10) have a unique continuous solution μ_b for any positive initial condition, i.e. $X_b(0) \equiv \mu_b(0) = b > 0$. Note that Theorems 7.1 and 7.2 remain true in this case. We study the problem of asymptotic stability with respect to the initial condition for the SDE (7.12).

THEOREM 7.4. *Under* (GS)*, assume condition* (7.7) *holds, and let* (7.8) *or at least one of the conditions* (i)–(v) *of Proposition* 7.1 *be satisfied for* g*. Then, for all positive* b_1 *and* b_2*, one has*

$$\lim_{t\to\infty} \frac{X_{b_1}(t)}{X_{b_2}(t)} = 1 \quad a.e.$$

Asymptotic behavior of generalized renewal processes. As above we assume that both functions g and σ are positive and continuous and such that problem (7.1) has a.e. a unique continuous solution X, as well as problem (7.2) has a unique continuous solution μ. Let us consider the following *generalized renewal processes* for the process X:

$$F(s) = \inf\{t \geqslant 0 : X(t) = s\}, \quad s > 1,$$

i.e. the *first* time when the stochastic process X crosses the level s,

$$L(s) = \sup\{t \geqslant 0 : X(t) = s\}, \quad s > 1,$$

the *last* time when the process X crosses the level s, and

$$T(s) = \operatorname{meas}(\{t \geqslant 0\colon X(t) \leqslant s\}) = \int_0^\infty I(X(t) \leqslant s)dt, \quad s > 1,$$

the *total time* spent by the process X in $(-\infty, s]$, where "meas" denotes the Lebesgue measure.

The next theorems describe the asymptotic behavior of the generalized renewal processes introduced above.

THEOREM 7.5. *Let* g *and* σ *be positive and continuous functions such that problem* (7.1) *has a.e. a unique continuous solution, as well as problem* (7.2) *has a unique continuous solution. Assume relation* (7.6)*. Then it holds, a.e. on* $\{\lim_{t\to\infty} X(t) = \infty\}$,

$$\lim_{t\to\infty} \frac{F(t)}{G(t)} = \lim_{t\to\infty} \frac{T(t)}{G(t)} = \lim_{t\to\infty} \frac{L(t)}{G(t)} = 1,$$

and

$$\lim_{t\to\infty}\frac{F(\mu(t))}{t}=\lim_{t\to\infty}\frac{T(\mu(t))}{t}=\lim_{t\to\infty}\frac{L(\mu(t))}{t}=1.$$

Moreover, if condition (7.7) *holds and* (7.8) *or at least one of the conditions* (i)–(v) *of Proposition* (7.1) *is satisfied, then, a.e. on* $\{\lim_{t\to\infty}X(t)=\infty\}$,

$$\lim_{t\to\infty}\frac{\mu(F(t))}{t}=\lim_{t\to\infty}\frac{\mu(T(t))}{t}=\lim_{t\to\infty}\frac{\mu(L(t))}{t}=1.$$

Theorem 7.6. *Under condition (GS) we have that*

$$\lim_{t\to\infty}\frac{F(t)}{G(t)}=\lim_{t\to\infty}\frac{T(t)}{G(t)}=\lim_{t\to\infty}\frac{L(t)}{G(t)}=1 \quad a.e.,$$

and

$$\lim_{t\to\infty}\frac{F(\mu(t))}{t}=\lim_{t\to\infty}\frac{T(\mu(t))}{t}=\lim_{t\to\infty}\frac{L(\mu(t))}{t}=1 \quad a.e.$$

Moreover, if condition (7.7) *holds and* (7.8) *or at least one of the conditions* (i)–(v) *of Proposition* (7.1) *is satisfied, then*

$$\lim_{t\to\infty}\frac{\mu(F(t))}{t}=\lim_{t\to\infty}\frac{\mu(T(t))}{t}=\lim_{t\to\infty}\frac{\mu(L(t))}{t}=1 \quad a.e.$$

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