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# ON SOME NEUTRIX PRODUCTS OF DISTRIBUTIONS

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**Abstract.** The neutrix product [**2**] of the distributions $x_+^\lambda L(x)$ and $x^\mu$ or $\delta^{(m)}$ is analysed and explicitly calculated, where $\lambda, \mu \notin \overline{\mathbf{Z}}$, $m \in \mathbf{N}_0$ and $L$ is a slowly varying function at both zero and infinity [**7**].

The neutrix product of distributions is defined with a fixed infinitely differentiable function $\rho : \mathbf{R} \to [0, \infty)$, which has the following properties:

(i) $\rho(x) = 0$, $|x| \geq 1$,

(ii) $\rho(x) = \rho(-x)$, $x \in \mathbf{R}$,

(iii) $\displaystyle\int_{-1}^{1} \rho(x)\, dx = 1$.

The sequence $\delta_n(x) = n \cdot \rho(nx)$, $n \in \mathbf{N}$, $x \in \mathbf{R}$, is a so-called *"delta sequence"* i.e. it is a sequence of functions from the space $\mathcal{D}$ which tends to the measure $\delta$ in the topology of $\mathcal{D}'$. Further on, for arbitrary $g \in \mathcal{D}'$ we put

$$g_n(x) = g * \delta_n(x), \qquad n \in \mathbf{N},\ x \in \mathbf{R}. \tag{1}$$

Then the sequence of infinitely differentiable functions $\{g_n\}$ tends to $g$ in the topology of $\mathcal{D}'$.

This leads to the following definition of the product of two distributions on an open interval $(a, b)$:

*Definition* 1. Let $f$ and $g$ be distributions and let $g_n$ be as in (1). We say that the product $f \circ g$ exists and is equal to the distribution $h$ on $(a, b)$ if for each $\varphi \in \mathcal{D}(a, b)$

$$\lim_{n\to\infty} \langle f \cdot g_n, \varphi\rangle = \lim_{n\to\infty} \langle f, g_n \cdot \varphi\rangle =: \langle h, \varphi\rangle.$$

It turns out that this definition gives an extension of the product of continuous functions (observed as regular distributions). However, the neutrix product of distributions, see [**2**], is even more general. In order to define it we need

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*Definition* 2. A *neutrix* $N$ is a commutative additive group of functions $\nu : N' \to N''$ (where the domain $N'$ is a set and the range $N''$ is a commutative additive group) with the property that if $\nu$ is in $N$ and $\nu(\xi) = \gamma$ for all $\xi$ in $N'$, then $\gamma = 0$. The functions in $N$ are said to be *negligible*. Now suppose that $N'$ is contained in a topological space with a limit point $b$ which is not in $N'$ and let $N$ be a commutative additive group of functions $\nu : N' \to N''$ with the property that if $N$ contains a function of $\xi$ which tends to a finite limit $\gamma$ as $\xi$ tends to $b$, then $\gamma = 0$. It follows that $N$ is a neutrix. If now $f : N' \to N''$ and there exists a constant $\beta$ such that $f(\xi) - \beta$ is negligible in $N$, then $\beta$ is called the *neutrix limit* of $f(\xi)$ as $\xi$ tends to $b$ and we write $N\text{-}\lim_{\xi\to b} f(\xi) = \beta$, where $\beta$ is always unique if it exists.

Now let $N' = \mathbf{N}$ and $N'' = \mathbf{R}$ and $\mathbf{N}$ be the neutrix whose neglibigle functions are all linear sums of functions that tend to zero and all functions of the form

$$n^\lambda,\ n^\lambda \ln^{m-1} n,\ \ln^m n \tag{2}$$

for all real $\lambda \neq 0$ and $m \in \mathbf{N}$. Then we have

*Definition* 3. Let $f$, $g$ and $g_n$ be as in Definition 1. We say that the neutrix product $f \cdot g$ exists and is equal to $h$ on the open interval $(a, b)$ if

$$N\text{-}\lim_{n\to\infty} \langle f \cdot g_n, \varphi\rangle = N\text{-}\lim_{n\to\infty} \langle f, g_n \cdot \varphi\rangle =: \langle h, \varphi\rangle$$

for each $\varphi \in \mathcal{D}(a, b)$.

It is important to note that if the product of two distributions exists by Definition 1 then so does the neutrix product and they are equal, see [**2**]. However, the converse does not hold as the following example shows:

*Example* 1. Let $f = g = \delta$. Then for arbitrary $\varphi \in \mathcal{D}$ we have $\langle \delta \cdot \delta_n, \varphi\rangle = \delta_n(0)\varphi(0) = n\rho(0)\varphi(0)$. It follows that the product $\delta \circ \delta$ does not exist by Definition 1 but it does by Definition 3 and then $\delta \cdot \delta = 0$.

The neutrix product (Definition 3) has some "expected" properties of a product. For instance, if $h = f \cdot g$ exists for $f, g, \in \mathcal{D}'$, then

$$\operatorname{supp} h \subseteq \operatorname{supp} f \cap \operatorname{supp} g, \quad \text{and} \tag{3.1}$$

$$\operatorname{sing\,supp} h \subseteq \operatorname{sing\,supp} f \cup \operatorname{sing\,supp} g. \tag{3.2}$$

Further, if $f \cdot g$ and $f' \cdot g$ (or $f \cdot g'$) exist, then $f \cdot g'$ (or $f' \cdot g$) exists too and the Leibniz rule holds: $(f \cdot g)' = f' \cdot g + f \cdot g'$. However, the neutrix product is not commutative as the following example shows.

*Example* 2. Let $f = \delta$ and $g = x^{-1}$. Then $(x^{-1})_n = x^{-1} * \delta_n$ is an odd function and so $(x^{-1})_n(0) = 0$. Thus for arbitrary $\varphi \in \mathcal{D}$ we have

$$\langle \delta \cdot (x^{-1})_n, \varphi\rangle = (x^{-1})_n(0)\varphi(0) = 0$$

implying $\delta \cdot x^{-1} \cdot \varphi = 0$, but

$$\langle x^{-1} \cdot \delta_n, \varphi\rangle = \int_0^\infty x^{-1}[\delta_n(x)\varphi(x) - \delta_n(-x)\varphi(-x)]\, dx$$

$$= \int_0^\infty x^{-1}\delta_n(x)[\varphi(x) - \varphi(-x)]\,dx$$
$$= \varphi'(0) + O(1/n)$$

implying $x^{-1} \cdot \delta = -\delta'$.

On using Definition 3, one can find several important (neutrix) products of distributions, like $x_+^\lambda \ln^j x \cdot x_-^\mu$ for different values of $\lambda$, $\mu$ and $j$ (see [**2**], [**3**], [**4**]). However, the more general cases, like $x_+^\lambda L(x) \cdot x_-^\mu$ or $x_+^\lambda L(x) \cdot \delta^{(m)}(x)$ cannot be obtained with the neutrix used in the mentioned papers. Here and also throughout this paper $L : (0,\infty) \to (0,\infty)$ is a given locally integrable function which satisfies the following conditions

$$\lim_{x\to 0+} \frac{L(kx)}{L(x)} = 1 \quad \text{for any } k > 0, \tag{4.1}$$

$$\lim_{x\to\infty} \frac{L(kx)}{L(x)} = 1 \quad \text{for any } k > 0. \tag{4.2}$$

A positive locally integrable function satisfying (4.1) (resp. (4.2.)) is called *slowly varying at zero* (resp. *slowly varying at infinity*). The first example of a function satisfying the relations (4) is the logarithm.

The distribution $x_+^\lambda L(x)$ in $\mathcal{S}'_+$ (tempered distributions with supports in $[0,\infty)$), is defined for different values of the real parameter $\lambda$ by:

$$\langle x_+^\lambda L(x), \varphi(x)\rangle = \int_0^\infty x^\lambda L(x)\varphi(x)\,dx \quad \text{if } \lambda > -1, \tag{5.1}$$

$$\langle x_+^\lambda L(x), \varphi(x)\rangle = \int_0^\infty x^\lambda L(x)\left[\varphi(x) - \sum_{j=0}^{l-1} \frac{x^j}{j!}\varphi^{(j)}(0)\right] dx \tag{5.2}$$
$$\text{if } -(l+1) < \lambda < -l \text{ and } l \in N,$$

$$\langle x_+^\lambda L(x), \varphi(x)\rangle = \int_0^1 x^\lambda L(x)\left[\varphi(x) - \sum_{j=0}^{-\lambda-1} \frac{x^j}{j!}\varphi^{(j)}(0)\right] dx + \int_1^\infty x^\lambda L(x)\left[\varphi(x) - \sum_{j=0}^{-\lambda-2} \frac{x^j}{j!}\varphi^{(j)}(0)\right] dx \tag{5.3}$$
$$\text{if } \lambda \in \mathbf{Z}_- = \{-1,-2,\dots\} \text{ and } \varphi \in \mathcal{S}.$$

(By definition if $\lambda = -1$, then the last summation is omitted.)

It is worth noting that if $L(x) = 1$ on $(0,\infty)$, then the distribution $x_+^\lambda$ defined by relations (5) coincides with the distribution $x_+^\lambda$ defined in [**6**] (see also [**2**]):

$$x_+^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+l+1)} D^l x_+^{\lambda+l} \tag{6}$$

if $-(l+1) < \lambda < -l$, $l \in \mathbf{N}$. The distribution $x_-^\mu$ is defined in an analogous way to (6); its support is $(-\infty, 0]$.

The aim of this paper is to analyse and explicitely calculate several neutrix products involving slowly varying functions. For that reason we replace the neutrix $N$ from Definition 3 with the one whose negligible functions are all linear sums of functions that tend to zero and all functions of the form

$$n^\lambda,\ n^\lambda L(1/n),\ L(1/n) \tag{7}$$

for all real $\lambda \neq 0$ (compare with (2)). Naturally, if $L(1/n)$ tends to a non zero limit as $n \to \infty$, then $L(1/n)$ is omitted in (7).

Before we turn to the announced neutrix products, we cite two statements that we need later on.

THEOREM 1. *Let $L$ be a slowly varying function at zero and let $f$ be a locally integrable function on the interval $[0, b]$ with the property that*

$$\int_0^b x^{-\delta}|f(x)|\,dx < \infty \quad \textit{for some } \delta > 0.$$

*Then the integral*

$$\Phi(\varepsilon) = \int_0^b f(x)L(\varepsilon x)\,dx$$

*exists and*

$$\Phi(\varepsilon) \sim L(\varepsilon)\int_0^b f(x)\,dx \quad \textit{as } \varepsilon \to 0^+.$$

THEOREM 2. *Let $x_+^\lambda L(x)$ be given by (5.2) for $-(l+1) < \lambda < -l$, $l \in \mathbf{N}$ and $L$ a slowly varying function both at zero and at infinity. Then there exists a locally integrable function $K : (0,\infty) \to \mathbf{R}$ which is both slowly varying at zero and at infinity and satisfies the following conditions:*

$$D^l(x_+^{l+\lambda}K(x)) = x_+^\lambda L(x), \qquad K_l(x) \sim ((\lambda+1)\cdot\ldots\cdot(\lambda+l))^{-1}L(x)$$

*as $x \to 0^+$ and as $x \to +\infty$.*

Theorem 1 is an easy consequence of Théorème 2 from [**1**, p. 82], while Theorem 2 was proved in [**8**, p. 180, Lemma 2].

Because of (3.1) and (3.2) it is clear that

$$x_+^\lambda L(x)\cdot x_-^\mu = \sum_{j=0}^n a_j D^j\delta(x) \tag{8}$$

for some constants $a_j$ and some $n \in \mathbf{N}_0$ whatever $\lambda$ and $\mu$ are, provided that the left hand side exists. The case $L(x) \equiv 1$ was analysed in [**3**]:

THEOREM 3. *The neutrix product of $x_+^\lambda$ and $x_-^\mu$ exists and $x_+^\lambda \cdot x_-^\mu = 0$, provided that $\lambda + \mu \notin \mathbf{Z}_- = \{-1,-2,\ldots\}$.*

We first of all prove the following generalization of (8):

THEOREM 4. *The neutrix product of* $x_+^\lambda L(x)$ *and* $x_-^\mu$ *exists and*

$$x_+^\lambda L(x) \cdot x_-^\mu = 0 \tag{9}$$

*provided that* $\lambda, \mu, \lambda+\mu \notin \mathbf{Z}_-$, *i.e. all* $a_j$ *from* (8) *are zero.*

*Proof*. We follow the lines of the proof of Theorem 6 from [**3**], giving the modifications necessary because of the slowly varying function $L$.

Suppose first of all that $\lambda, \mu, \lambda+\mu > -1$. Then the left hand side of (9) exists even in the sense of Definition 1, since by (5.1) and (1) we have for $\varphi \in \mathcal{S}$

$$\begin{aligned}
&\langle x_+^\lambda L(x), (x_-^\mu)_n \varphi(x)\rangle \\
&= \int_0^\infty x^\lambda L(x)\varphi(x)\left(\int_{-\infty}^\infty (t-x)_-^\mu \delta_n(t)\,dt\right)dx \\
&= \int_0^{1/n} x^\lambda L(x)\varphi(x)\left(\int_x^{1/n} (t-x)^\mu \delta_n(t)\,dt\right)dx \\
&= \int_0^{1/n} \delta_n(t)\left(\int_0^t x^\lambda L(x)(t-x)^\mu \varphi(x)\,dx\right)dt \\
&= n^{-\lambda-\mu-2}\int_0^1 s^{\lambda+\mu+1}\delta_n\left(\frac{s}{n}\right)\int_0^1 v^\lambda(1-v)^\mu L\left(\frac{sv}{n}\right)\varphi\left(\frac{sv}{n}\right)dvds.
\end{aligned}$$

The function $L$ is slowly varying at zero and so we can find a positive number $\varepsilon > 0$ such that

$$|L(x)| \le Cx^{-\varepsilon} \quad \text{for } 0 < x < 1 \text{ and } \lambda+\mu+1-\varepsilon > 0; \tag{10}$$

the constant $C$ depends on $\lambda$, $\mu$ and $\varepsilon$. Further on, the function $\varphi$ is fast decreasing, hence bounded and so

$$|\langle x_+^\lambda L(x), (x_-)_n \varphi(x)\rangle| \le C' n^{-\lambda-\mu-1+\varepsilon}\int_0^1 s^{\lambda+\mu+1}\rho(s)\left(\int_0^1 v^\lambda(1-v)^\mu\,dv\right)ds$$

for some constant $C' > 0$ and $\varepsilon > 0$ from (10).

Now let $\lambda > -1$, $-(m+1) < \mu < -m$, $m \in \mathbf{N}$, and $\lambda+\mu \notin \mathbf{Z}$. As in [**3**], we have for $j \in \mathbf{N}_0$

$$\begin{aligned}
&\frac{\Gamma(\mu+m+1)}{\Gamma(\mu+1)}\int_{-\infty}^\infty x_+^\lambda L(x)(x_-^\mu)_n x^j\,dx \\
&= n^{-\lambda-\mu-j-1}\int_0^1 s^{\lambda+\mu+j+m+1}\rho^{(m)}(s)\int_0^1 v^{\lambda+j}(1-v)^{\mu+m}L\left(\frac{sv}{n}\right)dvds.
\end{aligned}$$

On using Theorem 1, we see that the right hand side behaves as

$$L\left(\frac{1}{n}\right)n^{-\lambda-\mu j-1}\mathrm{B}(\lambda+j+1, \mu+m+1)\int_0^1 s^{\lambda+\mu+j+m+1}\rho^{(m)}(s)\,ds.$$

Hence by the suppositions on $\lambda, \mu, j$ and the function $L$ we have

$$N\text{-}\lim_{n\to\infty} \int_{-\infty}^{\infty} x_+^{\lambda} L(x)(x_-^{\mu})_n x^j \, dx = 0. \tag{11}$$

We note that if $j \in \mathbf{N}$ is chosen so that $j > -(\lambda + \mu + 1)$ then (11) holds even in the usual sense. The remainder of the proof for the case $\lambda > -1, \mu, \lambda + \mu \notin \mathbf{Z}$ is essentialy as in [**3**, pp. 324–325] and so we omit it here.

Let us now suppose that (9) is proved for any $\lambda$ such that $\lambda > -l$, $\lambda \notin \mathbf{Z}$, any $\mu$ such that $\mu, \lambda + \mu \notin \mathbf{Z}$, and any slowly varying function $L$ at both zero and at infinity. On using Theorem 2 for given $\lambda - 1$ and $L$, we can find a function $K$ which is slowly varying at both zero and at infinity and satisfies

$$D^{l+1}(x_+^{\lambda+l} K(x)) = x_+^{\lambda-1} L(x). \tag{12}$$

The Leibniz rule gives

$$D^{l+1}(x_+^{\lambda+l} K(x) \cdot x_-^{\mu})$$
$$= D^{l+1}(x_+^{\lambda+l} K(x)) \cdot x_-^{\mu} + \sum_{j=0}^{l} \binom{l+1}{j} D^j(x_+^{\lambda+l} K(x)) \cdot D^{l+1-j}(x_-^{\mu}),$$

or on using (12)

$$x_+^{\lambda-1} L(x) \cdot x_-^{\mu} = \sum_{j=0}^{l} C_j D^j(x_+^{\lambda+l} K(x)) \cdot x_-^{\mu-l-1+j}$$

for some constants $C_j$, provided that we can show the existence of the right hand side. In fact we will show that each term in the last sum is zero. It is obviously enough to show that

$$D^j(x_+^{\lambda+l} K(x)) \cdot x_-^{\mu'} = 0 \tag{13}$$

for $j = 0, 1, \ldots, l$ and $\lambda, \mu', \lambda + \mu' \notin \mathbf{Z}$. This has been proved already for $j = 0$. If (13) is true for some $j \in \{0, 1, \ldots, l-1\}$, then

$$D^{j+1}(x_+^{\lambda+l} K(x)) \cdot x_-^{\mu'} = D(D^j(x_+^{\lambda+l} K(x)) \cdot x_-^{\mu'}) + \mu' D^j(x_+^{\lambda+l} K(x)) \cdot x_-^{\mu'-1} = 0,$$

i.e. it is true for $j+1$ as well. We have thus proved (9) for $\lambda, \mu, \lambda + \mu \notin \mathbf{Z}$.

We now prove

THEOREM 5. *The neutrix product of $x_+^{\lambda} L(x)$ and $\delta^{(m)}(x)$ exists and*

$$x_+^{\lambda} L(x) \cdot \delta^{(m)}(x) = 0 \tag{14}$$

*for $m \in \mathbf{N}_0$ and $\lambda \neq 0, \pm 1, \ldots, \pm m, -m-1, -m-2, \ldots$.*

*Proof*. Assume first that $\lambda > -1$ and $\lambda \neq 0, 1, \ldots, m$. Then for $j = 0, 1, \ldots$ we have

$$\int_{-\infty}^{\infty} x_+^\lambda L(x)\delta_n^{(m)}(x)x^j\,dx = n^{m-\lambda-j}\int_0^1 t^{j+\lambda}L\left(\frac{t}{\mu}\right)\rho^{(m)}(t)\,dt \qquad (15)$$

$$\sim n^{m-\lambda-j}L\left(\frac{1}{n}\right)\int_0^1 t^{j+\lambda}\rho^{(m)}(t)\,dt \quad \text{as } n\to\infty.$$

Hence the functions $\int_{-\infty}^{\infty} x_+^\lambda L(x)\delta_n^{(m)}(x)x^j\,dx$ are negligible for $j = 0,1,\dots$ and $j \neq m-\lambda$. Then we have

$$\left|\int_{-\infty}^{\infty} x_+^\lambda L(x)\delta_n^{(m)}(x)x^{m+1}\,dx\right| \le CL\left(\frac{1}{n}\right)n^{-\lambda-1}\int_0^1 t^{\lambda+m+1}|\rho^{(m)}(t)|\,dt \qquad (16)$$

for some constant $C > 0$. Since $L$ is slowly varying at zero, the right hand side of the last inequality tends to zero as $n$ tends to infinity. Using Taylor's theorem for a test function $\varphi \in \mathcal{S}$, we have

$$\varphi(x) = \sum_{j=0}^{m}\frac{x^j}{j!}\varphi^{(j)}(0) + \frac{x^{m+1}}{(m+1)!}\varphi^{(m+1)}(\xi x)$$

for some $\xi = \xi(x) \in [0,1]$. It follows from (15) and (16) that

$$N\text{-}\lim_{n\to\infty}\langle x_+^\lambda L(x), \delta_n^{(m)}(x)\varphi(x)\rangle = 0$$

i.e. (14) follows for $\lambda > -1$ and $\lambda \neq 0,1,\dots,m$.

Now assume that $-2 < \lambda < -1$. On using Theorem 2 we can find a locally integrable function $K$ which is slowly varying both at zero and infinity such that $D(x_+^{\lambda+1}K(x)) = x_+^\lambda L(x)$. Then

$$0 = D(x_+^{\lambda+1}K(x)\cdot\delta^{(m)}(x)) = x_+^\lambda L(x)\cdot\delta^{(m)}(x) + x_+^{\lambda+1}K(x)\cdot\delta^{(m+1)}(x).$$

It follows from what we have just proved that (14) holds for $-2 < \lambda < -1$. More generally, it follows by induction that (14) holds for $m \in \mathbf{N}_0$ and $\lambda \neq 0,1,\dots,m,-m-1,-m-2,\dots$.

We are now going to consider the product $x_+^\lambda L(x)\cdot x_-^{-m}$ for $\lambda \notin \mathbf{Z}_-$. For this purpose we note that by definition

$$D^m \ln x_+ = -(m-1)!x_+^{-m}, \qquad m \in \mathbf{N}$$

and this is in accordance with (5.3) for $L(x) \equiv 1$. Further

$$D^m \ln x_- = -(m-1)!x_-^{-m}, \qquad m \in \mathbf{N}.$$

THEOREM 6. *The product $x_+^\lambda L(x)\cdot x_-^{-m}$ exists and $x_+^\lambda L(x)\cdot x_-^{-m} = 0$ for $m \in \mathbf{N}_0$ and $\lambda \notin \mathbf{Z}_-$.*

*Proof*. For $\lambda > -1$ we have

$$\langle x_+^\lambda L(x), (x_-^{-m})_n x^j\rangle = \int_0^{1/n} x^{\lambda+j}L(x)\int_x^{1/n}\ln(t-x)\delta_n^{(m)}(t)\,dtdx$$

$$= \int_0^{1/n} t^{\lambda+j+1}\delta_n(t) \int_0^1 v^{\lambda+j} \ln(t-tv) L(tv)\, dvdt.$$

Putting $t = s/n$ and using the method from the proof of Theorem 4 we find that the last double integral is negligible. The rest of the proof is as in [**3**, p. 326], with the already used modifications.

We will now use the statement and the proof of Theorem 4 for finding the $\alpha$-product of $x_+^\lambda L(x)$ and $x_-^\mu$. This product was analysed in [**5**] as a natural generalization of the neutrix product defined by a vector.

*Definition* 4. Let $f$ and $g$ be distributions and let $g_n$ be as in (1). We say that the *$\alpha$-neutrix product of $f$ and $g$*, denoted by $f \overset{\alpha}{\cdot} g$, exists and is equal to the distribution vector $\mathbf{h} = [h_0, h_1, \dots, h_r, \dots]$ on the open interval $(a,b)$ if

$$N\text{-}\lim_{n\to\infty} \langle f, g_n\varphi\rangle = \langle h_0, \varphi\rangle, \qquad N\text{-}\lim_{n\to\infty} n^{-\alpha-r}\langle f, g_n\varphi\rangle = \langle h_r, \varphi\rangle \tag{17}$$

for $r = 1, 2, \dots$ and all test functions $\varphi \in \mathcal{D}(a,b)$. It is supposed that $-1 < \alpha \le 0$.

The following generalization of Theorem 3 was proved in [**5**].

Theorem 7. *Let $\lambda, \mu$ be real numbers such that $\lambda, \mu, \lambda+\mu \notin \mathbf{Z}$ and $\lambda+\mu < -1$. Then the $\alpha$-product $x_+^\lambda \overset{\alpha}{\cdot} x_-^\mu$ exists and*

$$x_+^\lambda \overset{\alpha}{\cdot} x_-^\mu = \mathbf{h}(\lambda,\mu) = [0, h_1(\lambda,\mu), \dots, h_q(\lambda,\mu)],$$

*where $q = [-\lambda-\mu]$, $\alpha = -\lambda-\mu-q-1$,*

$$h_i(\lambda,\mu) = \frac{\mathrm{B}(\mu+1, \lambda+q-i+1)}{(q-i)!}(-1)^{q-i} a_{q-i}(\lambda,\mu)\delta^{(q-i)},$$

$$a_i(\lambda,\mu) = \frac{(-1)^p\Gamma(\lambda+\mu+i+2)}{\Gamma(\lambda+\mu+p+i+2)} \int_0^1 u^{\lambda+\mu+p+i+1}\rho^{(p)}(u)\, du$$

*for $i = 1, \dots, q$ and $\mathrm{B}$ and $\Gamma$ denote the beta and gamma functions respectively.*

In order to find the $\alpha$-product of $x_+^\lambda L(x)$ and $x_-^\mu$, we must slightly change Definition 4. In fact, replacing (17) by

$$N\text{-}\lim_{n\to\infty} n^{-\alpha-r} L(1/n)\langle f, g_n\varphi\rangle = \langle h_r, \varphi\rangle, \qquad r \in \mathbf{N},$$

we get a product which will be denoted by $f \overset{(\alpha,L)}{\cdot} g$; $L$ is a slowly varying function with properties (4.1) and (4.2). Similarly as Theorem 4 from [**5**], one can prove

Theorem 8. *Let $\lambda, \mu$ be real numbers such that $\lambda, \mu, \lambda+\mu \notin \mathbf{Z}$ and $\lambda+\mu < -1$. Then the $(\alpha, L)$ product*

$$x_+^\lambda L(x) \overset{(\alpha,L)}{\cdot} x_-^\mu = \mathbf{h}(\lambda,\mu) = [0, h_1(\lambda,\mu), \dots, h_q(\lambda,\mu)],$$

*where $\alpha, q$ and $h_i(\lambda,\mu)$ are as given in Theorem 7.*

*Thus we can write*

$$\langle x_+^\lambda L(x), (x_-^\mu)_n \varphi\rangle = \langle 0, \varphi\rangle + \sum_{r=1}^{q} \frac{n^{\alpha+r}}{L(1/n)} \langle h_r, \varphi\rangle + O\left(\frac{n^\alpha}{L(1/n)}\right)$$

*as* $n \to \infty$.

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