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CONDITIONAL PROBABILITY IN NONSTANDARD ANALYSIS

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Abstract. In this paper we apply the theory of Loeb measure to conditional probability for hyperfinite Loeb spaces. We show that conditional probability $^{\sim}P(\cdot/A)$ on a Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$ for $A \in^* \mathfrak{P}(V)$ $(P(A) > 0$ and $P(A) \approx' 0$[1] is a Loeb measure and for $A \in \mathfrak{M}(^{\sim}P)$ $(^{\sim}P(A) > 0)$ can be represented by a Loeb measure. For the case $A \in \mathfrak{M}(^{\sim}P)$ we prove that there exists a set $C \in^* \mathfrak{P}(V)$ such that $^{\sim}P(\cdot/A)$ is equal to the Loeb conditional probability $L(P(\cdot/C))$. We introduce internal conditional probability relative to an internal subalgebra $\mathfrak{A}$ of $^*\mathfrak{P}(V)$ as in case of finite standard probability spaces. We show, analogously to a well-known probability result, that internal conditional probability $P(A/\mathfrak{A})$, $A \in^* \mathfrak{P}(V)$, and internal conditional expectation $E(X/\mathfrak{A})$, X is S-integrable, are P-a. s. unique, in nonstandard sense, random variables on $(V, \mathfrak{A}, P)$. Finally, we give a nonstandard characterization of conditional probability $^{\sim}P(A/\mathfrak{M}(\mathfrak{A}))$, $A \in \mathfrak{M}(^{\sim}P)$ on a Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$. We prove that there exists a set $C \in^* \mathfrak{P}(V)$ such that $P(C/\mathfrak{A})$ is the lifting of $^{\sim}P(A/\mathfrak{M}(\mathfrak{A}))$.

Introduction. In this paper we concern ourselves with conditional probability for hyperfinite Loeb spaces. We use the well-known results from the theory of Loeb measure [**8**] and nonstandard probability [**2**], [**10**] and the methodology developed by P. Loeb, J. Keisler, R. Anderson and others.

In the first part we define internal conditional probability $P(\cdot/A)$, $A \in {}^*\mathfrak{P}(V)$ for a hyperfinite probability space $(V, {}^*\mathfrak{P}(V), P)$ and give the nonstandard representation of conditional probability $^{\sim}P(\cdot/A)$, $A \in \mathfrak{M}(^{\sim}P)$ on the Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$. We show that for $A \in^* \mathfrak{P}(V)$ with $P(A) > 0$ and $P(A) \approx' 0$ $^{\sim}P(\cdot/A)$ is a Loeb measure on $(V, \mathfrak{M}(^{\sim}P))$ and for $A \in \mathfrak{M}(^{\sim}P)$ with $^{\sim}P(A) > 0$ there exists a set $C \in^* \mathfrak{P}(V)$ such that $^{\sim}P(\cdot/A)$ can be represented by the Loeb conditional probability $L(P(\cdot/C))$.

In the second part we define, analogously to the definition of internal conditional expectation $E(X/\mathfrak{A})$, [**10**], internal conditional probability $P(A/\mathfrak{A})$ $A \in {}^*\mathfrak{P}(V)$, is an internal subalgebra of ${}^*\mathfrak{P}(V))$ for a hyperfinite probability space $(V, {}^*\mathfrak{P}(V), P)$. We show that so-introduced $P(A/\mathfrak{A})$ $(E(X/\mathfrak{A})$ as well) is

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[1] $\approx'$ is the negation of $\approx$

P-a. s. unique random variable on $(V, \mathfrak{A}, P)$. The P-a. s. uniqness of the $P(A/\mathfrak{A})$ $(E(X/\mathfrak{A}))$ is introduced in theorem 4, corresponding to the same concept from standard probability. Finally, we give a nonstandard characterization of conditional probability $P(A/\mathfrak{M}(\mathfrak{A}))$ $(A \in \mathfrak{M}(^{\sim}P)$, $(A \in \mathfrak{M}(^{\sim}P)$ $\mathfrak{M}(\mathfrak{A}$ is a sub-σ-algebra of $\mathfrak{M}(^{\sim}P))$ on a Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$. We show that for $A \in \mathfrak{M}(^{\sim}P)$ there exists a set $B \in^* \mathfrak{P}(V)$ such that $P(B/\mathfrak{A})$ is lifting of $^{\sim}P(A/\mathfrak{M}(\mathfrak{A}))$.

We assume $(V,^* \mathfrak{P}(V), P)$ to be a hyperfinite probability space and $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$ a Loeb space constructed from it, i.e. Loeb measure $^{\sim}P$ is defined by

$$\begin{aligned} ^{\sim}P(F) &= \inf\{\mathrm{st}(P(A)) \mid F \subseteq A \text{ and } A \in^* \mathfrak{P}(V)\} \\ &= \sup\{\mathrm{st}(P(A)) \mid A \subseteq F \text{ and } A \in^* \mathfrak{P}(V)\} \end{aligned} \qquad \text{for } F \subseteq V$$

and $\mathfrak{M}(^{\sim}P)$ is a σ-algebra of all $^{\sim}P$-measurable sets $F \subseteq V$.

According to standard probability, for $A \in^* \mathfrak{P}(V)$ with $P(A) > 0$ we define internal conditional probability $P(\cdot/A)$ of an internal event B relative to A by

$$P(B/A) = P(A \cap B)/P(A)$$

It is easy to show that $(V,^* B(V), P(\cdot/A))$ is a hyperfinite probability space, so it gives rise to a Loeb space denoted by $(V, \mathfrak{M}(L(P(\cdot/A))), L(P(\cdot/A)))$.

On the other hand, for a Loeb space $(V, \mathfrak{M}(^{\sim}P), ^{\sim}P)$, conditional probability $^{\sim}P(\cdot/A)$ of an event $B \in \mathfrak{M}(^{\sim}P)$ relative to $A \in \mathfrak{M}(^{\sim}P)$, $^{\sim}P(A) > 0$, is standardly defined by

$$^{\sim}P(B/A) = ^{\sim}P(A \cap B)/^{\sim}P(A)$$

It is well known that $^{\sim}P(\cdot/A)$ is a probability measure on $(V, \mathfrak{M}(^{\sim}P))$ but not necessary a Loeb measure. However, for $A \in^* \mathfrak{P}(V)$ with $P(A) > 0$ $P(A) \approx' 0$, $^{\sim}P(\cdot/A)$ is a Loeb measure and we shall show it in this paper.

Let $\sigma(^{\sim}P(\cdot/A))$ be σ-algebra of all $^{\sim}P(\cdot/A)$-measurable sets, i.e.

$$\sigma(^{\sim}P(\cdot/A)) = \{F \subseteq V \mid F \cap A \in \mathfrak{M}(^{\sim}P)\}$$

It is obvious that $\mathfrak{M}(^{\sim}P) \subset \sigma(^{\sim}P(\cdot/A))$. From the theory of Loeb measure we know that $^{\sim}P$ is a complete measure on $(V, \mathfrak{M}(^{\sim}P))$ and that $\mathfrak{M}(^{\sim}P)$ is a completion of $\c{L}(V)$ relative to $^{\sim}P$. We have the same for $^{\sim}P(\cdot/A)$:

LEMMA 1. *Probability measure* $^{\sim}P(\cdot/A)$ *is a complete measure on* $(V, \mathfrak{M}(^{\sim}P))$ *and* $\mathfrak{M}(^{\sim}P)$ *is a completion of* $\c{L}(V)$ *relative to* $^{\sim}P(\cdot/A)$ *in the sense that for* $F \in \mathfrak{M}(^{\sim}P)$ *there exist sets* $Z \in \c{L}(V)$ *and* $N \subseteq V$ *such that*

$$F = Z \cup N, \qquad N \subseteq U, \qquad \textit{and} \qquad ^{\sim}P(U/A) = 0.$$

Proof Let $F \in \mathfrak{M}(^{\sim}P)$, $^{\sim}P(F/A) = 0$ and $M \subseteq F$. Than is $F \cap A \in \mathfrak{M}(^{\sim}P)$, $M \cap A \subseteq F \cap A$ and $^{\sim}P(F \cap A) = 0$. Since $^{\sim}P$ is a complete measure, $M \cap A \in \mathfrak{M}(^{\sim}P)$ and $^{\sim}P(M \cap A) =)$. This implies that M is $^{\sim}P(\cdot/A)$ – measurable and $^{\sim}P(M/A) = 0$.

Let $F \in \mathfrak{M}(^{\sim}P)$. Then there exist sets $Z \in \mathfrak{L}(V)$ and $N \subseteq V$ such that $F = Z \cup N$, $N \subseteq U$ and $^{\sim}P(U) = 0$. Since $U \cap A \subseteq U$ and $^{\sim}P$ is a complete measure, it follows that $^{\sim}P(U \cap A) = 0$, i.e. $^{\sim}P(U/A) = 0$. Hence $\mathfrak{M}(^{\sim}P)$ is a completion of $\mathfrak{L}(V)$ relative to $^{\sim}P(\cdot/A)$.

The next theorem shows that conditional probability $^{\sim}P(\cdot/A)$ for $A \in^{*} \mathfrak{P}(V)$, $P(A) > 0$, and $P(A) \approx' 0$ is a Loeb measure on $(V, \mathfrak{M}(^{\sim}P))$.

THEOREM 1. *Let* $A \in^{*} \mathfrak{P}(V)$, $P(A) >)$ *and* $P(A) \approx' 0$. *Then* $(V, \mathfrak{M}(^{\sim}P), {}^{\sim}P(\cdot/A))$ *is a Loeb probability space.*

Proof. We show that $^{\sim}P(\cdot/A)$ is a Loeb measure obtained from internal conditional probability $P(\cdot/A)$. Using notations already defined we prove that

$$L(P(B/A)) = {}^{\sim}P(B/A) \qquad \text{for } B \in \mathfrak{M}(^{\sim}P) \tag{1}$$

Let $F \in^{*} \mathfrak{P}(V)$. Since $P(A) > 0$ and $P(A) \approx' 0$

$$\begin{aligned} {}^{\sim}P(F/A) &= {}^{\sim}P(F \cap A)/{}^{\sim}P(A) = \operatorname{st}(P(F \cap A))/\operatorname{st}(P(A)) = \\ &= \operatorname{st}(P(F \cap A)/P(A)) = L(P(F/A)). \end{aligned}$$

Let $F \in \mathfrak{L}(V)$. The definition of Loeb measure

$$\begin{aligned} &L(P(F/A)) = \\ &= \sup\{\operatorname{st}(P(C/A)) \mid C \in^{*} \mathfrak{P}(V), C \subseteq F\} = \inf\{\operatorname{st}(P(D/A)) \mid (D \in^{*} \mathfrak{P}(V), D \supseteq F\} \end{aligned}$$

implies that for $\varepsilon \in^{\sigma} R^{+}$ there exist sets $C, D \in^{*} \mathfrak{P}(V)$ such that

$$^{\sim}P(C/A) \leq {}^{\sim}P(F/A) {}^{\sim}P(D/A) \qquad \text{and} \tag{2}$$

$$^{\sim}P(D/A) - \varepsilon < L(P(F/A)) < {}^{\sim}P(C/A) + \varepsilon \tag{3}$$

Relations (2) and (3) imply

$$^{\sim}P(F/A) - \varepsilon < L(P(F/A)) < {}^{\sim}P(F/A) + \varepsilon \qquad \text{i.e.} \qquad L(P(F/A)) = {}^{\sim}P(F/A)$$

Let $F \in \mathfrak{M}(^{\sim}P)$. Then, according to [**8**], there exist sets $C, D \in \mathfrak{L}(V)$ such that $C \subseteq F \subseteq D$ and $^{\sim}P(C) = {}^{\sim}P(F) =^{\sim} P(D)$. We show that $F \in \mathfrak{M}(L(P(\cdot/A)))$ and that $L(P(F/A)) = {}^{\sim}P(F/A)$. Since

$$\begin{aligned} {}^{\sim}P(DUA) &= {}^{\sim}P(D) + {}^{\sim}P(A) - {}^{\sim}P(D \cap A) \leq {}^{\sim}P(D) + {}^{\sim}P(A) - {}^{\sim}P(C \cap A) = \\ &= {}^{\sim}P(C) + {}^{\sim}P(A) - {}^{\sim}P(C \cap A) = {}^{\sim}P(C \cup A) \leq {}^{\sim}P(D \cup A) \end{aligned}$$

it follows that $^{\sim}P(D \cap A) = {}^{\sim}P(C \cap A)$, whence, and from $^{\sim}P(C \cap A) \leq {}^{\sim}P(F \cap A) \leq {}^{\sim}P(D \cap A)$ we get

$$^{\sim}P(C/A) = {}^{\sim}P(F/A) = {}^{\sim}P(D/A) \tag{4}$$

From (1) and (4) it follows that

$$L(P(C/A)) = L(P(D/A)) = {}^{\sim}P(F/A) \tag{5}$$

and $L(P(D\mathbf{C}/A)) = 0$. For the set $F \in \mathfrak{M}(^{\sim}P)$ we have the following representation:

$$F = C \cup (F\mathbf{C}) \text{ where } C \in Ļ(V), F\mathbf{C} \subseteq D\mathbf{C} \text{ and } L(P(D\mathbf{C}/A)) = 0.$$

So $F \in \mathfrak{M}(L(P(\cdot/A)))$. Since $C \subseteq F \subseteq D$ and $C, D, F \in \mathfrak{M}(L(P(\cdot/A)))$ we have that $L(P(C/A)) \le L(P(F/A)) \le L(P(D/A))$ which, in view of (5), implies

$$^{\sim}P(F/A) \le L(P(F/A)) \le {}^{\sim}P(F/A) \qquad \text{i.e.} \qquad L(P(F/A)) = {}^{\sim}P(F/A)$$

Later on, whenever $A \in^* \mathfrak{P}(V)$, $P(A) > 0$ and $P(A) \approx' 0$, the conditional probability $^{\sim}P(\cdot/A)$ on $(V, \mathfrak{M}(^{\sim}P))$ will be denoted by $L(P(\cdot/A))$; assuming that it is a Loeb measure.

We now prove a representation theorem for conditional probability $^{\sim}P(\cdot/A)$ ($A \in \mathfrak{M}(^{\sim}P)$ and $^{\sim}P(A) > 0$) on $(V, \mathfrak{M}(^{\sim}P))$. We shall show that there exists a set $C \in^* \mathfrak{P}(V)$ with $P(C) > 0$ and $P(C) \approx' 0$ such that the conditional probability $^{\sim}P(\cdot/A)$ is equal to the Loeb conditional probability $L(P(\cdot/C))$.

THEOREM 2. *Let $A \in \mathfrak{M}(^{\sim}P)$ with $^{\sim}P(A) > 0$. Then, there exists a set $C \in^* \mathfrak{P}(V)$ with $P(C) > 0$ and $P(C) \approx' 0$ such that*

$$L(P(F/C)) = {}^{\sim}P(F/A), \qquad \textit{for any } F \in \mathfrak{M}(^{\sim}P)$$

Proof. According to [**8**] exists a set $C \in^* \mathfrak{P}(V)$ such that $^{\sim}P(C \triangle A) = 0$. We show that $P(C) > 0$ and $P(C) \approx' 0$: For sets $A, C \subseteq V$ we have that $C \setminus A \subseteq C \triangle A$, and $A \setminus C \subseteq C \triangle A$, so, by completeness of measure $^{\sim}P$

$$^{\sim}P(C \setminus A) = {}^{\sim}P(A \setminus C) = 0$$

Since $C = (C \setminus A) \cup (C \cap A)$ and $A = (A \setminus C) \cup (A \cap C)$ and sets A, C satisfy (1)

$$^{\sim}P(C) = {}^{\sim}P(C \setminus A) + {}^{\sim}P(C \cap A) = {}^{\sim}P(C \cap A) = {}^{\sim}P(A \setminus C) + (C \cap A) = {}^{\sim}P(A) \quad (2)$$

Hence $P(C) > 0$ and $P(C) \approx' 0$,

Let $F \in \mathfrak{M}(^{\sim}P)$. Then

$$F \cap C = (F \cap A \cap C) \cup ((C \setminus A) \cap F) \text{ and } F \cap A = (F \cap A \cap C) \cup ((A \setminus C) \cap F) \quad (3)$$

From (3), $(C \setminus A) \cap F \subseteq A \triangle C$, $(A \setminus C) \cap F \subseteq A \triangle C$, $^{\sim}P((C \setminus A) \cap F) = 0$ and $^{\sim}P((A \setminus C) \cap F) = 0$ it follows that

$$\begin{aligned} ^{\sim}P((A \setminus C) &= {}^{\sim}P(F \cap A \cap C) + {}^{\sim}P((C \setminus A) \cap F) = {}^{\sim}P(F \cap A \cap C) = \\ &= {}^{\sim}P(F \cap A \cap C) + {}^{\sim}P((A \setminus C) \cap F) = {}^{\sim}P(F \cap A) \end{aligned} \quad (4)$$

Finally, according to theorem 1 (2) and (4) imply

$$L(P(F/C)) = {}^{\sim}P(F/C) = {}^{\sim}P(F \cap C)/{}^{\sim}P(C) = {}^{\sim}P(F \cap A)/{}^{\sim}P(A) = {}^{\sim}P(F/A)$$

The following theorem is a simple consequence of the Loeb theorem [**8**], but can be quite useful when working in nonstandard probability.

THEOREM 3. *Let* $A \in \mathfrak{M}(^{\sim}P)$ *with* $^{\sim}P(A) > 0$. *Than, for any set* $F \in \mathfrak{M}(^{\sim}P)$ *there exists a set* $C \in^{*} \mathfrak{P}(V)$ *such that* $^{\sim}P(F/A) = {}^{\sim}P(C/A)$.

Proof. For $F \in \mathfrak{M}(^{\sim}P)$, by the Loeb theorem [**8**], there exists a set $C \in^{*} \mathfrak{P}(V)$ such that $^{\sim}P(F \triangle C) = 0$. Since $F \cap A = (F \cap A \cap C) \cup ((F \setminus C) \cap A)$ $C \cap A = (F \cap A \cap C) \cup ((C \setminus F) \cap A)$ $(F \setminus X) \cap A \subseteq F \triangle C$ and $(C \setminus F) \cap A \subseteq FtriangleC$, by the same arguments as in theorem 2, we get that $^{\sim}P(F \cap A) = {}^{\sim}P(C \cap A)$, i.e. $^{\sim}P(F/A) = {}^{\sim}P(C/A)$.

In the second part of this paper we are dealing with internal conditional expectation $E(X/\mathfrak{A})$ of an internal random variable $X : V \to^{*} R$ relative to $\mathfrak{A}$, where $\mathfrak{A}$ is an internal subalgebra of $^{*}\mathfrak{P}(V)$ and $P(A/\mathfrak{A})$ denotes the internal conditional probability of an event $A \in^{*} \mathfrak{P}(V)$ elative to $\mathfrak{A}$.

We consider a hiperfinite probability space $V,^{*} \mathfrak{P}(V), P)$, $A \in^{*} \mathfrak{P}(V)$ and internal subalgebra of $^{*}\mathfrak{P}(V)$. The hyperfinitness of $\mathfrak{A}$ implies, by transfer principle, that $\mathfrak{A}$ is generated by a hyperfinite partition $\{V_1, V_2, \dots V_H\}$ $(H \in^{*} N \setminus N)$ of the set V. It permits us to definer $P(A/\mathfrak{A})$ in the same way as m the case of finite standard probability spaces:

$$P(A/\mathfrak{A})(v) = \sum_{i=1}^{H} P(A/V_i) I_{V_i}(v) \qquad \text{for } v \in V \tag{1}$$

where $P(A/V_i) = P(A \cap V_i)/P(V_i) \qquad i = 1, 2, \dots, H$. Since

$$\begin{aligned} P(A/\mathfrak{A})(v) &= \sum_{i=1}^{H} P(A/V_i) I_{V_i}(v) \\ &= \sum_{i=1}^{H} (P(A \cap V_i)/P(V_i)) I_{V_i}(v) \\ &= \sum_{i=1}^{H} ((P(V_i)^{-1} \sum (P(u) I_A(u), u \in V_i)) I_{V_i}(v) \\ &= \sum_{i=1}^{H} E(I_A/V_i) I_{V_i}(v) \\ &= E(I_A/\mathfrak{A})(v) \qquad \text{i.e.} \end{aligned}$$

$$P(A/\mathfrak{A}) = E(I_A/\mathfrak{A}). \tag{2}$$

in the further work we shall use both (1) and (2) as definitions of internal conditional probability.

For internal random variable $X : V \to^{*} R$ on $(V,^{*} \mathfrak{P}(V), P)$ internal conditional expectation $E(X/\mathfrak{A})$ has already been defined [**6**]. In [**10**] it is proved that $E(X/\mathfrak{A})$ is an S-$\mathfrak{A}$-integrable random variable on $(V,\mathfrak{A},P)$ provided X is S-integrable. This result applied to $P(A/\mathfrak{A})$ implies that $P(A/\mathfrak{A})$ is an S-$\mathfrak{A}$-integrable random variable on $(V,\mathfrak{A},P)$ since I_A is S-integrable, [**9**]. In [**10**] it is proved that

$$E(E(X/\mathfrak{A}))=E(X). \tag{i}$$

Taking (2) as definition of $P(A/\mathfrak{A})$, from (i) it follows, [**9**], that

$$E(P(A/\mathfrak{A}))=P(A) \tag{ii}$$

Results (i) and (ii) make the Theorem of probability completeness for $E(X/\mathfrak{A})$ and $P(A/\mathfrak{A})$ hold for hyperfinite probability spaces.

We now prove a nonstandard version of the well known probability theorem, namely, that conditional probability and expectation relative to σ-subalgebra Ḅ are μ-a. s. unique random variables on $(V,Ḅ,\mu)$, [**11**].

THEOREM 4. *Let* $(V,^*\mathfrak{P}(V),P)$ *be a hyperfinite probability space,* $\mathfrak{A}\subseteq^*\mathfrak{P}(V)$ *an internal subalgebra generated by a hyperfinite partition* $\{V_1,V_2,\dots,V_H\}$ $(H\in^* N\mathbf{N})$ *of* V, $X:VV\to^* R$ *an S-integrable random variable on* $(V,^*\mathfrak{P}(V),P)$ *and* $A\in^*\mathfrak{P}(V)$. *Then*

(i) $\sum(X(v)P(v),v\in U)=\sum(E(X/\mathfrak{A})(v)P(v),v\in U)$ *for* $U\in\mathfrak{A}$

(ii) $E(X/\mathfrak{A})$ *is the P-a. s. unique internal random variable on* $(V,\mathfrak{A},P)$ *which satisfies* (i), *i.e. for any other S-integrable* $Y:V\to^* R$ *on* $(V,\mathfrak{A},P)$ *satisfying* (i)

$$Y(v)\approx E(X/\mathfrak{A})(v)\qquad P\text{-n. s.}$$

and for any S-integrable $H:V\to^* R$ *on* $V,\mathfrak{A},P)$ *with*

$$\sum(|(H(v)-E(X/\mathfrak{A})(v)|P(v),v\in V)\approx 0\qquad \textit{one has}$$
$$\sum(H(v)P(v),v\in U)\approx\sum(E(X/\mathfrak{A})(v)P(v),v\in U)\qquad \textit{for } U\in\mathfrak{A}.$$

(iii) *For any set* $B\in\mathfrak{A}$

$$P(A\cap B)=\sum(P(A/\mathfrak{A})(v)P(v),v\in B).$$

(iv) $P(A/\mathfrak{A})$ *is the P-a. s. unique internal random variable on* $(V,\mathfrak{A},P)$ *in the sense given in* (ii).

Proof (i) Since $U=\bigcup_{i=1}^{H}(U\cap V_i)$, for $v\in U\cap V_i E(X/\mathfrak{A})(v)=E(X/U\cap V_i)$

one has

$$\begin{aligned}\sum(E(X/\mathfrak{A})(v)P(v), v \in U) &= \sum_{i=1}^{H}(E(X/\mathfrak{A})(v)P(v), v \in \bigcup_{i=1}^{H}(U \cap V_i))\\ &= \sum_{i=1}^{H}(E(X/\mathfrak{A})(v)P(v), v \in U \cap V_i))\\ &= \sum_{i=1}^{H}(E(X/U \cap V_i)P(U \cap V_i)\\ &= \sum_{i=1}^{H}((P(U \cap V_i)^{-1}\sum(X(u)P(u), u \in U \cap V_i))P(U \cap V_i)\\ &= \sum(X(u)P(u), u \in \bigcup_{i=1}^{H}(U \cap V_i))\\ &= \sum(X(u)P(u), u \in U)\end{aligned}$$

(ii) Let $F(v) = E(X/\mathfrak{A})(v)$. Then, in view of the Projection Theorem for Integrabllity, [**1**], S-$\mathfrak{A}$-integrability of $F : V \to^* R$ implies that $^\sim F : V \to R$ is a $^\sim P$-integrable random variable on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$. If $Y : V \to {}^*R$ is any S-$\mathfrak{A}$-integrable random variable on $(V, \mathfrak{A}, P)$ which satisfies (i) then $^\sim Y : V \to R$ is a $^\sim P$-integrable random variable on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$ as well. Therefore, for $U \in \mathfrak{A}$

$$\int_U {}^\sim Y d^\sim P = \mathrm{st}(\sum(Y(u)P(u), u \in U)) = \mathrm{st}(E(F(u)P(u), u \in U)) = \int_U {}^\sim F d^\sim P$$

Let $M \in \mathfrak{M}(\mathfrak{A})$. Then, by [**8**], there exists a set $U \in \mathfrak{A}$ such that $^\sim P(U \triangle M) = 0$. Since U satisfies (1), we have

$$\int_M {}^\sim Y d^\sim P = \int_U {}^\sim Y d^\sim P = \int_U {}^\sim F d^\sim P = \int_M {}^\sim F d^\sim P$$

and hence $^\sim Y(v) = {}^\sim F(v)$ P-a. s.. This implies that

$$P\{v \in V| \, |Y(v) - F(v)| > n^{-1}\} \approx 0 \quad \text{for every } n \in N.$$

According to [**4**, Robinson's lemma about sequences] there exists $h \in^* N\mathbf{N}$ such that for every $k \in^* N$, $k \leq h$

$$P\{v \in V| \, |Y(v) - F(v)| > k^{-1}\} \approx 0$$

Therefore, the set $U = \{v \in V| \, |Y(v) - F(v)| > h^-\} \approx 0$ satisfies: $U \in \mathfrak{A}$, $P(U) \approx 0$, $U \supset \{v \in V| \, Y(v) \neq F(v)\}$ and $Y(v) \approx E(X/\mathfrak{A})(v)$ for $u \notin U$. Hence

$$Y(v) \approx E(X/\mathfrak{A})(v) \qquad P\text{-a. s.}$$

Let $H : V \to^* R$ be an S-$\mathfrak{A}$-integrable random variable on $(V, \mathfrak{A}, P)$ with $\sum(|H(v) - F(v)|P(v), v \in V) \approx 0$. Since ${}^\sim H$, ${}^\sim F$ are ${}^\sim P$-integrable random variables on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$ and

$$\int_V |{}^\sim H - {}^\sim F| d^\sim P = \mathrm{st}(\sum(|H(v) - F(v)|P(v), v \in V)) = 0$$

we have that ${}^\sim H(v) = {}^\sim F(v)$ P-a. s.. Therefore, for $U \in \mathfrak{A}$

$$\sum(H(v)P(v), v \in U) \approx \int_U {}^\sim H d^\sim P = \int_U {}^\sim F d^\sim P \approx \sum(F(v)P(v), v \in U) \quad \text{i.e.}$$

$$\sum(H(v)P(v), v \in U) \approx \sum(E(X/\mathfrak{A})(v)P(v), v \in U)$$

(iii) According to def (2) for $P(A/\mathfrak{A})$ and (i), for $B \in \mathfrak{A}$ we have

$$\begin{aligned}\sum(P(A/\mathfrak{A})(v)P(v), v \in B) &= \sum(E(I_A/\mathfrak{A})(v)P(v), v \in B) \\ &= \sum(I_A(v)P(v), v \in B) = P(A \cap B)\end{aligned}$$

(iv) Let $F(v) = P(A/\mathfrak{A})(v)$. Then, since F is S-$\mathfrak{A}$-integrable, ${}^\sim F : V \to R$ is a ${}^\sim P$-integrable random variable on $(V, \mathfrak{M}(\mathfrak{A}), {}^\sim P)$, so, for an S-$\mathfrak{A}$-integrable random variable $G : V \to {}^*R$ on $(V, \mathfrak{A}, P)$ which satisfies (iii), we have that for $B \in \mathfrak{A}$

$$\int_B {}^\sim G d^\sim P = \mathrm{st}(\sum(G(v)P(v), v \in B)) = \mathrm{st}(\sum(F(v)P(v), v \in B)) = \int_B {}^\sim F d^\sim P$$

Hence, by the same arguments as in proof of (ii)

$$G(v) \approx F(v) = P(A/\mathfrak{A})(v) \qquad P\text{-n. s.}$$

For $H : V \to {}^*R$ which is S-$\mathfrak{A}$-integrable and satisfies $\sum(|H(v) - F(v)|P(v), v \in V) \approx 0$, like in (ii), we have that ${}^\sim H(v) = {}^\sim F(v)$ P-a. s. implies that for any $B \in \mathfrak{A}$

$$\begin{aligned}\mathrm{st}(\sum(H(v)P(v), v \in B)) &= \int_B {}^\sim H d^\sim P = \int_B {}^\sim F d^\sim P \\ &= \mathrm{st}(\sum(F(v)P(v), v \in B)) = \mathrm{st}(P(A \cap B))\end{aligned}$$

i.e.

$$\sum(H(v)P(v), v \in B) \approx P(A \cap B)$$

In **[10]** it is proved that for S-integrable random variable $X : V \to {}^*R$ on $(V, {}^*\mathfrak{P}(V), P)$ the internal conditional expectation $E(X/\mathfrak{A})$ is a lifting of ${}^\sim E({}^\sim X/\mathfrak{M}(\mathfrak{A}))$, ${}^\sim E({}^\sim X/\mathfrak{M}(\mathfrak{A}))$ being the conditional expectation of ${}^\sim X : V \to R$ relative to sub-σ-algebra $\mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{M}({}^\sim P)$. From this result we derive the following nonstandard characterization of the conditional probability ${}^\sim P(A/\mathfrak{M}(\mathfrak{A}))$, $A \in \mathfrak{M}({}^\sim P)$ on a Loeb space.

THEOREM 5. *Let* $(V, {}^*\mathfrak{P}(V), P)$ *be a hyperfinite probability space,* $\mathfrak{A}$ *an internal subalgebra of* ${}^*\mathfrak{P}(V)$ *and* ${}^\sim P(A/\mathfrak{M}(\mathfrak{A}))$ *the conditional probability of* $A \in \mathfrak{M}({}^\sim P)$ *relative to sub-*σ*-algebra* $\mathfrak{M}(\mathfrak{A}) \subseteq \mathfrak{M}({}^\sim P)$. *Then there exists a set* $B \in {}^*\mathfrak{P}(V)$ *such that*

$$\mathrm{st}(P(B/\mathfrak{A})) = {}^\sim P(A/\mathfrak{M}(\mathfrak{A})) \qquad P\text{-a. s.}$$

Proof. For $A \in \mathfrak{M}({}^\sim P)$ there is a set $B \in {}^*\mathfrak{P}(V)$ such that ${}^\sim P(A \triangle B) = 0$. The indicator function I_B

$$I_B(v) = \begin{cases} 1, v \in B \\ 0, v \notin B \end{cases}$$

is an internal, S-integrable random variable on $(V, {}^*\mathfrak{P}(v), P)$. Since

$$P\{v|\, I_A(v) neq I_B(v)\} = P(A \triangle B) = 0$$

I_B is an S-integrable lifting of I_A. In view of [**10**], this implies

$$ {}^\sim E(I_B/\mathfrak{A}) = {}^\sim E(I_A/\mathfrak{M}(\mathfrak{A})) \qquad P\text{-a. s.} \quad \text{and so}$$
$$\mathrm{st}(P(B/\mathfrak{A})) = {}^\sim P(A/\mathfrak{M}(\mathfrak{A})) \qquad P\text{-a. s.}$$

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