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CHARACTERIZATION OF SOME SUBSPACES OF (D′) BY S-ASYMPTOTIC

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Abstract. We characterize by the S-asymptotic some subspaces of the space (D') of distributions, as (E'), (O'_c) and (B'). We give also, using S-asymptotic, sufficient conditions and necessary conditions that a distribution belongs to a subspace of (D')b

1. Introduction

One can find several notions connected with the asymptotic behavior of a distribution (see for example [1], [2] and [3]). In this article we use another definition of the asymptotic behavior of a distribution-S, asymptotic [4].

2. Notations and definitions

Let Γ be a cone in R^n with vertex at zero. By $\Sigma(\Gamma)$ we denote the set of all real valued functions $c(h)$, $h \in \Gamma$, which are different from zero when $h \in \Gamma$ $\|h\| \geq \beta_c$. $B(a,r)$ will be the ball $\{x \in R^n, \|a-x\| < r\}$.

We shall deal with the following subspaces of (D') (see [7]):

(E') the space of distributions with the compact support;

(S') the space of tempered distributions;

(O'_c) the space of distributions with a fast descent;

(D_{L^p}) the space of all functions $\varphi \in C^\infty$ which belong with all derivatives to $L^p(R^n)$, $1 \leq p \leq \infty$.

(D'_{L^p}) the space of continuous linear functionals on (D_{L_q}),

$$1 < p \leq \infty, \quad 1 \leq q < \infty, \quad q = p/(p-1);$$

$(B') = (D'_{L^\infty})$;

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(K_p), $p \geq 1$, the space of all function $\varphi \in C^\infty$ such

$$\text{that } \nu_k(\varphi) = \sup_{x \in R^n, |a| \leq k} \exp(k\|x\|^p)|D^\alpha \varphi(x)| < \infty, \quad k = 1, 2, \ldots$$

(K'_p) the space of continuous linear functionals on (K_p) (see [6]).

Definition 1. A distribution $T \in (D')$ has the S-asymptotic in the cone Γ related to some $c(h) \in \Sigma(\Gamma)$ and with the limit $U \in (D')$ if there exists

(1) $$lim_{h \in \Gamma, \|h\| \to \infty} \langle T(x+h)/c(h), \varphi(x) \rangle = \langle U, \varphi \rangle, \quad \varphi \in (D).$$

Then we write $T(x+h) \overset{s}{\sim} c(h)U(x)$, $h \in \Gamma$.

Remark. We can give another expression for $\langle T(x+h), \varphi(x) \rangle$. We know that

(2) $$\langle T(x+h), \varphi(x) \rangle = \langle T(x), \varphi^{\smile}(h-x) \rangle = (T * \varphi^*)(h)$$

where $\varphi^{\smile}(x) = \varphi(-x)$. It is well known [7 T. II, p. 22] that $T * \varphi^*(h)$ is a function which has all derivatives (in the usual sense) and

(3) $$\frac{\partial}{\partial h_k}(T * \varphi^{\smile})(h) = (T * \frac{\partial}{\partial x_k}\varphi^{\smile})(h).$$

3. Characterization of some subspaces of distributions by the S-asymptotic

PROPOSITION 1. *Let Γ be a cone. A necessary and sufficient condition that the support of $T \in (D')$ has the property: For every $r > 0$ there exists β_r such that the sets $\{\operatorname{supp} T \cap B(h,r)\}$, $h \in \Gamma$, $\|h\| \leq \beta_r$, are empty is that $T(x+h) \overset{s}{\sim} c(h) \cdot 0$, $h \in \Gamma$ for every $c(h) \in \Sigma(\Gamma)$.*

The proof of Proposition 1 will be based on the following

LEMMA 1. *Necessary and sufficient that for every $c(h) \in \Sigma(\Gamma)$*

(4) $$\lim_{h \in \Gamma, \|h\| \to \infty} T(x+h)/c(h) = 0 \ \text{ in } \ (D')$$

is that for every $\varphi \in (D)$ there exists a $\beta(\varphi)$ such that

(5) $$\langle T(x+h), \varphi(x) \rangle = 0, \ \ h \in \Gamma, \ \ \|h\| \geq \beta(\varphi).$$

Proof of Lemma 1. From our relation (4) it follows that for every $\varepsilon > 0$ there exists a $\beta(\varphi, c, \varepsilon)$ such that

$$|\langle T(x+h)/c(h), \varphi(x) \rangle| < \varepsilon, \quad h \in \Gamma, \ \ \|h\| \geq \beta(\varphi, c, \varepsilon).$$

We denote by $\beta_0(\varphi, c, \varepsilon)$ the infimum of all numbers $\beta(\varphi, c, \varepsilon)$ for a fixed φ, c and ε. First we prove that the set $\{\beta_0(\varphi, c, \varepsilon), \varepsilon > 0\}$is bounded by a $\beta_0(\varphi, c) < \infty$. That means that $\langle T(x+h)/c(h), \varphi(x) \rangle = 0 \ h \in \Gamma$, $\|h\| \geq \beta_0(\varphi, c)$. Assume the contrary. Then there exists a sequence $\{h_k\} \in \Gamma$, $\|h_k\| \to \infty$ such that

$\langle T(x+h_k)/c(h_k), \varphi(x)\rangle = a_k \neq 0$. We choose $c_1(h) \in \Sigma(\Gamma)$, $c_1(h_k) = a_k$. Now $T(x+h)/c(h)c_1(h)$ does not converge to zero in (D') as $h \in \Gamma$ and $\|h\| \to \infty$. Hence, such a sequence $\{h_k\}$ does not exist.

To prove that the set $\{\beta_0(\varphi, c), c(h) \in \Sigma(\Gamma)\}$ is bounded by a $\beta_0(\varphi)$ we assume the contrary. Then there exists a sequence $\{h_k\} \subset \Gamma, \|h_k\| \to \infty$ and the sequence $\{c_k(h)\} \subset \Sigma(\Gamma)$ so that $\langle T(x+h_k)/c_k(h_k), \varphi(x)\rangle = d_k \neq 0$.

Now for $c_2'(h), c_2'(h_k) = c_k(h_k) \cdot d_k$, $\langle T(x+h)/c_2'(h), \varphi(x)\rangle$ does not converge to zero when $h \in \Gamma$, $\|h\| \to \infty$.

So we have proved that (5) follows from (4) The converse is trivial.

Proof of Proposition 1. Assume that the support of $T \in (D')$ has the property given in Proposition 1. We know that supp $T(x+h) = \operatorname{supp} T - h$, $h \in \Gamma$. Hence the sets $\{\operatorname{supp} T(x+h) \cap B(0,r)\}$, $h \in \Gamma$, $\|h\| \geq \beta_r$ are empty.

For every $\varphi \in (D)$ there exists $r > 0$ such that $\operatorname{supp} \varphi \subset B(Or)$. That gives

$$\langle T(x+h), \varphi(x)\rangle = 0, \quad h \in \Gamma, \quad \|h\| \geq \beta_r$$

By Lemma 1 we get (4).

Suppose now that the limit in relation (4) exists for every $c(h) \in \Sigma(\Gamma)$. By Lemma 1, relation (5) is true. Let $\beta_0(\varphi) = \inf \beta(\varphi)$, where $\beta(\varphi)$ are numbers from relation (5). We prove that the set $\{\beta_0(\varphi), \varphi(D_K)\}$ for every compact set $K \subset R^n$ is bounded. Assume the contrary. Then there exists a sequence $\{h_k\}$, $h_k \in \Gamma$, $\|h_k\| \to \infty$ and the sequence $\{\Phi_k(x)\}$, $Phi_k \in (D_K)$ such that

$$\langle T(x+h_k), \psi(x)\rangle = A_{k,p} = \begin{cases} a_k \neq 0, & p = k \\ 0, & p < k. \end{cases}$$

The construction of the sequences $\{h_k\}$ and $\{\Phi_k\}$ can go as follows. Let $\Phi_k \in (DK)$ be such that $\{\beta_0(\Phi_K)\}$ is a strict monotone sequence which tends to infinity. Then there exist $\{h_k\} \subset \Gamma$ and $\varepsilon_k > 0$, $k \in N$ such that $\beta_0(\Phi_{k-1}) + \varepsilon_k \leq \|h_k\| \leq \beta_0 \Phi_k) - \varepsilon_k$.

Now, we construct the sequence $\{\psi_p(x)\} \subset (DK)$ such that

$$\langle T(x+h_k), \psi_p(x)\rangle = \begin{cases} a_k, & p = k \\ 0, & p \neq k \end{cases}$$

Let $\psi_p(t) = \Phi_p(t) - \lambda_1^p \Phi_1(t) - \cdots - \lambda_{p-1}^p \Phi_{p-1}(t)$, $p > 1$. The numbers $\{\lambda_i^p\}$ can be found so that $\psi_p(t)$ satisfies the desired property.

It is easy to see that $\langle T(t+h_k), \psi_p(t)\rangle = ak$ and $\langle T(t+h_k), \psi_p(t)\rangle = 0$, $k > p$. For a fixed p and $k < p$ we can find $\lambda_i^p, i = 1, \ldots, p-1$ so that

$$0 = \langle T(t+h_k), \psi_p(t)\rangle = A_{k,p} - \lambda_1^p A_{k,1} - \cdots - \lambda_{p-1}^p A_{k,p-1}, k = 1, \ldots, p-1.$$

Hence

$$\lambda_1^p A_{k,1} + \ldots \lambda_{p-1}^p A_{k,p-1} = A_{k,p} k = 1, \ldots, p-1, \;\; p > 1.$$

As $A_{k,k} \neq 0$ for every k, this system has always a solution.

We introduce now a sequence of numbers $\{b_k\}$, $b_k = \text{supp}\,\{2^k|\psi_k^(i)(t), i \leq k\}$. Then

$$\psi(t) = \sum_{p=1}^{\infty} \psi_p(t)/b_p \in (D_k)$$

and the series converges in (D), thus in (DK) as well

$$\langle T(t+h_k), \psi(t)\rangle = \sum_{p=1}^{\infty} T(t+h_k), \quad \psi_p(t)/b_p = a_k/b_k$$

Now we choose $c(h)$, $c(h_k) = a_k/b_k$; $\langle T(t+h)/c(h), \psi(t)\rangle$ does not converge as $\|h\| \to \infty$, $h \in T$.

This proves that for every compact set K there exists $\beta_0(K)$ such that

$$\langle T(t+h), \Phi(t)\rangle = 0, \ \ \|h\| \geq \beta_0(K), \ \ h \in \Gamma, \ \ \Phi \in (D_k).$$

It follows that $T(t+h) = 0$ on $B(0,r)$, $\|h\| \geq \beta(r)$, $h \in \Gamma$, and with this $T(t) = 0$ on $B(h,r)$, $\|h\| \geq \beta(r)$, $h \in \Gamma$.

A consequence of Proposition 1 is the following

PROPOSITION 2. *A necessary and sufficient condition that a distribution T belongs to (E') is that $T(x+h) \overset{s}{\sim} c(h)\cdot 0$, $h \in R^n$, for every $c(h) \in \sum(R^n)$.*

Reimarks. In Proposition 1 the support of $T \in (D')$ has to have the following property: the distance from the supp$\,T$ and a point $h \in \Gamma$, $d(\text{supp}\,T, h)$ tends to infinity when $h \in \Gamma$, $\|h\| \to \infty$.

As a consequence of Proposition 1 and Lemma 1 we have a result on the support of a factor of the convolution. Let G be the set of all functions $f \in C^{\infty}$ so that supp$\,f$ lies in the complement of the set $\Gamma \cap \{h \in R^n, \|h\| \geq \beta_f\}$.

COROLLARY 1. *For a fixed $T \in (D')$ the convolution $T * \varphi$ maps (D) into G if and only if the support of T has the property given in Proposition 1.*

Proof. We have only to combine Lemma 1 and Proposition 1. From Proposition 1 it follows that the S-asymptotic is a local property.

COROLLARY 2. *A necessary and sufficient condition that two distributions T_1 and T_2 coincide on an open set A, $C_{R^n}A$ having the property of the supp T from Proposition 1, is that $T_1(x+h) - T_2(x+t) \overset{s}{\sim} c(h)\cdot 0$, $h \in \Gamma$, for every $c(h) \in \Sigma(\Gamma)$.*

Proof. If $T_1 = T_2$ on A, then supp$\,(T_1 - T_2)$ has the property from Proposition 1.

PROPOSITION 3. *Necessary and sufficient condition that $T \in (D')$ belongs to (O_c') is that T has S-asymptotic zero related to every $c(h) = \|h\|^{-\alpha}$, $\alpha \in R^+$.*

Proof. We have only to use Theorem IX, T. II, p. 100 of [7] which says: The necessary and sufficient condition that a distribution T belongs to (O'_c) is that for every $\varphi \in (D)$ the function $(T * \varphi)(h)$ be continuous and of fast descent at infinity. Now, Proposition 3 follows from (2) and the definition of a function of fast descent.

PROPOSITION 4. *Necessary and sufficient condition that a distribution T belongs to (B') is that T has the S-asymptotic zero related to every $c(h) \in \Sigma(R^n)$, $c(h) \to \infty$, as $\|h\| \to \infty$.*

Proof. By (2) $\langle T(x+h)/c(h), \varphi(x)\rangle = (T * \varphi^{\smile})(h)/c(h)$. Theorem XXV, T, I I, p. 57 of [7] says that $(T * \varphi)(h) \in L^\infty(R^n)$ for a $T \in (B')$ and every $\varphi \in (D)$. Hence $(T * \varphi^{\smile})(h)/c(h) \to 0$, when $\|h\| \to \infty$ and $c(h) \to \infty$.

Suppose that $(T * \varphi^{\smile})(h)/c(h) \to 0$, $h \to \infty$, for every $\varphi(D)$ and $c(h) \to \infty$ as $\|h\| \to \infty$. We show that $(T * \varphi^{\smile})(h) \in L^\infty(R^n)$ for every $\varphi(D)$. Then, by the same theorem, it follows that $T \in (B')$. To prove this let us assume the contrary, i.e. that $(T * \varphi^{\smile})(h)$ is not bounded for a $\varphi_0 \in (D)$. Then for the sequence of balls $\{B(0,n),\, n \in N\}$ we can find two sequences $\{h_n\} \subset R^n$ and $\{c_n\} \subset R$ such that $|c_n| \to \infty$ as $n \to \infty$; $\|h_n\| \geq n$ and $(T * \varphi_0^{\smile})(h_n) = c_n$. Now, for $c_0(h)$ such that $c_0(h_n) = \sqrt{|c_n|}$ the limit $\langle T(x+h)/c_0(h), \varphi(x)\rangle$ does not exist when $\|h\| \to \infty$. This is in contradiction with our assumption that T has S-asymptotic related to every $c(h)$ which tends to infinity as $\|h\| \to \infty$.

PROPOSITION 5. *Let for every $c(h) \in \Sigma(R^n)$, which has a fast descent, $T(x+h) \overset{s}{\sim} c^{-1}(h)U_c(t)$, $h \in R^n$. Then $T \in (S')$. (U_c can be the distribution zero as well).*

Proof. For a fixed $c(h)$ and $\|h\| \geq \beta_0$, for every $\varphi(D)$ we have:

$$|\langle T(x+h) \cdot c(h), \varphi(x)\rangle| \leq |\langle u, \varphi\rangle| + \varepsilon_\varphi \leq M_\varphi + \varepsilon_\varphi.$$

Therefore the set $\{T(x+h) \cdot c(h), h \geq \beta_0\}$ is weakly bounded and thus bounded in (D') [7, Theorem IX, T. I, p. 72]. Using Theorem VI of [7, T. II, d. 95] we obtain that $T \in (S')$.

A similar proposition can be proved for the space (K'_1) using the following theorem [5] :

Let $T \in (D')$. If for every rapidly decreasing function $r(x)$ the set $\{r(h)T(x+h), h \in R^n\}$ is bounded in (D'), then $T \in (K'_1)$.

A function $r(x)$, defined on R^n, is called rapidly exponenttially decreasing function if for every $k > 0$ $r(x)\exp(k\|x\|) \to 0$ as $\|x\| \to \infty$.

PROPOSITION 6. *Let for every rapidly exponentially decreasing function $r(h) \in \Sigma(R^n)$ $T(x+h) \overset{s}{\sim} r^{-1}(h)U_r$, $h \in R^n$, then $T \in (K'_1)$.*

The next propositions do not give a full characterization of some subspaces of distributions, but the property of the S-asymptotic their members.

PROPOSITION 7. *Every distribution which belongs to (D'_{L^p}), $1 \leq p < \infty$ has S-asymptotic related to $c(h) \equiv 1$ just zero.*

Proof. We use relation (2). By theorem XXV of [7] T. II, p. 57 it follows that $(T * \varphi^{\smile})(h)$ is in $L^p(R^n)$ for every $\varphi^{\smile} \in (D)$. By relation (3) we know that every derivative of $(T * \varphi)(h)$ is also in $L^p(R^n)$. Hence $(T * \varphi^{\smile})(h) \in (D_{L^p})$. We know that every element of (D_{L^p}), $1 \leq d < \infty$ is bounded over R^n and tends to zero when $\|h\| \to \infty$ ([7] T. II, p. 55).

PROPOSITION 8. *If* $\in (S')$ *then there exists a real number* k_0 *such that* T *has S-asymptotic zero related to* $c(h)\|h\|^{k_0}$ *where* $c(h)$ *tends to infinity when* $\|h\| \to \infty$.

Proof. By Theorem VI of [7 T.II, p. 75] there exists a number k_0 such that the set of distributions $\{T(x+h)/(1+\|h\|^2)^{k_0/2}, h \in R^n\}$ is bounded in (D'). Hence this set is weakly bounded and

$$\left\langle T(x+h)/(c(h)\|h\|^{k_0}, \varphi(x)) \right\rangle = \frac{1+\|h\|^2)^{k_0/2}}{c(h)\|h\|^{k_0}} \left\langle \frac{T(x+h)}{(1+\|h\|^2)^{k_0/2}}, \varphi(x) \right\rangle$$

tends to zero when $\|h\| \to \infty$.

PROPOSITION 9. *If* $T \in (K_p')$ *then there exists a* k_0 *such that* T *has S-asymptotic zero related to* $c(h)\exp(k_0\|h\|^p)$, *where* $c(h)$ *tends to infinity when* $\|h\| \to \infty$.

First we prove a lemma which is implicit in the proof of Theorem I [6].

LEMMA 2. *Let* $T \in (K_p')$. *There exists a positive integer* k, *such that* $\{T(x+h)\exp(-k\|h\|^p), h \in R^n\}$ *is a bounded set in* (D').

Proof. We start by giving a bound for $\nu_k(\varphi(x-h))$, $\varphi \in Kp$:

$$\begin{aligned} \nu_k[\varphi(x-h)] &= \sup_{x\in R^n, |a|\leq k} \exp(k\|x\|^p)|D^a\varphi(x-h)| \\ &= \sup_{x\in R^n, |a|\leq k} \exp(k\|x+h\|^p)|D^a\varphi(x)| \\ &= \exp(2^pk\|h\|^p) \sup_{x\in R^n, |a|\leq k} \exp(2^pk\|x\|^p)|D^a\varphi(x)| \\ &\leq \exp(2^pk\|h\|^p)\nu_2p_k(\varphi) \end{aligned}$$

By our assumption, T is continuous linear functional on (Kp). Then there exist $\varepsilon > 0$ and k_0 such that

$$|\langle T, \varphi\rangle| \leq 1 \text{ for } \varphi \in (K_p), \nu_{k_0}(\varphi) \leq \varepsilon \tag{6}$$

Since the seminorms ν_k are increasing, relation (6) holds for all $k \geq k_0$. Let φ be any element of $(K_p) \cdot \varphi^1 = \varepsilon\varphi/\nu_k(\varphi)$ satisfies $\nu_k(\varphi^1) \leq \varepsilon$, $k \geq k_0$ and $|\left\langle T, \varphi^1\right\rangle| \leq 1$. Hence

$$|\langle T, \varphi\rangle| \leq \varepsilon^{-1}\nu_k(\varphi), \;\; k \geq k_0 \text{ for every } \varphi(K_p). \tag{7}$$

We know that $(D) \subset (K_p)$ and that this injection is continuous. Let us suppose that $\varphi \in (D)$, then

$$\begin{aligned} |\langle \exp(-2^p k\|h\|^p)T(x+h), \varphi(x)\rangle| &= |\langle T(x), \exp(-2^p k\|h\|^p)\varphi(x-h)\rangle| \\ &\leq \varepsilon^{-1}\exp(-2^p k\|h\|^p)\nu_k[\varphi(x-h)] \leq \varepsilon^{-1}\nu_2 p_k(\varphi). \end{aligned}$$

Proof of Proposition 9. Now, we use Lemma 2 for the proof.

We can choose $k_0 \geq 2^p k$. The set $\{\exp(-k_0\|h\|^p)T(x+h),\ h \in R^n\}$ is bounded in (D') and weakly bounded in (D').

For every $\varphi \in (D)$:

$$\lim_{\|h\|\to\infty} \langle \exp(-k_0\|h\|^p)T(x+h)/c(h), \varphi(x)\rangle =$$

$$= \lim_{\|h\|\to\infty} \frac{1}{c(h)} \langle \exp(-k_0\|h\|^p)T(x+h), \varphi(x)\rangle = 0.$$

REFERENCES

[1] Ю.А. Бричков, Ю.М. Широков, *Об асимптотическом поведении преобразований Фурье*, Теорет. Мат. Физ. **4** (1970), 301–309.

[2] Б.И. Завьялов, *Автомодельная асимптотика электромагнитных форм-факторов и поведение их Фурье-образов в окресности световотго конуса*, Теорет. Мат. Физ. **17** (1973), 178–188.

[3] J. Lavoine and O.P. Misra, *Théorme Abélians pour la transformation de Stieltjes des distributions*, C.R. Acad. Sci. Paris, A **279** (1974), 99–102.

[4] S. Pilipović and B. Stanković, *S-asymptotic of a distribution*, Pliska Stud. Math. Bulgar. (in print).

[5] S. Pilipović, *S-asymptotic of tempered and $K-1'$ distributions Part I*, Univ. u Novom Sadu Zb. Rad. Prirod Mat. Fak. Ser. Mat. **15** (1985), 47–58.

[6] G. Sampson, Z. Zielezny, *Hypoelliptic convolution equations in K_p', $p \geq 1$*, Trans. Amer. Math. Soc. **223** (1976), 133–154. .

[7] L. Schwartz, *Théorie des distributions*, Herman, Paris, T. I (1957), T. II (1951).

[8] B. Stanković, *Applications of the S-asymptotic*, Univ. u Novom Sadu Zb. Rad. Prir. Mat. Fak. Ser Mat. **15** (1985), 1–9.

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