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NOTE ON THE CIRCUITS OF A PERFECT MATROID DESIGN

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Abstract. For a perfect matroid design $M(E, r)$ on a finite set E with r as a rank function and $B \subseteq E$ a basis of $M(E, r)$, the number of circuits of cardinality $r(E) + 1$ containing B is given.

Preliminaries. Throughout this paper we use some notions and results according to the standard literature on matroid theory (e.g., see [1, 2]). Let E be a finite set and $M(E, r)$ a *matroid* on E with r as a *rank function* ($r: \mathcal{P}(E) \to N$, where $\mathbf{N}$ is the set of non-negative integers and $\mathcal{P}(E)$ the power set of E). A subset $S \subseteq E$ is called *independent* if $r(S) = |S|$, where $|S|$ denotes the cardinality of S, a *basis* of $M(E, r)$ being a maximal independent subset of E. A subset $S \subseteq E$ is called *dependent* if $r(S) \leq |S|$, a circuit of $M(E, r)$ being a minimal dependent subset of E. The *span* S of a subset $S \subseteq E$ is

$$\bar{S} = \{e \in E: r(S \cup \{e\}) = r(S)\}.$$

For any integer $1 \leq k \leq r(E)$ we consider the set

$$CL[M(E, r), k] = \{S \subseteq E: S = \bar{S},\ \ r(S) = k\},$$

and $M(E, r)$ is called a *perfect matroid design* if every set of $CL[M(E, r), k]$ has a common cardinal $c(k)$, $1 \leq k \leq r(E)$. In the sequel we shall use without proofs (e.g., see [1, 2]) the following well-known results from matroid theory:

(a) $r(S) = r(\bar{S})$ for each $S \subseteq E$,

(b) if C is a circuit of $M(E, r)$, and $e \in C$, then $e \in \overline{C - \{e\}}$,

(c) if B is a basis of $M(E, r)$, then $\bar{B} = E$,

(d) if C is a circuit of $M(E, r)$, then $r(C) = |C| - 1$,

(e) if B is a basis of $M(E, r)$ and $e \in E - B$, then there exists a unique circuit $C(e, B)$ such that $e \in C(e, B) \subseteq B \cup \{e\}$.

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The main result. Throughout, $M(E,r)$ will be a perfect matroid design on E and $B \subseteq E$ an arbitrary fixed basis of $M(E,r)$, the following sets of pairs being used:

$$A(B,k) = \{(F,e) \colon F \subseteq B,\ \ |F| = k,\ \ e \in \bar{F}\},\ \text{ for any }\ 1 \leq k \leq r(E),$$
$$A(B) = \bigcup_{k=1}^{r(E)} A(B,k),$$
$$A(B,e) = \{(F,e) \colon (F,e) \in A(B)\},\ \text{ for each }\ e \in E,$$
$$A(B,k,e) = \{(F,e) \colon (F,e) \in A(B), |F| = k\},\ \text{ for each }\ e \in E\ \text{ and any }\ 1 \leq k \leq r(E).$$

Obviously, according to (a) and (c) we have

$$|A(B,k)| = \binom{r(E)}{k} c(k),\ \text{ for any }\ 1 \leq k \leq r(E). \tag{1}$$

Considering the function $\varepsilon \colon A(B) \to \{-1,1\}$ defined by $\varepsilon[(F,e)] = (-1)^{r(E)-k}$, where $(F,e) \in A(B,k)$, we obtain from (1)

$$\sum_{(F,e) \in A(B)} \varepsilon[(F,e)] = \sum_{k=1}^{r(E)} (-1)^{r(E)-k} \binom{r(E)}{k} c(k). \tag{2}$$

For each $e \in E$ let $\alpha(e,B) = \displaystyle\sum_{(F,e) \in A(B,e)} \varepsilon[(F,e)]$.

Lemma 1. *If $e \in B$, then $\alpha(e,B) = 0$.*

Proof. By (a), (c) and the definition of $A(B,k,e)$, if $e \in B$, then

$$|A(B,k,e)| = \binom{r(E)-1}{k-1},\ \text{ for any }\ 1 \leq k \leq r(E).$$

Thus

$$\alpha(e,B) = \sum_{k=1}^{r(E)} (-1)^{r(E)-k} \binom{r(E)-1}{k-1} = (1-1)^{r(E)-1} = 0.$$

Lemma 2. *If $e \in E - B$, then $\alpha(e,B) \in \{0,1\}$.*

Proof. Let $C(e,B)$ be as in (e) and $|C(e,B)| = p$. Obviously, $p \leq r(E)+1$ by (a), (c) and (e). Therefore, if $F \subseteq B$, then by (b) and (d) we have

$$(F,e) \in A(B) \Leftrightarrow C(e,B) \subseteq F \cup \{e\}.$$

Hence

$$|A(B,k,e)| = \begin{cases} \binom{r(E)-p+1}{k-p+1}, & \text{if }\ p-1 \leq k \leq r(E), \\ 0, & \text{if }\ 0 \leq k \leq p-2, \end{cases}$$

that is,

$$\alpha(e,B)=\sum_{k=p-1}^{r(E)}(-1)^{r(E)-k}\binom{r(E)-p+1}{k-p+1} \text{ or } \alpha(e,B)=\sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m},$$

where $n=r(E)-p+1$ and $m=k-p+1$. Consequently

$$\alpha(e,B)=\begin{cases}(1-1)^n=0, & \text{if } n>0,\\ 1, & \text{if } n=0.\end{cases}$$

Remark. From the above lemmas it follows that $\alpha(e,B)=1$ iff $e\notin B$ and $|C(e,B)=r(E)+1$, that is, iff $B\cup\{e\}$ is a circuit of $M(E,r)$ by (d), (a) and (c).

Let us denote by $\omega[B,r(E)+1]$ the number of circuits of cardinality $r(E)+1$ containing B.

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$$\omega[B,r(E)+1]=\sum_{k=1}^{r(E)}(-1)^{r(E)-k}\binom{r(E)}{k}c(k).$$

Proof. It follows from (2) and remark since

$$\sum_{e\in E}\alpha(e,B)=\sum_{e\in E}\sum_{(F,e)\in A(B,e)}\varepsilon[(F,e)]=\sum_{(F,e)\in A(B)}c[(F,e)]$$

REFERENCES

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