PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 39 (53), 1986, pp. 79–82

# EXTENDING DERIVATIONS AND ENDOMORPHISMS TO SKEW POLYNOMIAL RINGS

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**Abstract.** We treat the problem of extending derivations and endomorphisms of a given ring $R$ to a skew polynomial ring $R[x, f, d]$ over $R$. As an application we obtain the general conditions for the existence of such rings in finitely many variables over $R$. We also prove that under suitable conditions, the $d$ (or the $f$)-simplicity of $R$ implies the $f$-simplicity of $R[x, f, d]$.

In [**3**] we have obtained the general conditions for the existence of skew polynomial rings of the form $R[x_1, d_1] \dots [x_n, d_n]$ over $R$ and we have treated the problem of the simplicity of such rings.

**1. Preliminaries.** All the rings considered in this paper are with identities. Let $R$ be a ring and let $f$ be an endomorphism of $R$. Then a map $d : R \to R$, such that $d(a + b) = d(a) + d(b)$ and $d(ab) = ad(b) + d(a)f(b)$ for all $a, b$ in $R$, is called a $f$-derivation of $R$; when $f$ is the identity map of $R$, then $d$ is a derivation of $R$ (inner or outer, cf. [**3**, Section 1]).

Given an automorphism $f$ of $R$, an ideal $I$ is said to be a $f$-ideal of $R$ if $f(I) = I$ and $R$ is said to be a $f$-simple ring if it has no non-zero proper $f$-ideals. The notion of a $d$-simple ring $R$ (where $d$ is a derivation of $R$) is defined in a similar way (cf. [**3**, Section 1]).

Assume next that $f$ is an endomorphism and that $d$ is a $f$-derivation of $R$. Consider the set $S$ of all polynomials in one variable, say $x$, over $R$ and define in $S$ addition in the usual way and multiplication by the rule $xr = f(r)x + d(r)$, for all $r$ in $R$. Then it is well known that $S$ becomes a ring denoted by $R[x, f, d]$ and called a skew polynomial ring or an Ore extension over $R$ (e.g. [**1**, p. 35]).

Applying induction on $n$ one finds that $x_n r = \sum_{i=0}^{n} \binom{n}{i} f^{n-i}(d^i(r))x^{n-i}$, for all $r$ in $R$. When $f$ is the identity map of $R$ we write $S = R[x, d]$ and when $d$ is the zero derivation of $R$ we write $S = R[x, f]$.

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*AMS Subject Classification* (1980): Primary 16A05

**2. Main results.** First we need the following lemma:

LEMMA 2.1. *Let $R$ be a ring and let $S = R[x, f, d]$ be a skew polynomial ring over $R$. Assume that $f$ is a monomorphism of $R$. Then $x$ is a regular element of $S$.*

*Proof.* It suffices to show that if $r$ is in $R$ and $xr = 0$, then $r = 0$. Assume the contrary, then $f(r)x = -d(r)$ is in $R$, therefore $f(r) = 0$, which is absurd.

We are now ready to prove

THEOREM 2.2. *Let $R$ and $S$ be as in the previous Lemma. Then an endomorphism $g$ of $R$ extends to an endomorphism of $S$ by $g(x) = x$ if, and only if, $g$ commutes with $f$ and $d$.*

*Proof.* Assume that $g$ extends as above. Then for all $r$ in $R$ $g(xr) = g(f(r)x)+ g(d(r))$, or $xg(r) = g(f(r))x + g(d(r))$. But $xg(r) = f(g(r))x + d(g(r))$, therefore $[g(f(r)) - f(g(r))]x = d(g(r)) - g(d(r))$ is in $R$. Hence, by Lemma 2.1, $f \circ g = g \circ f$ and $d \circ g = g \circ d$. Conversely assume that $g$ commutes with $f$ and $d$. Then $g$ extends to an endomorphism of $S$ if $g(x)$ can be defined in a way compatible with the multiplication in $S$. But the previous relations give that $g(xr) - xg(r) = g(f(r)x) - f(g(r))x$, therefore we can put $g(x) = x$.

THEOREM 2.3. *Let $R$, $S$ and $g$ be as in the previous statement and let $d'$ be a $g$-derivation of $R$. Then $d'$ extends to a $g$-derivation of $S$ by $d'(x) = 0$ if and only if, $d'$ commutes with $f$ and $d$.*

*Proof.* By the previous theorem g extends to an endomorphism of S by $g(x) = x$. Assume first that $d'$ extends as above. Then for all $r$ in $R$ $d'(xr) = d'(f(r)x) + d'(d(r))$ (1), or $xd'(r) = d'(f(r)x + d'(d(r))$. But $xd'(r) = f(d'(r))x + d(d'(r))$ (2), therefore $[f(d'(r)) - d'(f(r))]x = d'(d(r)) - d(d'(r))$ is in $R$. Thus $f \circ d' = d' \circ f$ and $d \circ d' = d' \circ d$. Conversely assume that $d'$ commutes with $d$ and $f$, then $d'$ extends to a $g$-derivation of S if $d'(x)$ can be defined so that it satisfies relation (1). Namely, if $d'(x) = h$, we should be able to write $xd'(r) + hg(r) - f(r)h + d'(f(r))x + d'(d(r))$. Then relation (2) gives that $hg(r) = f(r)h$, hence we can put $d'(x) = 0$.

Our next result generalizes Theorem 2.2 of [**3**].

THEOREM. 2.4. *Let $R$ be a ring, let $f_1, \ldots, f_n$ be monomorphisms of $R$ and let $d_1$ be a $f_i$-derivation of $R$, for each $i = 1, \ldots, n$. Consider the set $S_n$ of all polynomials in $n$ variables, say $x_1, \ldots, x_n$, over $R$. Define in $S_n$ addition in the usual way and define multiplication by the relations $x_i r = f_i(r)x_i + d_i(r)$ and $x_i x_j = x_j x_i$ for all $r$ in $R$ and all $i, j = 1, \ldots, n$. Then $S$, is an Ore extension over $S_{i-1}$ (where $S_0 = R$) if, and only if, $d_i \circ d_j = d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $f_i \circ d_j = d_j \circ f_i$, for all $i, j = 1, \ldots, n$.*

*Proof.* Use Theorems 2.2 and 2.3 and apply induction on $n$.

We call the ring constructed above a skew polynomial ring in finitely many variables over $R$ and we denote it by $S_n = R[x_1, f_1, d_1] \ldots [x_n, f_n, d_n]$.

Our last Theorem gives a method of constructing $f$ simple skew polynomial rings over $R$.

THEOREM 2.5. *Let $R$ be an integral domain of characteristic zero, let $f$ be an automorphism of $R$ and let $d$ be a $f$ derivation of $R$ commuting with $f$. Assume that there exists a central element $r_0$ of $R$ such that $f(r_0) = r_0$ and $d(r_0) \neq 0$. Then, if $R$ is either a $d$-simple or a $f$-simple ring, $S = R[x, f, d]$ is a $f$-simple ring.*

*Proof.* By Theorem 2.2 $f$ extends to an automorphism of $S$ by $f(x) = x$. Assume first that $R$ is a $d$-simple ring and let $I$ be a non zero $f$-ideal of $S$. Then, for all $r$ in $I \cap R$, $d(r) = xr - f(r)x$ is in $I \cap R$, therefore $I \cap R$ is a $d$-ideal of $R$.

Let $g = \sum_{i=0}^{n} a_i x^i$ be element of $I$, then $r_0 g - g f^{-n}(r_0) = r_0 a_{n-1} x^{n-1} - [n a_n f^{-1}(d(r_0)) + a_{n-1} f^{-1}(r_0)] x^{n-1} +$ terms of lower degree $= -n a_n d(r_0) x^{n-1} +$ terms of lower degree, and therefore $I$ contains an element of degree $n-1$. Repeating the same argument we finally get that $I \cap R \neq \{0\}$, therefore $I \cap R = R$ and $I = S$. Assume next that $R$ is a $f$-simple ring, then obviously $I \cap R$ is a $f$-ideal of $R$. Show as before that $I \cap R \neq \{0\}$ to get the required result.

**3. Remarks and Examples.** 1) Let $R = T[y]$ be a polynomial ring over a given ring $T$. Put $d/dy = d$ and consider the first Weyl Algebra $A_1(T) = R[x, d]$ over $T$. Since $y$ is a central element of $R$ and $dy = 1$, $d$ is an outer derivation of $R$. but its extension to $A_1(T)$ by $d(x) = 0$ is an inner derivation induced by $x$.

2) In general, if $d$ is a derivation of a given ring $R$ and $d'$ is another derivation of $R$ extending to an inner derivation of $S = R[x, d]$ induced by an element $h = \sum_{i=0}^{n} a_i x^i$ of $S$, then $a_n$ is a central element of $R$ and $r a_k - a_k r = \sum_{i=k+1}^{n} a_i \binom{i}{i-k} d^{i-k}(r)$, for all $r$ in $R$ and each $k = 0, 1, \ldots, n-1$. This is a straightforward consequence of the fact that $d'(r) = hr - rh$ is in $R$, for all $r$ in $R$.

3) Let $S = R[x, f]$ be a skew polynomial ring defined with respect to an endomorphism $f$ of a given ring $R$.

Assume that there exists an outer derivation $d$ of $R$ extending to an inner derivation of $S$ induced by $h = \sum_{i=0}^{n} a_i x^i$. Then, for all $r$ in $R$, $d(r) = \sum_{i=1}^{n} [a_i f^i(r) - r a_i] x^i + (a_0 r - r a_0)$ is in $R$, therefore $d(r) = a_0 r - r a_0$, which contradicts our hypothesis.

4) Let $g$ be an endomorphism and let $d$ be a $g$-derivation of a given ring $T$. Then $g$ extends to an endomorphism of $A_1(T)$ by $g(x) = x$ and $d$ extends to a $g$-derivation of $A_1(T)$ by $d(x) = 0$. For this, it is clear that $g$ extends to an endomrphism of $T[y]$ by $g(y) = y$ and that $d$ extends to a $g$-derivation of $T[y]$ by $d(y) = 0$. It is easy also to check that $d/dy$ commutes with $g$ and $d$, therefore the result follows by Theorems 2.2 and 2.3.

5) Let $R = T[y_1, \ldots, y_n]$ be a polynomial ring over 2 given ring $T$. Define the $T$-automorphism $f_1$ of $R$ by the relations $f_i(y_j) = y_j + 1$ if $i \neq j$ and $f_i(y_i) = y_i$

for each $i = 1, \dots, n$, and let $d_i$ be the $f$-derivation of $R$ defined by the relations $d_i(t) = 0$ for all $t$ in $T$, $d_i(y_j) = 0$ if $i \neq j$ and $d_i(y_i) = 1$. It is easy to check that $d_i \circ d_j = d_j \circ d_i$, $f_i \circ f_j = f_j \circ f_i$ and $f_i \circ d_j = d_j \circ f_i$ for all $i, j = 1, \dots, n$, therefore we can construct the skew polynomial ring $S_n = R[x_1, f_1, d_1] \dots [x_n, f_n, d_n]$ (cf. Theorem 2.4). If we replace $f_i$ with the identity map of $R$ for each $i$, then $S_n$ is the $n$-th Weyl Algebra $A_n(T)$ over $T$.

6) Notice that if $f$ is the identity map of $R$, Theorem 2.5 gives that if $R$ is $d$-simple then $S$ is simple, but this is a straightforward consequence of Theorem 3.4 of [**3**].

7) Let $k(y)$ be the field of rational functions over a field $k$ of characteristic zero and let $R = k(y)[t]$ be a polynomial ring over $k(y)$. Define a $k(y)$-automorphism of $R$ by $f(t) = t + 1$. Then $R$ is an $f$-simple ring (cf. [**2**, Theorem 2.1.1]).

Extend the derivation $d = d/dy$ of $k(y)$ to a $f$-derivation of $R$ by $d(t) = 0$. Each element of $R$ can be written in the form $g/b$ with $g$ in $k[y][t]$ and $b$ in $k[y]$. Then it is easy to check that $f(d(g/b)) = g^*(t+1)b - g(t+1)b'/b^2$, where $g^*$ denotes the polynomial obtained from $g$ by replacing its coefficients (which are polynomials in $y$) with their usual derivatives and $b'$ denotes the derivative of $b$. Hence $d(f(g/b)) = d(g(t+1)/b) = f(d(g/b))$, therefore $f$ commutes with $d$. Furthermore $f(y) = y$, $d(y) = 1$ and $R$ is a commutative integral domain, therefore, by Theorem 2.5, $R[x, f, d]$ is a $f$-simple ring.

8) Let $k(y)$ and $R$ be as in the previous example. Define a $k$-automorphism of $k(y)$ by $f(y) = y + 1$ and extend $f$ to an automorphism of $R$ by $f(t) = t$. Let $d$ be the $f$ derivation of $R$ defined by the relations $d(c) = 0$ for all $c$ in $k$, $d(y) = 0$ and $d(t) = 1$. Then it is easy to check that $R$ is a $d$-simple ring and that $f$ commutes with $d$. Therefore, since $f(t) = t$ and $d(t) \neq 0$, Theorem 2.5 gives that $R[x, f, d]$ is $f$-simple ring.

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(Received 10 06 1985)