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INFINITE CLASSES OF SIMPLE AND MULTIPLE ANTISYMMETRY FOUR-DIMENSIONAL POINT GROUPS

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Abstract. Using complete P-symmetry groups G'_{30}, G^p_{20}, $G^{p'}_{20}$, infinite classes of four-dimensional point groups G_{40}, are derived. For groups obtained, presentations, antisymmetric characteristics, numbers of corresponding simple and multiple antisymmetry groups and possibility for their cataloguation, are given.

The 227 four-dimensional crystallographic point groups G_{40} are discussed in works [**1**, **2**, **3**, **4**, **5**]. In this paper we consider the four-dimensional point groups G_{40} without the crystallographic restriction which belong to the infinite classes of groups G_{40} including the 137 crystallographic groups stated in papers [**3**; **4**, Tab. 1; **5**, Tab. 1C, classes 1–23]. The remaining four-dimensional point groups G_{40} are the polyhedral symmetry groups or their subgroups, so there is a finite number of such groups.

1. Infinite classes of four-dimensional point groups. Different colored symmetry groups and the groups of simple and multiple cryptosymmetry [**6**, **7**, **8**] are included in the concept of P-symmetry (permutation symmetry), introduced by A. M. Zamorzaev [**9**, **10**, **11**, **12**] and defined as follows. If P is a subgroup of the symmetry group, every transformation $C = cS = Sc$, $c \in P$, $S \in G$ is a P-symmetry transformation. Every group G^P derived from G by such a substitution of symmetries by P-symmetries is a P-symmetry group. If the substitutions included in G^P exhaust the group P, G^P is a complete P-symmetry group. Every complete P-symmetry group G^P can be derived from its generating group G by means of searching in G and P for normal subgroups H and Q for which the isomorphism $G/H \cong P/Q$ holds, by paired multiplication of the cosets corresponding in this isomorphism and by the unification of the products obtained. The groups of complete P-symmetry fall into senior ($G = H$ and $G^P = G \times P$), junior ($G/H \cong P$ and $G^P \cong G$) and middle groups for $Q = P$, $Q = I$ and $I \subset Q \subset P$, respectively [**9**, **10**, **11**, **12**].

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In particular, for $P = {C_2}^l$ we have the simple ($l = 1$) and multiple ($l \geq 2$) antisymmetry groups, for $P = C_p$ the Belov colored (p)-symmetry groups, for $P = D_{p(2p)}$ the Pawley (p')-symmetry groups, etc. [**9**, **10**, **11**, **12**].

Bohm symbols [**13**] are used to denote the corresponding categories of the isometric symmetry groups. In the symbol $G_{nst\ldots}$ the first subscript n represents the maximal dimension of space in which the transformations of the symmetry group act, while the following subscripts $st\ldots$ represent the maximal dimensions of subspaces which are invariant with respect to the action of the symmetry group and which are properly included in each other. The corresponding categories of P-symmetry groups will be denoted by additional P-superscripts.

The indices ascribed to the points of a figure with the P-symmetry group have an extrageometric sense with respect to the space in which the figure is considered. In additional dimensions such index permutations can be geometrically interpreted, making possible the investigation of multi-dimensional symmetry groups by means of P-symmetry groups. The simplest example illustrating this is the derivation of the infinite classes of point groups G_{30} from the two-dimensional point groups G_{20} using antisymmetry. The 7 infinite classes of three-dimensional point groups can be derived from two infinite classes of two-dimensional point groups **n** and **nm** using antisymmetry, as **n**, **nm**, $\mathbf{n}\times\underline{\mathbf{1}} \longrightarrow$ **n:m**, $(\underline{\mathbf{2n}}) \longrightarrow (\mathbf{2\tilde{n}})$, $\mathbf{n}\underline{\mathbf{m}} \longrightarrow$ **n:2**, $(\underline{\mathbf{2n}})\mathbf{m} \longrightarrow (\mathbf{2\tilde{n}})\mathbf{m}$, $\mathbf{nm}\times\underline{\mathbf{1}} \longrightarrow$ **mn:m** ($n \in \mathbf{N}$), where the antiidentity transformation $\underline{\mathbf{1}}$ is identified with the reflection in the invariant plane [**14**]. The remaining three-dimensional point groups G_{30} are the polyhedral symmetry groups $[3, q]$ and their subgroups $[3, q]^+$ ($q = 3, 4, 5$) [**15**].

There are the 7 infinite classes of three-dimensional point groups G_{30}: I) **n**, II) **nm**, III) **n:m**, IV) $(\mathbf{2\tilde{n}})$, V) **n:2**, VI) $(\mathbf{2\tilde{n}})\mathbf{m}$, VII) **mn:m** ($n \in \mathbf{N}$). Using complete antisymmetry, (p)-symmetry and (p')-symmetry, we can derive from them all the infinite classes of four-dimensional point groups G_{40}.

If $P = {C_2}^1$ we have the simple ($l = 1$) and multiple ($l \geq 2$) antisymmetry groups. Let a symmetry group G be given by the representation [**15**]:

$$\{S_1, S_2, \ldots, S_r\}, \quad g_k(S_1, S_2, \ldots, S_r) = I, \qquad k = 1, 2, \ldots, s,$$

be antiidentity transformations of the first, second, $\ldots$, l-th kind satisfying the relations:

$${e_i}^2 = I, \quad e_i e_j = e_j e_i, \quad e_i S_q = S_q e_j, \qquad i, j \in \{1, 2, \ldots, l\},\ q \in \{1, 2, \ldots, r\}.$$

Every transformation $S' = e'S$, $S \in G$, where e' is an antiidentity transformation or their product is called a (multiple) antisymmetry transformation. Any group G' derived from G, which contains at least one (multiple) antisymmetry transformation is called a (multiple) antisymmetry group, and the group G is called its generating group. A (multiple) antisymmetry group G' is called junior iff $G' \cong G$. A junior m-multiple antisymmetry group ($1 \leq m \leq l$) is the M^m-type group if every particular e_i-transformation ($i = 1, 2, \ldots, m$) belongs to the G'. In line with the existence criterion for the M^m-type groups, a group of simple or multiple antisymmetry G' will be of the M^m-type

(a) if all the relations given within the presentation of its generating group G remain satisfied after the generators have been substituted by antigenerators, and

(b) if the antisymmetry of an arbitrary kind can be obtained in G' as an independent antisymmetry transformation. In the theory of simple and multiple antisymmetry only the derivation of junior simple and multiple antisymmetry groups of the M^m-type is non-trivial.

In particular, at $l = 1$ we have the (simple) antisymmetry. Among (simple) antisymmetry groups we can distinguish the senior antisymmetry groups of the form $G \times \{e_1\}$, with the structure $G \times C_2$, where $\{e_1\}$ denotes the group generated by e_1, with the structure C_2, and the junior antisymmetry groups isomorphic to their generating group G. Since the antiidentity transformation e_1 can be identified with the (hyper)plane reflection T_1 [**3**, **4**, **5**, **9**, **10**, **11**, **12**, **14**, **16**, **17**, **18**], to the antisymmetry groups of the category G'_{30} correspond symmetry groups of the category G_{430}, belonging to the larger category G_{40} [**17**, **18**]. Moreover, if $G \in G_{30}$, to every senior antisymmetry group $G \times \{e_1\}$ corresponds the symmetry group $G \times \{T_1\}$, where $\{T_1\}$ denotes the symmetry group with the structure D_1 ($D_1 \cong C_2$), generated by a (hyper) plane reflection T_1, and to every junior antisymmetry group G' given by the representation:

$$\{S'_1, S'_2, \dots, S'_r\}, \quad g_k(S'_1, S'_2, \dots, S'_r) = I, \qquad k = 1, 2, \dots, s,$$

where the set $\{S'_1, S'_2, \dots, S'_r\}$ consists of the (anti)generators S'_q ($S'_q = S_q$ or $S'_q = e_1 S_q$, $q = 1, 2, \dots, r$), corresponds the four-dimensional symmetry group generated by the (four-dimensional) symmetries $S'_q = S_q$ or $S'_q = T_1 S_q$.

In a similar way, the remaining infinite classes of four dimensional point groups G_{40} can be derived from the two-dimensional point groups G_{20} using complete (p)- and (p')-symmetry groups [**8**; **10**, pp. 92; **12**]. Namely, all the symmetry groups of the category G_{420} can be interpreted by the groups of the categories G_{20}, G'_{20}, G^{p}_{20}, $G^{p'}_{20}$. Since all the groups of the categories G_{20} and G'_{20} are included, respectively, in the categories G_{30} and G'_{30}, for completing the derivation of all the infinite classes of the four-dimensional point groups G_{40} we need only consider the categories G^{p}_{20}, $G^{p'}_{20}$ and to them corresponding symmetry groups of the category G_{40}.

In the case of (p)-symmetry, the group $P = C_p$ is generated by the permutation $c_1 = (1\ 2\ \dots\ p)$ satisfying the relations

$$c_1{}^p = I, \quad c_1 S = S c_1, \qquad S \in G,$$

which can be identified with a p-fold rotation commuting with all the symmetries of the group G. In the case of (p')-symmetry, the group $P = D_{p(2p)}$ is the regular dihedral permutation group generated by the permutations c_1 and $e_1 = (11')$ satisfying the relations:

$$c_1{}^p = e_1{}^2 = (c_1 e_1)^2 = I, \quad c_1 S = S c_1, \quad e_1 S = S e_1, \qquad S \in G,$$

which can be identified, respectively, with the p-fold rotation and (hyper) plane reflection T_1 commuting with all the symmetries of the group G. All the complete

(p)- and (p')-symmetry groups can be divided into the senior, middle and junior groups [**9**, **10**, **11**, **12**].

Because all the antisymmetry groups of the category G'_{30} are already derived in the articles [**19**, **20**], and the crystallographic groups of the categories G_{20}^{p}, $G_{20}^{p'}$ $(p = 3, 4, 6)$ are given in the monographs [**9**, **10**], we need only derive complete (p)- and (p')-symmetry groups of these categories, without the crystallographic restriction.

There are two infinite classes of two-dimensional point symmetry groups G_{20}: I) **n** and II) **nm** $(n \in \mathbf{N})$ [**14**]. In order to obtain corresponding infinite classes of four-dimensional point groups we need only the following complete (p)- and (p')-symmetry groups derived from the generating groups **n** and **nm**:

(a) (p)-symmetry groups derived from the group **n**:

(a_1) senior groups $\mathbf{n} \times \mathbf{1}^{(p)}$ $(n \geq p \geq 3)$ of the structure $C_n \times C_p$;

(a_2) middle groups $\mathbf{n}^{(p)}$ $((n,p) \neq 1, n \not\equiv 0 \pmod p, n > p \geq 4)$ of the structure $C_{/n,p/}$ and $\mathbf{n}^{(p)} \times \mathbf{1}^{(q)}$ $(n \equiv 0 \pmod p$, $p \equiv 0 \pmod q$, $p \neq q$, $q \neq 1$, $n \geq p \geq 4)$ of the structure $C_n \times C_q$,

(a_3) junior groups $\mathbf{n}^{(p)}$ $(n \equiv 0 \pmod p, n \geq p \geq 3)$ of the structure C_n, where by (n, p) is denoted the maximal common divisor of the numbers n and p, and $/n, p/$ is the least common multiple of n and p;

(b) (p)-symmetry groups derived from the symmetry group **mn**:

(b_1) senior groups $\mathbf{nm} \times \mathbf{1}^{(p)}$ $(n \geq 3,\ p \geq 3)$ of the structure $D_n \times C_p$;

(b_2) middle groups $\mathbf{nm}^{(2)} \times \mathbf{1}^{(2q+2)}$ $(p = 4q + 2)$ of the structure $D_n \times C_{2q+1}$, $\mathbf{n}^{(2)}\mathbf{m} \times \mathbf{1}^{(2q+1)}$ $(p = 4q + 2,\ n \equiv 0 \pmod 2)$ of the structure $D_n \times C_{2q+1}$, $\mathbf{nm}^{(p)}$ $(p \equiv 0 \pmod 4)$ and $\mathbf{n}^{(p)}\mathbf{m}$ $(p \equiv 0 \pmod 4,\ n \equiv 0 \pmod 2)$;

(c) (p')-symmetry groups derived from the symmetry group **nm**:

(c_1) senior groups $\mathbf{nm} \times \mathbf{1}^{(p}\mathbf{1'}^{)}$ of the structure $D_n \times D_p$;

(c_2) middle groups $\mathbf{n}^{(2}\mathbf{m} \times \mathbf{1}^{2q+1}\mathbf{1'}^{)}$, $(n \geq p \geq 6,\ n \equiv 0 \pmod 2,\ p = 4q + 2)$ of the structure $D_n \times D_{2q+1}$, $\mathbf{nm}^{(2} \times \mathbf{1}^{(2q+1}\mathbf{1'}^{)}$ $(n \geq p \geq 6,\ p = 4q + 2)$ of the structure $D_n \times D_{2q+1}$, $\mathbf{n}^{(p}\mathbf{m'}^{)} \times \mathbf{1}^{(2}$ $(n \equiv 0 \pmod p,\ p \equiv 0 \pmod 2,\ p \geq 4)$, of the structure $D_n \times C_2$, $\mathbf{n}^{(p}\mathbf{m'}^{)}$ $(n > p \geq 3)$ of the structure $D_{/n,p/}$, $\mathbf{nm'}^{)}\mathbf{1}^{(p}$ $(n \geq p \geq 3)$, $\mathbf{n}^{(p}\mathbf{m1'}^{)}$ $(n \geq p \geq 4,\ (n,p) \neq 1,\ p \equiv 0 \pmod 4)$, $\mathbf{n'}^{)}\mathbf{m1}^{(p}$ $(n > p \geq 3,\ n \equiv 0 \pmod 2)$, $\mathbf{nm}^{(p}\mathbf{1'}^{)}$ $(n \geq p \geq 4,\ p \equiv 0 \pmod 4)$, $\mathbf{n}^{(2}\mathbf{m'}^{)}\mathbf{1}^{(2q+1}$ $(n \geq p \geq 6,\ n \equiv 0 \pmod 2,\ p = 4q + 2)$, $\mathbf{n'}^{)}\mathbf{m}^{(2}\mathbf{1}^{(2q+1}$ $(n \geq p \geq 6,\ n \equiv 0 \pmod 2,\ p = 4q + 2)$ and $\mathbf{n'}^{)}\mathbf{m}^{(p}$ $(n \geq p \geq 4,\ n \equiv 0 \pmod 2,\ p \equiv 0 \pmod 4)$;

(c3) junior groups $\mathbf{n}^{(p}\mathbf{m'}^{)}$, $(p \geq 3,\ n \equiv 0 \pmod p)$ of the structure D_n.

Definition 1. Let all products of the generators of the group G, within which all generators occur at most once, be formed and then separate the subsets of transformations that are equivalent with respect to symmetry. The resulting system is called the antisymmetric characteristics of the group G $(AC(G))$ [**21**].

THEOREM 1. *Two groups of simple or multiple antisymmetry G_1^l and G_2^l of the M^m-type for m fixed, with common generating group G, are equal iff they possess equal AC* [**21**].

THEOREM 2. *Symmetry groups that possess isomorphic AC generate the same number of simple and multiple antisymmetry groups of the M^m-type for every fixed m $(1 \leq m \leq l)$, which correspond to each other with regard to structure* [**21**].

All the four-dimensional point groups G_{40} derived, belonging to the corresponding infinite classes, are given by one or several different presentations, together with their antisymmetric characteristics (AC) making possible the derivation of all the infinite classes of simple and multiple antisymmetry groups G_{40}^l, the computation of the numbers of such groups, and their cataloguation based on paper [**21**].

The catalogue obtained contains the infinite classes of four-dimensional point groups G_{40} derived using antisymmetry groups G_{30}'. The following elements are given:

1) the symbol of a generating group G (class I–VII) [**14**], its presentation and structure symbol, and its antisymmetric characteristic (AC) followed by the existence conditions for the M^m-type simple and multiple antisymmetry groups and by the symbol of equivalency class [**21**] to which this AC belongs. In the derivation of M^m-type simple and multiple antisymmetry groups, a generator S_q of G not participating in $AC(G)$ cannot be replaced by an antigenerator or must be replaced by the antigenerator of the same antisymmetry kind as an other generator S_w belonging to the $AC(G)$ $(q, w \in \{1, 2, \ldots, r\})$. This is denoted by the relationship $S_q = S_w$. There is also the catalogue of crystallographic groups [**1**, **2**, **3**, **4**, **5**] belonging to the infinite class discussed, where every such a group is given by its catalogue numbers from papers [**5**] and [**1**], the generators [**1**, **16**, **20**], the structure symbol, and followed by its antisymmetric characteristic AC and by the symbol of the equivalency class [**21**] that AC belongs.

In the part 2) and 3), respectively, senior and junior antisymmetry groups derived in the same family of antisymmetry groups with the generating group G (class I–VII) and corresponding four-dimensional point groups (infinite classes), are stated. Every such a group (infinite class) is followed by the same data as in the part 1).

The infinite classes of four-dimensional point groups G_{40} derived using complete (p)- and (p')-symmetry groups of the categories G_{20}^p, $G_{20}^{p'}$, are treated in the same way in the parts 1), 2), 3), 4), dealing, respectively, with the generating, senior, middle and junior groups of these categories and with the corresponding four-dimensional point groups.

In the catalogue are omitted all the repeating infinite classes of four-dimensional point groups G_{40} and the crystallographic groups belonging to them, already derived from certain infinite class of three-dimensional point groups G_{30} or two-dimensional point groups G_{20}. For example, all the generating symmetry groups of the classes III–VII are omitted, since the same groups are already derived from the antisymmetry groups of the classes I and II. Hence, the catalogue consists of all the different infinite classes of four-dimensional point groups G_{40} and to them belonging different crystallographic groups G_{40}.

2. Catalogue. The way in which the catalogue is formed will be illustrated by the following example: every generating group of the class I) **n** possesses the representation: 1) $\{S\}$, $S^n = I$ and the structure C_n. If the existence condition for the M^m-type simple and multiple antisymmetry groups $n \equiv 0 \pmod 2$ is satisfied, its antisymmetric characteristic AC is $\{S\}$, so it belongs to the equivalency class 1.1 [**21**]. To this infinite class of four-dimensional point groups belong the following crystallographic groups, obtained at $n = 1, 2, 3, 4, 6$: the group denoted by the catalogue numbers 1/01 [**5**] and $(2a, 1)$ [**1**], obtained at $n = 1$, generated by the identity transformation I [**1**, **16**], with the structure C_1, and without AC since it does not satisfies the existence condition for the M^m-type simple and multiple antisymmetry groups $n \equiv 0 \pmod 2$. In the same way, the other four crystallographic groups are described:

3/01 (2a,2)	$\{E\}$	C_2,	AC: $\{E\}$,	1.1;
8/01 (2a,3)	$\{K\}$	C_3;		
7/02 (2a,4)	$\{R\}$	C_4,	AC: $\{R\}$,	1.1;
9/01 (2a,7)	$\{Z\}$	C_6,	AC: $\{Z\}$,	1.1.

In the same family of antisymmetry groups with the generating group **n** (class I) is derived the senior antisymmetry group 2) $\{S\} \times \{e_1\}$ isomorphic to the four-dimensional group $\{S\} \times \{T_1\}$, where $\{S\}$ and $\{T_1\}$ denote, respectively, the symmetry groups generated by n-fold rotation S and (hyper)plane reflection T_1 commuting with it. The four-dimensional groups obtained, at $n \equiv 0 \pmod 2$ possess the antisymmetric characteristic AC: $\{S\}\{T_1\}$ belonging to the equivalency class 2.1 [**21**] or, at $n \equiv 1 \pmod 2$ the antisymmetric characteristic AC: $\{T_1\}$ belonging to the equivalency class 1.1 [**21**]. At $n = 1, 2, 3, 4, 6$, we have the crystallographic groups:

2/01 (2b,1)	$\{T_1\}$	D_1,	AC: $\{T_1\}$,	1.1;
4/02 (2b,4)	$\{E\} \times \{T_1\}$	$C_2 \times D_1$,	AC: $\{E\}\{T_1\}$,	2.1;
14/02 (2b,9)	$\{K\} \times \{T_1\}$	$C_3 \times D_1$,	AC: $\{T_1\}$,	1.1;
13/01 (2b,–)	$\{R\} \times \{T_1\}$	$C_4 \times D_1$,	AC: $\{R\}\{T_1\}$,	2.1;
15/01 (2b,15)	$\{Z\} \times \{T_1\}$	$C_6 \times D_1$,	AC: $\{Z\}\{T_1\}$,	2.1.

Finally, in the case of the junior antisymmetry group $\{e_1 S\}$ existing at $n \equiv 0 \pmod 2$, isomorphic with the four-dimensional point group $\{\widetilde{S}_1\}$ generated by the rotational reflection $\widetilde{S}_1 = ST_1$ at the same n, with the structure C_{2n} and with AC: $\{\widetilde{S}_1\}$ belonging to the equivalency class 1.1 [**21**], we have at $n = 1, 2, 3$, the following crystallographic groups:

2/02 (2b,2)	$\{T'\}$	C_2,	AC: $\{T'\}$,	1.1;
12/01 (2b,6)	$\{F\}$	C_4,	AC: $\{F\}$,	1.1;
14/01 (2b,10)	$\{N\}$	C_6,	AC: $\{N\}$,	1.1.

In line with this, the complete catalogue reads as follows:

I) **n**

1) $\{S\}$ $S^n = I$, C_n, AC: $\{S\}$, $n \equiv 0 \pmod 2$, 1.1.

1/01 (2a,1)	$\{I\}$	C_1;		
3/01 (2a,2)	$\{E\}$	C_2,	AC: $\{E\}$,	1.1;
8/01 (2a,3)	$\{K\}$	C_3,		
7/02 (2a,4)	$\{R\}$	C_4,	AC: $\{R\}$,	1.1;
9/01 (2a,7)	$\{Z\}$	C_6,	AC: $\{Z\}$,	1.1.

2) $\{S\} \times \{e_1\} \cong \{S\} \times \{T_1\}$, $C_n \times D_1$, AC: $\{T_1\}$, $n \equiv 1 \pmod 2$, 1.1;
$\{S\}\{T_1\}$, $n \equiv 0 \pmod 2$, 2.1;

2/01 (2b,1)	$\{T_1\}$	D_1,	AC: $\{T_1\}$,	1.1;
4/02 (2b,4)	$\{E\} \times \{T_1\}$	$C_2 \times D_1$,	AC: $\{E\}\{T_1\}$,	2.1;
14/02 (2b,9)	$\{K\} \times \{T_1\}$	$C_3 \times D_1$,	AC: $\{T_1\}$,	1.1;
13/01 (2b,–)	$\{R\} \times \{T_1\}$	$C_4 \times D_1$,	AC: $\{R\}\{T_1\}$,	2.1;
15/01 (2b,15)	$\{Z\} \times \{T_1\}$	$C_6 \times D_1$,	AC: $\{Z\}\{T_1\}$,	2.1.

3) $\{e_1 S\} \cong \{\widetilde{S}_1\}$, $n = 2k$, C_{2k}, AC: $\{\widetilde{S}_1\}$, 1.1.

2/02 (2b,2)	$\{T'\}$	C_2,	AC: $\{T'\}$,	1.1;
12/01 (2b,6)	$\{F\}$	C_4,	AC: $\{F\}$,	1.1;
14/01 (2b,10)	$\{N\}$	C_6,	AC: $\{N\}$,	1.1.

II) **nm**

1) a) $\{S, T\}$ $S^n = T^2 = (ST)^2 = I$, D_n, $n > 1$,
AC: $\{T\}$, $n \equiv 1 \pmod 2$, 1.1;
$\{T, ST\}$, $n \equiv 0 \pmod 2$, 2.2;
b) $\{T, T_2\}$, $T^2 = {T_2}^2 = (TT_2)^2 = I$,
AC: $\{T\}$, $n \equiv 1 \pmod 2$, $T = T_2$;
$\{T, T_2\}$, $n \equiv 0 \pmod 2$.

4/01 (2b,3)	a) $\{E, T\}$	D_2,	AC: $\{T, ET\}$,	2.2;
	b) $\{T, T_2\}$		AC: $\{T, T_2\}$;	
8/03 (2b,7)	a) $\{K, T\}$	D_3,	AC: $\{T\}$,	1.1;
	b) $\{T, T_2\}$		AC: $\{T\}$, $T = T_2$;	
7/06 (2b,11)	a) $\{R, T\}$	D_4,	AC: $\{T, RT\}$,	2.2;
	b) $\{T, T_2\}$		AC: $\{T, T_2\}$;	
9/04 (2b,13)	a) $\{Z, T\}$	D_6,	AC: $\{T, ZT\}$,	2.2;
	b) $\{T, T_2\}$		AC: $\{T, T_2\}$.	

2) a) $\{S, T\} \times \{e_1\} \cong \{S, T\} \times \{T_1\}$, $D_n \times D_1$, $n > 1$,
AC: $\{T\}\{T_1\}$, $n \equiv 1 \pmod 2$, 2.1;

$\{T,T_1,ST\}$, $n=2$, 3.7;

b) $\{T,T_2\}\times\{e_a\}\cong\{T,T_2\}\times\{T_1\}$,

AC: $\{T\}\{T_1\}$, $n\equiv 1\ (\mathrm{mod}\,2)$, $T=T_2$;

$\{T,T_1,T_2\}$, $n=2$;

$\{T_1\}\{T,T_2\}$, $n>2$, $n\equiv 0\ (\mathrm{mod}\,2)$.

6/01 (2b,32)	a) $\{E,T\}\times\{T_1\}$	$D_2\times D_1$,	AC: $\{T,T_1,ET\}$,	3.7;
	b) $\{T,T_2\}\times\{T_1\}$		AC: $\{T,T_1,T_2\}$;	
14/08 (2b,36)	a) $\{K,T\}\times\{T_1\}$	$D_3\times D_1$,	AC: $\{T\}\{T_1\}$,	2.1;
	b) $\{T,T_2\}\times\{T_1\}$		AC: $\{T\}\{T_1\}$, $T=T_2$;	
13/09 (2b,52)	a) $\{R,T\}\times\{T_1\}$	$D_4\times D_1$,	AC: $\{T_1\}\{T,RT\}$,	3.2;
	b) $\{T,T_2\}\times\{T_1\}$		AC: $\{T_1\}\{T,T_2\}$;	
15/06 (2b,40)	a) $\{Z,T\}\times\{T_1\}$	$D_6\times D_1$,	AC: $\{T_1\}\{T,ZT\}$,	3.2;
	b) $\{T,T_2\}\times\{T_1\}$		AC: $\{T_1\}\{T,T_2\}$.	

3) a) $\{S,e_1T\}\cong\{S,E_1\}$, D_n, $n>1$,

AC: $\{E_1\}$, $n\equiv 1\ (\mathrm{mod}\,2)$, 1.1;

$\{S,E_1,SE_1\}$, $n=2$, 2.3;

$\{E_1,SE_1\}$, $n>2$, $n\equiv 0\ (\mathrm{mod}\,2)$, 2.2;

b) $\{e_1T,e_1T_2\}\cong\{E_1,E_2\}$,

AC: $\{E_1\}$, $n\equiv 1\ (\mathrm{mod}\,2)$, $E_1=E_2$;

$\{E_1,E_2,E_1E_2\}$, $n=2$;

$\{E_1,E_2\}$, $n>2$, $n\equiv 0\ (\mathrm{mod}\,2)$.

5/01 (2a,17)	$\{E_1,E_2\}$	D_2,	AC: $\{E_1,E_2,E_1E_2\}$,	2.3;
14/03 (2a,18)	a) $\{K,E_2\}$	D_3,	AC: $\{E_1\}$,	1.1;
	b) $\{E_1,E_2\}$		AC: $\{E_1\}$, $E_1=E_2$;	
13/04 (2a,19)	a) $\{R,E_1\}$	D_4,	AC: $\{E_1,RE_1\}$,	2.2;
	b) $\{E_1,E_2\}$		AC: $\{E_1,E_2\}$;	
15/04 (2a,23)	a) $\{Z,E_1\}$	D_6,	AC: $\{E_1,ZE_1\}$,	2.2;
	b) $\{E_1,E_2\}$		AC: $\{E_1,E_2\}$.	

3) a) $\{e_1S,e_1T\}=\{\widetilde{S}_1,E_1\}$, $n=2k$, $k>1$, D_{2k}, AC: $\{\widetilde{S}_1\}\{E_1\}$, 2.1;

a′) $\{e_1S,T\}\cong(\widetilde{S}_1,T)$, AC: $\{\widetilde{S}_1\}\{T\}$;

b) $\{e_1T,T_2\}\cong\{E_1,T_2\}$, AC: $\{E_1\}\{T_2\}$.

12/03 (2b,34)	a) $\{F,E_1\}$	D_4	AC: $\{F\}\{E_1\}$,	2.1;
	a′) $\{F,T\}$		AC: $\{F\}\{T\}$;	
	b) $\{E_1,T_2\}$		AC: $\{E_1\}\{T_2\}$;	
14/06 (2b,38)	a) $\{N,E_1\}$	D_6	AC: $\{N\}\{E_1\}$,	2.1;
	a′) $\{N,T\}$		AC: $\{N\}\{T\}$;	
	b) $\{E_1,T_2\}$		AC: $\{E_1\}\{T_2\}$.	

III) **n:m**

1) $\{S,T\}$, $S^n=T^2=I$, $ST=TS$, $C_n\times D_1$,

AC: $\{T\}$ $n \equiv 1 \pmod 2$, 1.1;
$\{S\}\{T\}$, $n \equiv 0 \pmod 2$, 2.1.

2) $\{S,T\} \times \{e_1\} \cong \{S,T\} \times \{T_1\}$, $n > 1$, $C_n \times D_2$,
AC:$\{T,T_1\}$, $n \equiv 1 \pmod 2$, 2.2;
$\{S\}\{T,T_1\}$, $n \equiv 0 \pmod 2$, 3.2.

4/04 (1b, XXXIII,2)	$\{E,T\} \times \{T_1\}$	$C_2 \times D_2$,	AC: $\{E\}\{T,T_1\}$,	3.2;
15/02 (2b,17)	$\{K,T\} \times \{T_1\}$	$C_3 \times D_2$,	AC: $\{T,T_1\}$,	2.2;
13/05 (1b, XXXIII,6)	$\{R,T\} \times \{T_1\}$	$C_4 \times D_2$,	AC: $\{R\}\{T,T_1\}$,	3.2;
15/08 (1b,XXXIII,9)	$\{Z,T\} \times \{T_1\}$	$C_6 \times D_2$,	AC: $\{Z\}\{T,T_1\}$,	3.2.

3) $\{e_1S,T\} \cong \{\widetilde{S}_1,T\}$, $n = 2k$, $C_{2k} \times D_1$, AC: $\{\widetilde{S}_1\}\{T\}$, 2.1.

2/03 (1b,XXXIII,1)	$\{T',T\}$	$C_2 \times D_1$,	AC: $\{T'\}\{T\}$,	2.1;
13/02 (2b,21)	$\{F,T\}$	$C_4 \times D_1$,	AC: $\{F\}\{T\}$,	2.1;
14/04 (1b,XXXIII,4)	$\{N,T\}$	$C_6 \times D_1$,	AC: $\{N\}\{T\}$,	2.1.

3) $\{S,e_1T\} \cong \{S,E_1\}$, $n > 1$, $C_n \times C_2$,
AC: $\{E_1\}$, $n \equiv 1 \pmod 2$, 1.1;
$\{S,E_1\}$, $n = 1$, 2.2;
$\{S\}\{E_1\}$, $n > 2$, $n \equiv 0 \pmod 2$, 2.1.

3/02 (1a,I,2)	$\{E,E_1\}$	$C_2 \times C_2$,	AC: $\{E,E_1\}$,	2.2;
9/02 (2a,8)	$\{K,E_1\}$	$C_3 \times C_2$,	AC: $\{E_1\}$,	1.1;
7/03 (1a,I,4)	$\{R,E_1\}$	$C_4 \times C_2$,	AC: $\{R\}\{E_1\}$,	2.1;
9/03 (1a,I,6)	$\{Z,E_1\}$	$C_6 \times C_2$,	AC: $\{Z\}\{E_1\}$,	2.1.

3) $\{e_1S,e_1T\} \cong \{\widetilde{S}_1,E_1\}$, $n = 2k$, $C_{2k} \times C_2$, AC: $\{\widetilde{S}_1,\widetilde{S}_1E\}$, 2.2.

4/03 (2b,5)	$\{T',E_1\}$	$C_2 \times C_2$,	AC: $\{T',T'E_1\}$,	2.2;
12/02 (1b,XXXIII,3)	$\{F,E_1\}$	$C_4 \times C_2$,	AC: $\{F,FE_1\}$,	2.2;
15/03 (2b,18)	$\{N,E_1\}$	$C_6 \times C_2$,	AC: $\{N,NE_1\}$,	2.2.

IV) $(\mathbf{2\tilde{n}})$

1) $\{\widetilde{S}\}$, $\widetilde{S}^{2n} = I$, C_{2n}, AC: $\{\widetilde{S}\}$.

3) $\{e_1\widetilde{S}\} \cong \{\widetilde{S}'\}$, C_{2n}, AC: $\{\widetilde{S}'\}$, 1.1.

1/02 (1a,I,1)	$\{I'\}$	C_2,	AC: $\{I'\}$,	1.1;
7/01 (2a,5)	$\{R'\}$	C_4,	AC: $\{R'\}$,	1.1;
8/02 (1a,I,3)	$\{K'\}$	C_6,	AC: $\{K'\}$,	1.1.

V) **n:2**

1) a) $\{S,E\}$, $S^n = E^2 = (SE)^2 = I$, D_n,
AC: $\{E\}$, $n \equiv 1 \pmod 2$, 1.1;
$\{S,E,SE\}$, $n = 2$, 2.3;

$\{E, SE\}$, $n > 2,\ n \equiv 0 \pmod 2$, 2.2.

b) $\{E, E_2\},\ E^2 = E_2{}^2 = (EE_2)^n = I$,

AC: $\{E\},\ E = E_2$, $n \equiv 1 \pmod 2$;

$\{E, E_2, EE_2\}$, $n = 2$;

$\{E, E_2\}$, $n > 2,\ n \equiv 0 \pmod 2$.

2) a) $\{S, E\} \times \{e_1\} \cong \{S, E\} \times \{T_1\},\ n > 1,\ D_n \times D_1$,

AC: $\{E\}\{T_1\}$, $n \equiv 1 \pmod 2$, 2.1;

$\{T_1\}\{S, E, SE\}$, $n = 2$, 3.9;

$\{T_1\}\{E, SE\}$, $n > 2,\ n \equiv 0 \pmod 2$, 3.2;

b) $\{E, E_2\} \times \{e_1\} \cong \{E, E_2\} \times \{T_1\}$,

AC: $\{E\}\{T_1\},\ E = E_2$, $n \equiv 1 \pmod 2$;

$\{T_1\}\{E, E_2, EE_2\}$, $n = 2$;

$\{T_1\}\{E, E_2\}$, $n > 2,\ n \equiv 0 \pmod 2$.

6/02 (2b,33)	$\{E, E_2\} \times \{T_1\}$	$D_2 \times D_1$,	AC: $\{T_1\}\{E, E_2, EE_2\}$,	3.9;
14/09 (2b,39)	a) $\{K, E\} \times \{T_1\}$	$D_3 \times D_1$,	AC: $\{E\}\{T_1\}$,	2.1;
	b) $\{E, E_2\} \times \{T_1\}$		AC: $\{E\}\{T_1\},\ E = E_2$;	
13/08 (2b,53)	a) $\{R, E\} \times \{T_1\}$	$D_4 \times D_1$,	AC: $\{T_1\}\{E, RE\}$,	3.2;
	b) $\{E, E_2\} \times \{T_1\}$		AC: $\{T_1\}\{E, E_2\}$;	
15/09 (2b,41)	a) $\{Z, E\} \times \{T_1\}$	$D_6 \times D_1$,	AC: $\{T_1\}\{E, ZE\}$,	3.2;
	b) $\{E, E_2\} \times \{T_1\}$		AC: $\{T_1\}\{E, E_2\}$.	

3) a) $\{S, e_1E\} \cong \{S, T_1{}'\},\ n > 2,\ D_n$,

AC: $\{T_1{}'\}$, $n \equiv 1 \pmod 2$, 1.1;

$\{T_1{}', ST_1{}'\}$, $n \equiv 0 \pmod 2$, 2.2;

b) $\{e_1E, e_1E_2\} \cong \{T_1{}', T_2{}'\}$

AC: $\{T_1{}'\},\ T_1{}' = T_2{}'$, $n \equiv 1 \pmod 2$;

$\{T_1{}', T_2{}'\}$, $n \equiv 0 \pmod 2$.

8/04 (2b,8)	a) $\{K, T_1{}'\}$	D_3,	AC: $\{T_1{}'\}$,	1.1;
	b) $\{T_1{}', T_2{}'\}$		AC: $\{T_1{}'\},\ T_1{}' = T_2{}'$;	
7/05 (2b,12)	a) $\{R, T_1{}'\}$	D_4,	AC: $\{T_1{}', RT_1{}'\}$,	2.2;
	b) $\{T_1{}', T_2{}'\}$		AC: $\{T_1{}', T_2{}'\}$;	
9/05 (2b,14)	a) $\{Z, T_1{}'\}$	D_6,	AC: $\{T_1{}', ZT_1{}'\}$,	2.2;
	b) $\{T_1{}', T_2{}'\}$		AC: $\{T_1{}', T_2{}'\}$.	

3) a) $\{e_1S, E\} \cong \{\widetilde{S}_1, E\},\ n = 2k,\ k > 1,\ D_{2k}$, AC: $\{\widetilde{S}_1\}\{E\}$, 2.1;

a′) $\{e_1S, e_1E\} \cong \{\widetilde{S}_1, T_1{}'\}$, AC: $\{\widetilde{S}_1\}\{T_1{}'\}$;

b) $\{e_1E, E_2\} \cong \{T_1{}', E_2\}$, AC: $\{T_1{}'\}\{E_2\}$.

12/04 (2b,35)	a) $\{F, E\}$	D_4,	AC: $\{F, E\}$,	2.1;
	a′) $\{F, T_1{}'\}$		AC: $\{F\}\{T_1{}'\}$;	
	b) $\{T_1{}', E_2\}$		AC: $\{T_1{}'\}\{E_2\}$;	
14/07 (2b,37)	a) $\{N, E\}$	D_6,	AC: $\{N\}\{E\}$,	2.1;

a$'$) $\{N, T_1{}'\}$ AC: $\{N\}\{T_1{}'\}$;
b) $\{T_1{}', E_2\}$ AC: $\{T_1{}'\}\{E_2\}$.

VI) $(\mathbf{2\tilde{n}})\mathbf{m}$

1) a) $\{\widetilde{S}, T\}$, $\widetilde{S}^{2n} = T^2 = (\widetilde{S}T)^2 = I$, D_{2n}, AC: $\{\widetilde{S}\}\{T\}$, 2.1;
b) $\{\widetilde{S}, E\}$, $\widetilde{S}^{2n} = E^2 = (\widetilde{S}E)^2 = I$, AC: $\{\widetilde{S}\}\{E\}$;
2) a) $\{\widetilde{S}, T\} \times \{e_1\} \cong \{\widetilde{S}, T\} \times \{T_1\}$, $n > 1$, $D_{2n} \times D_1$,
AC: $\{\widetilde{S}\}\{T\}\{T_1\}$, 3.1;
b) $\{\widetilde{S}, E\} \times \{e_1\} \cong \{\widetilde{S}, E\} \times \{T_1\}$, AC: $\{\widetilde{S}\}\{E\}\{T_1\}$.

13/07 (2b,46)	a) $\{F, T\} \times \{T_1\}$	$D_4 \times D_1$,	AC: $\{F\}\{T\}\{T_1\}$,	3.1;
	b) $\{F, E\} \times \{T_1\}$		AC: $\{F\}\{E\}\{T_1\}$;	
14/10 (1b,XXXV,3)	a) $\{N, T\} \times \{T_1\}$	$D_6 \times D_1$,	AC: $\{N\}\{T\}\{T_1\}$,	3.1;
	b) $\{N, E\} \times \{T_1\}$		AC: $\{N\}\{E\}\{T_1\}$.	

3) a) $\{e_1\widetilde{S}, T\} \cong \{\widetilde{S}', T\}$, $n > 1$, D_{2n}, AC: $\{\widetilde{S}'\}\{T\}$, 2.1;
b) $\{e_1\widetilde{S}, e_1E\} \cong \{\widetilde{S}', T_1{}'\}$, AC: $\{\widetilde{S}'\}\{T_1{}'\}$.

7/04 (2b,20)	a) $\{R', T\}$	D_4,	AC: $\{R'\}\{T\}$,	2.1;
	b) $\{R', T_1{}'\}$		AC: $\{R'\}\{T_1{}'\}$;	
8/05 (1b,XXXIII,5)	a) $\{K', T\}$	D_6,	AC: $\{K'\}\{T\}$,	2.1;
	b) $\{K', T_1{}'\}$		AC: $\{K'\}\{T_1{}'\}$.	

3) $\{e_1\widetilde{S}, e_1T\} \cong \{\widetilde{S}', E_1\}$, $n > 1$, D_{2n}, AC: $\{E_1, \widetilde{S}'E_1\}$, 2.2;

13/03 (2a,20)	$\{R', E_1\}$	D_4,	AC: $\{E_1, \widetilde{R}'E_1\}$,	2.2;
14/05 (1a,XI,2)	$\{K', E_1\}$	D_6,	AC: $\{E_1, \widetilde{K}'E_1\}$,	2.2.

1) a) $\{S, T, T_2\}$, $S^n = T^2 = (ST)^2 = I$, $T_2{}^2 = I$, $ST_2 = T_2S$, $TT_2 = T_2T$,
$D_n \times D_1$,
AC: $\{T, T_2\}$, $n = 1$, 2.2;
$\{T\}\{T_2\}$, $n > 1$, $n \equiv 1 \pmod 2$, 2.1;
$\{T, T_2, ST\}$, $n = 2$, 3.7;
$\{T_1\}\{T, ST\}$, $n > 2$, $n \equiv 0 \pmod 2$, 3.2.
b) $\{T, T_2, T_3\}$, $T^2 = T_2{}^2 = (TT_2)^n = I$, $T_3{}^2 = I$, $TT_3 = T_3T$, $T_2T_3 = T_3T_2$,
AC: $\{T, T_3\}$, $T = T_2$, $n = 1$;
$\{T\}\{T_2\}$, $n > 1$, $n \equiv 1 \pmod 2$;
$\{T, T_2, T_3\}$, $n = 2$;
$\{T_3\}\{T, T_2\}$, $n > 2$, $n \equiv 0 \pmod 2$.
2) a) $\{S, T, T_2\} \times \{e_1\} \cong \{S, T, T_2\} \times \{T_1\}$, $n > 1$, $D_n \times D_2$,
AC: $\{T, T_1, T_2, ST\}$, $n = 2$, 4.22;
$\{T\}\{T_1, T_2\}$, $n \equiv 1 \pmod 2$, 3.2;
$\{T, ST\}\{T_1, T_2\}$, $n > 2$, $n \equiv 0 \pmod 2$, 4.6;
b) $\{T, T_2, T_3\} \times \{e_1\} \cong \{T, T_2, T_3\} \times \{T_1\}$,

AC: $\{T, T_1, T_2, T_3\}$, $n = 2$;
$\{T\}\{T_1, T_3\}$, $T = T_2$, $n > 1$, $n \equiv 1 \pmod 2$;
$\{T, T_2\}\{T_1, T_3\}$, $n > 2$, $n \equiv 0 \pmod 2$.

6/03 (1b,XXXV,1)	a) $\{E, T, T_2\} \times \{T_1\}$,	$D_2 \times D_2$,	AC: $\{T, T_1, T_2, ET\}$,	4.22;
	b) $\{T, T_2, T_3\} \times \{T_1\}$,		AC: $\{T, T_1, T_2, T_3\}$;	
15/07 (2b,42)	a) $\{K, T, T_2\} \times \{T_1\}$,	$D_3 \times D_1$,	AC: $\{T\}\{T_1, T_3\}$,	3.2;
	b) $\{T, T_2, T_3\} \times \{T_1\}$,		AC: $\{T\}\{T_1, T_3\}$, $T = T_2$;	
13/10 (1b,XXXV,4)	a) $\{R, T, T_2\} \times \{T_1\}$,	$D_4 \times D_1$,	AC: $\{T, RT\}\{T_1, T_2\}$,	4.6;
	b) $\{T, T_2, T_3\} \times \{T_1\}$,		AC: $\{T, T_2\}\{T_1, T_3\}$;	
15/12 (1b,XXXV,6)	a) $\{Z, T, T_2\} \times \{T_1\}$,	$D_6 \times D_1$,	AC: $\{T, ZT\}\{T_1, T_2\}$,	4.6;
	b) $\{T, T_2, T_3\} \times \{T_1\}$,		AC: $\{T, T_2\}\{T_1, T_3\}$.	

3) a) $\{S, e_1T, e_1T_2\} \cong \{S, E_1, E_2\}$, $n > 1$, $D_n \times D_1$,
AC: $\{S, E_1, E_2\}$, $n = 2$, 3.7;
$\{E_1, E_1E_2\}$, $n \equiv 1 \pmod 2$, 2.2;
$\{(E_1, SE_1), (E_1E_2, SE_1E_2)\}$, $n > 2$, $n \equiv 0 \pmod 2$, 3.4;
b) $\{e_1T, e_1T_2, e_1T_3\} \cong \{E_1, E_2, E_3\}$, $n \equiv 1 \pmod 2$;
AC: $\{E_1, E_1E_3\}$, $E_1 = E_2$, $n \equiv 1 \pmod 2$;
$\{(E_1, E_2), (E_1E_3, E_2E_3)\}$, $n > 2$, $n \equiv 0 \pmod 2$.
AC: $\{S, E_1, E_2\}$, $n = 2$, 3.7

5/02 (1a,XI,1)	$\{E_1, E_2, E_3\}$	$D_2 \times D_1$,	AC: $\{E_1, E_2, E_3\}$,	3.7;
15/05 (2a,22)	a) $\{K, E_1, E_2\}$	$D_3 \times D_1$,	AC: $\{E_1, E_1E_2\}$,	2.2;
	b) $\{E_1, E_2, E_3\}$		AC: $\{E_1, E_1E_3\}$, $E_1 = E_2$,	2.2;
13/09 (1a,XI,3)	a) $\{R, E_1, E_2\}$	$D_4 \times D_1$,	AC: $\{(E_1, RE_1), (E_1E_2, RE_1E_2)\}$,	3.4;
	b) $\{E_1, E_2, E_3\}$		AC: $\{(E_1, E_2), (E_1E_3, E_2E_3)\}$;	
15/11 (1a,XI,5)	a) $\{Z, E_1, E_2\}$	$D_6 \times D_1$,	AC: $\{(E_1, ZE_1), (E_1E_2, ZE_1E_2)\}$,	3.4;
	b) $\{E_1, E_2, E_3\}$		AC: $\{(E_1, E_2), (E_1E_3, E_2E_3)\}$.	

3) a) $\{S, T, e_1T_2\} \cong \{S, T, E_2\}$, $n > 2$, $D_n \times D_1$,
AC: $\{T\}\{E_2\}$, $n \equiv 1 \pmod 2$, 2.1;
$\{E_2\}\{T, ST\}$, $n \equiv 0 \pmod 2$, 3.2;
b) $\{T, T_2, e_1T_3\} \cong \{T, T_2, E_3\}$,
AC: $\{T\}\{E_3\}$, $T = T_2$, $n \equiv 1 \pmod 2$;
$\{E_3\}\{T, T_2\}$, $n \equiv 0 \pmod 2$.

9/06 (2b,16)	a) $\{K, T, E_2\}$	$D_3 \times D_1$,	AC: $\{T\}\{E_2\}$,	2.1;
	b) $\{T, T_2, E_3\}$		AC: $\{T\}\{E_3\}$, $T = T_2$;	
7/07 (1b, XXXIII,7)	a) $\{R, T, E_2\}$	$D_4 \times D_1$,	AC: $\{E_2\}\{T, RT\}$,	3.2;
	b) $\{T, T_2, E_3\}$		AC: $\{E_3\}\{T, T_2\}$;	
9/07 (1b,XXXIII,10)	a) $\{Z, T, E_2\}$	$D_6 \times D_1$,	AC: $\{E_2\}\{T, ZT\}$,	3.2;
	b) $\{T, T_2, E_3\}$		AC: $\{E_3\}\{T, T_2\}$.	

3) a) $\{e_1S, e_1T_2\} \cong \{\widetilde{S}_1, T, E_2\}$, $n = 2k$, $k > 1$, $D_{2k} \times D_1$,

AC: $\{T\}\{\widetilde{S}_1, E_2\widetilde{S}_1\}$, 3.2;

a′) $\{e_1S, e_1T, e_1T_2\} \cong \{\widetilde{S}_1, E_1, E_2\}$, AC: $\{E_1\}\{\widetilde{S}_1, \widetilde{S}_1E_2\}$;

b) $\{T, e_1T_2, e_1T_3\} \cong \{T, E_2, E_3\}$, AC: $\{T\}\{TE_2, TE_2E_3\}$.

12/05	(1b,XXXV,2)	a) $\{F, T, E_2\}$	$D_4 \times D_1$,	AC: $\{T\}\{F, FE_2\}$,	3.2;
		a′) $\{F, E_1, E_2\}$		AC: $\{E_1\}\{F, FE_2\}$;	
		b) $\{T, E_2, E_3\}$		AC: $\{T\}\{TE_2, TE_2E_3\}$;	
15/10	(2b,43)	a) $\{N, T, E_2\}$	$D_6 \times D_1$,	AC: $\{T\}\{N, NE_2\}$,	3.2;
		a′) $\{N, E_1, E_2\}$		AC: $\{E_1\}\{N, NE_2\}$;	
		b) $\{T, E_2, E_3\}$		AC: $\{T\}\{TE_2, TE_2E_3\}$,	

I) **n**

1) $\{S\}$, $S^n = I$, C_n,

2) $\mathbf{n} \times \mathbf{1}^{(p)} = \{S\} \times \{c_1\} \cong \{S\} \times \{S_1\}$, $n \geq p \geq 3$, $C_n \times C_p$,

AC: $\{S\}$, $n \equiv 0 \pmod 2$, $p \equiv 1 \pmod 2$, 1.1;

$\{S_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 0 \pmod 2$, 1.1;

$\{S, S_1\}$, $n \equiv 0 \pmod 2$, $n = p$, 2.2;

$\{S\}\{S_1\}$, $n \equiv 0 \pmod 2$, $p \equiv 0 \pmod 2$, $n \neq p$, 2.1.

22/01	(2a,13)	$\{K\} \times \{K_1\}$	$C_3 \times C_3$;		
20/01	(2a,10)	$\{R\} \times \{K\}$	$C_4 \times C_3$,	AC: $\{R\}$,	1.1;
19/02	(1a,I,13)	$\{R\} \times \{R_1\}$	$C_4 \times C_4$,	AC: $\{R, R_1\}$,	2.2;
23/01	(2a,14)	$\{Z\} \times \{K\}$	$C_6 \times C_3$,	AC: $\{Z\}$,	1.1;
20/05	(1a,I,7)	$\{Z\} \times \{R\}$	$C_6 \times C_4$,	AC: $\{Z\}\{R\}$,	2.1;
23/02	(1a,I,16)	$\{Z\} \times \{Z_1\}$	$C_6 \times C_6$,	AC: $\{Z, Z_1\}$,	2.2.

3) $\mathbf{n}^{(p)} = \{c_1S\} \cong \{\bar{S}_1\}$, $n > p \geq 4$, $(n,p) \neq 1$, $n \not\equiv 0 \pmod p$, $C_{/n,p/}$,

AC: $\{\bar{S}_1\}$, $/n,p/ \equiv 0 \pmod 2$, 1.1

20/02	(2a,9)	$\{M\}$	C_{12},	AC: $\{M\}$,	1.1.

3) $\mathbf{n}^{(p)} \times \mathbf{1}^{(q)} = \{c_1S\} \times \{c_1^{p/q}\} \cong \{\bar{S}_1\} \times \{S_2\}$, $n \equiv 0 \pmod p$, $p \equiv 0 \pmod q$, $p \neq q$, $q \neq 1$, $n \geq p \geq 4$, $C_n \times C_q$,

AC: $\{\bar{S}_1\}$, $n \equiv 0 \pmod 2$, $q \equiv 1 \pmod 2$, 1.1;

$\{\bar{S}_1, \bar{S}_1S_2\}$, $p = 4$, $n \equiv 0 \pmod 2$, $q \equiv 0 \pmod 2$, 2.2;

$\{\bar{S}_1\}\{S_2\}$, $p \neq 4$, $n \equiv 0 \pmod 2$, $q \equiv 0 \pmod 2$, 2.1.

18/01	(1a,I,12)	$\{D\} \times \{E\}$	$C_4 \times C_2$,	AC: $\{D, DE\}$,	2.2;
21/02	(1a,I,11)	$\{S\} \times \{E\}$	$C_6 \times C_2$,	AC: $\{S, E\}$,	2.1;
22/02	(1a,I,15)	$\{S\} \times \{K\}$	$C_6 \times C_3$,	AC: $\{S\}$,	1.1.

4) $\mathbf{n}^{(p)} = \{c_1S\} \cong \{\bar{S}_1\}$, $n \equiv 0 \pmod p$, $n \geq p \geq 3$, C_n,

AC: $\{\bar{S}_1\}$, $n \equiv 0 \pmod 2$, 1.1.

11/01	(2a,11)	$\{S'\}$	C_3;		
10/01	(1a,I,8)	$\{D\}$	C_4,	AC: $\{D\}$,	1.1;
21/01	(2a,12)	$\{B\}$	C_6,	AC: $\{B\}$,	1.1;
11/02	(1a,I,10)	$\{S\}$	C_6,	AC: $\{S\}$,	1.1.

II) **nm**

1) $\{S,T\}$, $S^n = T^2 = (ST)^2 = I$, D_n,

2) $\mathbf{nm} \times \mathbf{1}^{(p)} = \{S,T\} \times \{c_1\} \cong \{S,T\} \times \{S_1\}$, $n \geq 3$, $p \geq 3$, $D_n \times C_p$,

AC: $\{T\}$, $n \equiv 1 \pmod 2$, $p \equiv 1 \pmod 2$, 1.1;
$\{T\}\{S_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 0 \pmod 2$, 2.1;
$\{T, ST\}$, $n \equiv 0 \pmod 2$, $p \equiv 1 \pmod 2$, 2.2;
$\{S_1\}\{T, ST\}$, $n \equiv 0 \pmod 2$, $p \equiv 0 \pmod 2$, 3.2.

22/03 (2b,22)	$\{K,T\} \times \{K_1\}$	$D_3 \times C_3$,	AC: $\{T\}$,	1.1;
20/04 (2b,26)	$\{K,T\} \times \{R\}$	$D_3 \times C_4$,	AC: $\{T\}\{R\}$,	2.1;
23/06 82b,29)	$\{K,T\} \times \{Z\}$	$D_3 \times C_3$,	AC: $\{T\}\{Z\}$,	2.1;
20/08 (2b,24)	$\{R,T\} \times \{K\}$	$D_4 \times C_3$,	AC: $\{T\}\{RT\}$,	2.2;
19/03 (1b,XXXIII,15)	$\{R,T\} \times \{R_1\}$	$D_4 \times C_4$,	AC: $\{R_1\}\{T, RT\}$,	3.2;
20/18 (1b,XXXIII,13)	$\{R,T\} \times \{Z\}$	$D_4 \times C_6$,	AC: $\{Z\}\{T, RT\}$,	3.2;
22/04 (2b,27)	$\{Z,T\} \times \{K\}$	$D_6 \times D_3$,	AC: $\{T\}\{ZT\}$,	2.2;
20/15 (2b,27)	$\{Z,T\} \times \{R\}$	$D_6 \times C_4$,	AC: $\{R\}\{T, ZT\}$,	3.2;
23/07 (1b,XXXIII,12)	$\{Z,T\} \times \{Z_1\}$	$D_6 \times C_6$,	AC: $\{Z_1\}\{T, ZT\}$,	3.2.

3) $\mathbf{nm}^{(2)} \times \mathbf{1}^{(2q+1)} = \{S, {c_1}^{2q+1}T\} \times \{{c_1}^2\} = \{S, T'\} \times \{S_1\}$, $p = 4q+2$,
$D_n \times C_{2q+1}$, AC: $\{T'\}$ $n \equiv 1 \pmod 2$, 1.1;
$\{T', ST'\}$ $n \equiv 0 \pmod 2$, 2.2.

22/04 (2b,23)	$\{K,T'\} \times \{K_1\}$	$D_3 \times C_3$,	AC: $\{T'\}$,	1.1;
20/06 (2b,25)	$\{R,T'\} \times \{K\}$	$D_4 \times C_3$,	AC: $\{T', RT'\}$,	2.2;
23/03 (2b,28)	$\{Z,T'\} \times \{K\}$	$D_6 \times C_3$,	AC: $\{T', ZT'\}$,	2.2.

3) $\mathbf{n}^{(2)}\mathbf{m} \times \mathbf{1}^{(2q+1)} = \{{c_1}^{2q+1}S, T\} \times \{{c_1}^2\} \cong \{\bar{S}_1, T\} \times \{S_2\}$, $p = 4q+2$,
$n \equiv 0 \pmod 2$, $D_n \times C_{2q+1}$, AC: $\{\bar{S}_1\}\{T\}$, 2.1.

20/10 (2b,31)	$\{R',T\} \times \{K\}$	$D_4 \times C_3$,	AC: $\{R'\}\{T\}$,	2.1;
22/06 (1b,XXXIII,16)	$\{K',T\} \times \{K\}$	$D_6 \times C_3$,	AC: $\{K'\}\{T\}$,	2.1.

3) $\mathbf{nm}^{(p)} = \{S, c_1T\} \cong \{S, \widetilde{S}_1\}$, $p \equiv 0 \pmod 4$, $n > 2$,
AC: $\{\widetilde{S}_1\}$, $n \equiv 1 \pmod 2$, 1.1;
$\{\widetilde{S}_1, S\widetilde{S}_1\}$, $n \equiv 0 \pmod 2$, 2.2.

20/03 (2b,19)	$\{K,F\}$	Q_3,	AC: $\{F\}$,	1.1;
19/01 (1b,XXXIII,8)	$\{R,F\}$	$\langle 2.2\|4;2\rangle$,	AC: $\{F, RF\}$,	2.2;
20/14 (1b,XXXIII,11)	$\{Z,F\}$	$Q_3 \times C_2$,	AC: $\{F, ZF\}$,	2.2.

3) $\mathbf{n}^{(p)}\mathbf{m} = \{c_1S, T\} \cong \{\bar{S}_1, T\}$, $p \equiv 0 \pmod 4$, $p \equiv 0 \pmod 2$,
AC: $\{\bar{S}_1\}\{T\}$, 2.1.

18/02 (1b,XXXIII,14)	$\{D,T\}$	$(4.4\|2.2)$,	AC: $\{D\}\{T\}$,	2.1;
20/09 (2b, 30)	$\{M,T\}$	$D_3 \times C_4$,	AC: $\{M\}\{T\}$,	2.1.

II) **nm**

1) $\{S,T\}$, $S^n = T^2 = (ST)^2 = I$, D_n.

2) $\mathbf{nm} \times \mathbf{1}^{((p)}\mathbf{1'}^{)} = \{S,T\} \times \{c_1, e_1\} \cong \{S,T\} \times \{S_1, T_1\}$, $n \geq p \geq 3$, $D_n \times D_p$,

AC: $\{T\}\{T_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 1 \pmod 2$, $n \neq p$, 2.1;
$\{T, T_1\}$, $n \equiv 1 \pmod 2$, $n = p$, 2.2;
$\{T\}\{T_1, S_1T_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 0 \pmod 2$, 3.2;
$\{T_1\}\{T, ST\}$, $n \equiv 0 \pmod 2$, $p \equiv 1 \pmod 2$, 3.2;
$\{T, ST\}\{T_1, S_1T_1\}$, $n \equiv 0 \pmod 2$, $p \equiv 0 \pmod 2$, $n \neq p$, 4.6;
$\{\{T, ST\}\{T_1, S_1T_1\}\}$, $n \equiv 0 \pmod 2$, $n = p$, 4.16.

20/08 (2b,47) $\{K,T\} \times \{K_1,T_1\}$ $D_3 \times D_3$, AC: $\{T,T_1\}$, 2.2;
20/17 (2b, 50) $\{R,T\} \times \{K_1,T_1\}$ $D_4 \times D_3$, AC: $\{T_1\}\{T,RT\}$, 3.2;
23/10 (2b, 54) $\{Z,T\} \times \{K_1,T_1\}$ $D_6 \times D_3$, AC: $\{T_1\}\{T,ZT\}$, 3.2;
19/06 (1b,XXXV,10) $\{R,T\} \times \{R_1,T_1\}$ $D_4 \times D_4$, AC: $\{\{T,RT\}\{T_1,R_1T_1\}\}$, 4.16;
20/22 (1b,XXXV,8) $\{Z,T\} \times \{R_1,T_1\}$ $D_6 \times D_4$, AC: $\{T,ZT\}\{T_1,R_1T_1\}$, 4.6;
23/11 (1b, XXXV,12) $\{Z,T\} \times \{Z_1,T_1\}$ $D_6 \times D_6$, AC: $\{\{T,ZT\}\{T_1,Z_1T_1\}\}$, 4.16.

3) $\mathbf{n}^{(2}\mathbf{m} \times \mathbf{1}^{(2q+1}\mathbf{1'}^{)} = \{{c_1}^{2q+1}S, T\} \times \{{c_1}^2, e_1\} \cong \{\bar{S}_1, T\} \times \{S_2, T_2\}$,
$n \geq p \geq 6$, $n \equiv 0 \pmod 2$, $p = 4q+2$, $D_n \times D_{2q+1}$,
AC: $\{\bar{S}_1\}\{T\}\{T_2\}$, $n \neq p$, 3.1;
$\{\bar{S}_1\}\{T, T_2\}$, $n = p$, 3.2.

22/11 (1b,XXXV,11) $\{K',T\} \times \{K_2,T_2\}$ $D_6 \times D_3$, AC: $\{K'\}\{T,T_1\}$, 3.2.

3) $\mathbf{nm}^{(2} \times \mathbf{1}^{(2q+1}\mathbf{1'}^{)} = \{S, {c_1}^{2q+1}T\} \times \{{c_1}^2, e_1\} \cong \{S, T'\} \times \{S_1, T_1\}$,
$n \geq p \geq 6$, $p = 4q+2$, AC: $\{T_1\}\{T', ST'\}$, 3.2.

23/09 (2b,55) $\{Z,T'\} \times \{K_1,T_1\}$ $D_6 \times D_3$, AC: $\{T_1\}\{T', ZT'\}$, 3.2.

3) $\mathbf{n}^{(p}\mathbf{m'}^{)} \times \mathbf{1}^{(2} = \{c_1S, e_1T\} \times \{{c_1}^{p/2}\} \cong \{\bar{S}_1, E_1\} \times \{E_2\}$, $n \equiv 0 \pmod p$,
$p \equiv 0 \pmod 2$, $p \geq 4$, $D_n \times C_2$, AC: $\{E_2\}\{\bar{S}_1E_1\}$, 3.2.

18/04 (1a,XI,9) $\{D, E_1\} \times \{E_2\}$ $D_4 \times C_4$, AC: $\{E_2\}\{E_1, DE_1\}$, 3.2;
21/04 (1a,XI,8) $\{S, E_1\} \times \{E_2\}$ $D_6 \times C_2$, AC: $\{E_2\}\{E_1, SE_1\}$, 3.2.

3) $\mathbf{n}^{(p}\mathbf{m'}^{)} = \{c_1S, e_1T\} \cong \{\bar{S}_1, E_1\}$, $n > p \geq 3$, $D_{/n,p/}$,
AC: $\{E_1\}$, $/n,p/ \equiv 1 \pmod 2$, 1.1;
$\{E_1, \bar{S}_1E_1\}$, $/n,p/ \equiv 0 \pmod 2$, 2.2.

20/11 (2a, 24) $\{M, E_1\}$ D_{12}, AC: $\{E_1, ME_1\}$, 2.2.

3) $\mathbf{nm'}^{)}\mathbf{1}^{(p} = \{S, e_1T, c_1\} \cong \{S, E_1, S_1\}$, $n \geq p \geq 3$,
AC: $\{E_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 1 \pmod 2$, 1.1;
$\{E_1, S_1E_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 0 \pmod 2$, 2.2;
$\{E_1, SE_1\}$, $n \equiv 0 \pmod 2$, $p \equiv 1 \pmod 2$, 2.2;
$\{(E_1, SE_1), (S_1E_1, SS_1E_1)\}$, $n \equiv 0 \pmod 2$, $p \equiv 0 \pmod 2$, 3.4;
$\{\{SE_1, S_1E_1\}, \{E_1, SS_1E_1\}\}$, $n \equiv 0 \pmod 2$, $n = p$, 3.7.

22/05 (2a,27)	$\{K, E_1, K_1\}$	$D(C_3 \times C_3)$,	AC: $\{E_1\}$,	1.1;
20/07 (2a,25)	$\{R, E_1, K_1\}$	D_{12},	AC: $\{E_1, RE_1\}$,	2.2;
23/05 (2a,28)	$\{K, E_1, K_1\}$	$D(C_3 \times C_3) \times C_2$,	AC: $\{E_1, ZE_1\}$,	2.2;
19/05 (1a,XI,10)	$\{R, E_1, R_1\}$	$D(C_4 \times C_4)$,	AC: $\{\{RE_1, R_1E_1\}, \{E_1, E_1RR_1\}\}$,	3.7;
20/19 (1a,XI,6)	$\{K, E_1, K_1\}$	$D_{12} \times C_2$,	AC: $\{(E_1, ZE_1), (R_1E_1, R_1ZE_1)\}$,	3.4;
23/08 (1a,XI, 13)	$\{Z, E_1, Z_1\}$	$D(C_6 \times C_6)$,	AC: $\{(ZE_1, Z_1E_1), (E_1, E_1ZZ_1)\}$,	3.7.

3) $\mathbf{n}^{(p}\mathbf{m1'}^{)} = \{c_1S, T, e_1\} \cong \{S_1, T, T_1\}$, $n \geq p \geq 4$, $(n,p) \neq 1$, $p \equiv 0 \pmod 4$,
AC: $\{\bar{S}_1\}\{T\}\{T_1\}$, $n \neq p$, 3.1;
$\{\bar{S}_1\}\{T, T_1\}$, $n = p$, 3.2.

18/05 (1b, XXXV,9)	$\{D, T, T_1\}$	$(C_2 \times C_2) \uparrow S_2$,	AC: $\{D\}\{T, T_1\}$,	3.2;
20/20 (2b,56)	$\{M, T, T_1\}$	$D_4 \times D_3$,	AC: $\{M\}\{T\}\{T_1\}$,	3.1.

3) $\mathbf{n'}^{)}\mathbf{m1}^{(p} = \{e_1S, T, c_1\} \cong \{\widetilde{S}, T, S_1\}$, $n > p \geq 3$, $n \equiv 0 \pmod 2$,
AC: $\{\widetilde{S}\}\{T\}$, $p \equiv 1 \pmod 2$, 2.1;
$\{T\}\{\widetilde{S}, S_1\widetilde{S}\}$, $p \equiv 0 \pmod 2$, 3.2.

20/12 (2b,44)	$\{F, T, K_1\}$	$(4,6\|2,2)$,	AC: $\{F\}\{T\}$,	2.1;
22/09 (2b,49)	$\{N, T, K_1\}$	$D_3 \times D_3$,	AC: $\{N\}\{T\}$,	2.1;
20/16 (2b,51)	$\{N, T, R\}$	$D_4 \times D_3$,	AC: $\{T\}\{N, RN\}$,	3.2.

3) $\mathbf{nm}^{(p}\mathbf{1'}^{)} = \{S, c_1T, e_1\} = \{S, \widetilde{S}, T_1\}$, $n \geq p \geq 4$, $p \equiv 0 \pmod 4$,
AC: $\{\widetilde{S}\}\{T_1\}$, $n \equiv 1 \pmod 2$, 2.1;
$\{T_1\}\{\widetilde{S}, S\widetilde{S}\}$, $n \equiv 0 \pmod 2$, 3.2.

19/04 (1b,XXXV,5)	$\{R, F, T_1\}$	32.36,	AC: $\{T_1\}\{F, RF\}$,	3.2;
20/21 (1b,XXXV,7)	$\{Z, F, T_1\}$	$(4,6\|2,2) \times C_2$,	AC: $\{T_1\}\{F, ZF\}$,	3.2.

3) $\mathbf{n}^{(2}\mathbf{m'}^{)}\mathbf{1}^{(2q+1} = \{{c_1}^{2q+1}S, e_1T, {c_1}^2\} \cong \{\bar{S}_1, E_1, S_2\}$, $n \geq p \geq 6$,
$n \equiv 0 \pmod 2$, $p = 4q+2$, AC: $\{E_1, \bar{S}_1E_1\}$, 2.2.

22/07 (1a,XI, 12)	$\{K', E_1, K\}$	$D(C_3 \times C_3) \times C_2$,	AC: $\{E_1, K'E_1\}$,	3.2.

3) $\mathbf{n'}^{)}\mathbf{m}^{(2}\mathbf{1}^{(2q+1} = \{e_1S, {c_1}^{2q+1}T, {c_1}^2\} \cong \{\widetilde{S}, T', S_1\}$, $n \geq p \geq 6$,
$n \equiv 0 \pmod 2$, $p = 4q+2$, AC: $\{\widetilde{S}\}\{T'\}$, 2.1.

22/10 (2b,48)	$\{N, T', K\}$	$D_3 \times D_3$,	AC: $\{N\}\{T'\}$,	2.1.

3) $\mathbf{n'}^{)}\mathbf{m}^{(p} = \{e_1S, c_1T\} \cong \{\widetilde{S}, \widetilde{S}_1\}$, $n \geq p \geq 4$, $n \equiv 0 \pmod 2$, $p \equiv 0 \pmod 4$,
AC: $\{\widetilde{S}\}\{\widetilde{S}_1\}$, $n \neq p$, 2.1;
$\{\widetilde{S}, \widetilde{S}_1\}$, $n = p$, 2.2.

18/03 (1b,XXXVI,1)	$\{F, F_1\}$	$(4,4\|2,2)$,	AC: $\{F, F_1\}$,	2.2;
20/13 (2b,45)	$\{N, F_1\}$	$(4,6\|2,2)$,	AC: $\{N\}\{F_1\}$,	2.1.

4) $\mathbf{n}^{(p}\mathbf{m'}^{)} = \{c_1S, e_1T\} \cong \{\bar{S}_1, E_1\}$, $p \geq 3$, $n \equiv 0 \pmod p$, D_n,
AC: $\{E_1\}$, $n \equiv 1 \pmod 2$, $p \equiv 1 \pmod 2$, 1.1;
$\{E_1, \bar{S}_1E_1\}$, $p \equiv 0 \pmod 2$, 2.2.

17/01 (2a,15)	$\{S', E_1\}$	D_3,	AC: $\{E_1\}$,	1.1;
16/01 (2a,III,1)	$\{D, E_1\}$	D_4,	AC: $\{E_1, DE_1\}$,	2.2;
21/03 (2a,26)	$\{B, E_1\}$	D_6,	AC: $\{E_1, BE_1\}$,	2.2;
17/02 (1a,III,2)	$\{S, E_1\}$	D_6,	AC: $\{E_1, SE_1\}$,	2.2.

The numbers $N_m(G)$ of the M^m-type simple and multiple antisymmetry groups, corresponding to AC-equivalency classes, and the possibility for their cataloging, is given in article [**21**]. According to the isomorphism of AC, all the AC obtained in this work can be distributed into the 13 equivalency classes, given together with the corresponding numbers N_m:

AC	N_1	N_2	N_3	N_4
1.1	1			
2.1	3	6		
2.2	2	3		
3.1	7	42	168	
3.2	5	24	84	
3.4	4	15	42	
3.7	3	10	28	
3.9	2	4	7	
3.10	1	1	1	
4.6	8	75	714	5040
4.16	5	39	357	2520
4.22	4	22	147	840

Except for the crystallographic four-dimensional point groups 1/01, 8/01, 11/01, 22/01 without AC, all the remaining 133 crystallographic four-dimensional point groups G_{40} belonging to the derived infinite classes of four-dimensional point groups G_{40}, are distributed into the following AC-equivalency classes:

1.1 1/02, 2/01, 2/02, 3/01, 7/01, 7/02, 8/02, 8/03, 8/04, 9/01, 9/02, 10/01, 11/02, 12/01, 14/01, 14/02, 14/03, 17/02, 20/01, 30/02, 20/03, 21/01, 22/02, 22/03, 22/04, 22/05, 23/01;

2.1 2/03, 4/02, 7/03, 7/04, 8/05, 9/03, 9/06, 12/03, 12/04, 13/01, 13/02, 14/04, 14/06, 14/07, 14/08, 14/09, 15/01, 18/02, 20/04, 20/09, 20/10, 20/12, 20/13, 21/02, 22/05, 22/06, 22/09, 22/10, 23/06;

2.2 3/02, 4/01, 4/03, 7/05, 7/06, 9/04, 9/05, 12/02, 13/03, 13/04, 14/05, 15/02, 15/03, 15/04, 15/05, 16/01, 17/02, 18/01, 18/03, 18/01, 19/02, 20/06, 20/07, 20/08, 20/11, 20/14, 21/03, 22/08, 23/02, 23/03, 23/04, 23/05;

2.3 5/01;

3.1 20/20;

3.2 4/04, 7/01, 9/07, 12/05, 13/05, 13/08, 13/09, 15/06, 15/07, 15/08, 15/09, 15/10, 18/04, 18/05, 19/03, 19/04, 20/15, 20/16, 20/17, 20/18, 20/21, 21/04, 22/07, 22/11, 23/07, 23/09, 23/10;

3.4 13//09, 15/11, 20/19;

3.7 5/02, 6/01, 19/05, 23/08;

3.9 6/20;

3.10 13/10, 15/12, 20/22;

4.16 19/06, 23/11,

4.22 6/03.

Hence, we can conclude that from the 137 crystallographic groups belonging to the infinite classes of four-dimensional point groups G_{40} are derived 387 M^1-type simple antisymmetry groups, and 1377 M^2-type, 5686 M^3-type, 21000 M^4-type multiple antisymmetry groups.

In the same way, using complete antisymmetry, (p)- and (p')-symmetry groups G'_{40}, G^p_{30} and $G^{p'}_{30}$ derived from all the infinite classes of four-dimensional point groups G_{40}, it is possible to obtain all the infinite classes of five-dimensional point groups G_{50} and to proceed with the same idea aiming to obtain all the infinite classes of n-dimensional point groups G_{n0} [**10**, **11**, **12**].

3. General procedure in higher dimensions. Every category G_{n0} contains a finite number of the polyhedral symmetry groups and their subgroups, and some infinite classes of n-dimensional point groups belonging to the categories $G_{n(n-1)0} = G_{n10}$, $G_{n(n-2)0} = G_{n20}$, $\dots$, $G_{nk0} = G_{n(n-k)0}$ ($[n/2] \leq k \leq n-1$). In the geometric classification of P-symmetries to every symmetry group of the category G_{k0} corresponds exactly one P-symmetry and vice versa. If P_{k0} denotes the set of all the P-symmetries corresponding to the category G_{k0}, the category G_{nk0} can be completely described by ${G_{n(n-k)0}}^{P_{k0}}$. After interpreting geometrically all the P-symmetry groups of the categories ${G_{n(n-k)0}}^{P_{k0}}$ ($k = 1, 2, \dots, [n/2]$), treating them as the symmetry groups for the larger category G_{n0}, and after excluding repeating groups, we can obtain all the n-dimensional point groups belonging to the infinite classes. In every such a case, the AC-method can serve as an efficient tool for deriving the corresponding simple and multiple antisymmetry groups of the M^m-type.

REFERENCES

[**1**] A. C., Hurley, *Finite rotation gorups and crystal classes in four dimensions*, Proc. Camb. Phil. Soc. **47** (1951), 650–661.

[**2**] J. Morzymas, A. Solecki, *R^4 point groups*, Rep. Mat. Physics **3** (1975), 363–394.

[**3**] A. C., Hurley, J. Neubüser, H. Wondratschek, *Crystal classes of four-dimensional space R^4*, Acta Cryst. **22** (1967), 605.

[**4**] H. Wondratschek, R. Bülow, J. Neubüser, *On crystallography in higher dimensions, III, Results in R_4*, Acta Cryst. **A27** (1971), 523–535.

[**5**] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, H. Zassenhaus, *Crystallographic Groups of Four-Dimensional Space*, Wiley-Interscience, New York, Chichester-Brisbane-Toronto, 1978.

[**6**] B. L. Van der Waerden, J. J. Burckhardt, *Farbgruppen*, Z. Kristallogr. **115** (1961), 231–234.

[**7**] A. Niggli, H. Wondratschek, *Eine Verallgemeinerung der Punktgrupen, I, Die einfachen Kryptosymmetrien*, Z. Kristallogr. **114** (1960), 215–231.

[**8**] H. Wondratschek, A. Niggli, *Eine Verallgemeinerung der Punktgruppen, I, Die mehrfachen Kryptosymmetrien*, Z. Kristallogr. **115** (1961), 1–20.

[**9**] A. M. Zamorzaev, E. I. Galyarskij, A. F. Palistrant, *Tsvetnaya simmetriya, eyo obobscheniya i prilozeniya*, Shtiintsa, Kishinev, 1978.

[**10**] A. M. Zamorzaev, Yu. S Karpova, A. P. Lungu, A. F. Palistrant, *P-simmetriya i eyo dal'neishee razvitie*, Kishinev, Shtiintsa, 1986.

[**11**] A. F. Palistrant, *Application of crystallographic groups of P-symmetry for studying* 5- *and* 6-*dimensional groups of symmetry*, Collected Abstracts of Paul Niggli Symp. (1984), 40–42.

[**12**] A. F. Palistrant, *Application of two-dimensional and three-dimensional crystallographic P-symmetry groups to the study of multi-dimensional symmetry groups*, Colloq. Math. Soc. János Bolyai **48**. Intutive Geometry, Siófok, 1985, 465–479.

[**13**] J. Bohm, K. Dornberger-Schiff, *The nomenclature of crystallographic symmetry groups*, Acta Cryst. **21** (1966), 1004–1007.

[**14**] A. V., Shubinkow, N. V. Belov et. al., *Colored Symmetry*, Pergamon Press, Oxford-London-New York-Paris, 1964.

[**15**] H. S. M. Coxeter, W. O. J. Moser, *Generators and Relations for Dicrete Groups*, 4th ed., Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[**16**] T. S. Kuncevich, N. V. Belov, *Geometricheskaya interpretaciya tochechnyh elementov simmetrii i reshetok Brave v chetyrehmernom prostranstve*, Acta Cryst. **A24** (1968), 42–51.

[**17**] A. M. Zamorzaev, *Teoriya prostoj i kratnoj antisimmetriji*, Shtiintsa, Kishinev, 1976.

[**18**] A. M. Zamorzaev, A. F. Palistrant, *Antisymmetry its generalizations and geometrical applications*, Z. Kristall. **151** (1980), 231–248.

[**19**] A. M. Zamorzaev, E. I. Sokolov, *Simmetriya i razlichnogo roda antisimmetrya konechnyh figur*, Kristallografiya **2** (1957), 9–14.

[**20**] S. V. Jablan, *Simple and multiple antisymmetry point groups* G^l_{30}, Mat. Vesnik **39** (1987), 301–308.

[**21**] S. V. Jablan, *Algebra of antisymmetric characteristics*, Publ. Inst. Math. **47 (61)** (1990), 39–55.

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