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ON THE AUTOMORPHISM GROUP OF AN INFINITE GRAPH

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Abstract. In this paper, a specially defined automorphism group $\Gamma(G)$ of a connected countable simple infinite graph is considered. As the main result, we prove that $\Gamma(G)$ contains at most one non-trivial element. All infinite graphs with a non-trivial automorphism group are completely described.

Finally, for graphs with odd, or with a small even number (2 or 4) of non-zero eignevalues, the corresponding automorphism groups are characterized.

1. Introduction. Throughout the paper, G is a connected infinite countable graph without loops or multiple edges, which we briefly call–a graph. Its vertex set is $V(G) = N$, and its adjacency matrix $A = [a_{ij}]$ is an infinite $N \times N$ matrix, where

$$a_{ij} = \begin{cases} a^{i+j-2} & \text{if } i, j, \quad \text{are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

(a is a fixed positive constant, $0 < a < 1$).

Hence, the whole graph G is labelled and the "weight" of vertex $v_i = i$ is $a^{i-1} (i \in N)$.

For other definitions and results concerning spectra of infinite gaphs, one can see [**3, 4, 5**].

2. Results. The automorphism group $\Gamma(G)$ of an infinite grapf G defined here, depends on the matrix A, thus especially depends on the way of labelling of the vertex set $V(G)$.

Namely, we put $P \in \Gamma(G)$ of and only if

$$AP = PA, \tag{1}$$

where $P = [p_{ij}]$ is an infinite permutation matrix of the set $V(G)$,

$$p_{ij} = \begin{cases} 1, & j = \omega(i) \\ 0, & j \neq \omega(i) \end{cases}$$

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and ω is the corresponding permutation of the set N.

In the sequel, we identify any automorphism $P \in \Gamma(G)$ with the corresponding permutation of the set N.

Obviously, each permutation $P \in \Gamma(G)$ is a unitary operator in the corresponding Hilbert space H, and

$$Pe_i = e_{\omega(i)} \quad (i \in N),$$

for a fixed orthonormal basis $\{e_i\}_1^\infty$ of H.

Relation (1) is equivalent to

$$a_{\omega(i)\omega(j)} = a_{ij} \quad (i, j \in N), \tag{2}$$

so that vertices i, j are adjacent if and only if $\omega(i)$, $\omega(j)$ are adjacent. In this case (2) gives

$$a^{\omega(i)+\omega(j)-2} = a^{i+j-2}, \text{ or}$$
$$\omega(i) - i = -[\omega(j) - j] \quad (i, j, \text{–adjacent}). \tag{3}$$

The last relation is very restrictive, and it is the main difference in comparison to the finite case.

Lemma 1. *(i) For every $\omega \in \Gamma(G)$ there is a unique integer $d = d(\omega)$ such that*

$$| \omega(i) - i | = d \quad (i \in N). \tag{4}$$

(ii) If $\omega(i) = i$ for an $i \in N$, then $\omega =$id.

(iii) If G has at least one odd cycle, then $\Gamma(G)$ is trivial.

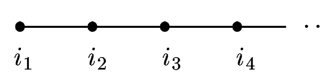
Proof. (i) For any two adjacent vertices $i, j \in V(G)$, relation (3) yields

$$| \omega(i) - i | = | \omega(j) - j |,$$

and the connectivity of G ends the proof.

The last two statements are then immediate by (i). □

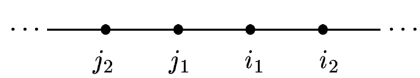
Examples. (1) The automorphism group of the one-way infinite path



is always trivial.

Indeed, for any $\omega \in \Gamma(G)$ we must have $\omega(i_1) = i_1$, thus $\omega =$id.

(2) The corresponding group of an infinite two-way infinite path



is either trivial or contains exactly one non-trivial element. If, for example, $i_p = 2p-1$ and $j_p = 2p (p \in N)$, then it is non-trivial.

The following property is one of the most important properties of the groups considered.

THEOREM 1. *In each case* $| \Gamma(G) | \leq 2$.

Proof. Let $\omega \in \Gamma(G)$ and $d = \omega(1) - 1$. Then $| \omega(i) - i | = d \ (i \in N)$, and the only possibilities we have are

$$\omega(1) = d+1, \quad 1 = \omega(d+1), \dots, \quad \omega(d) = 2d, \quad d = \omega(2d), \quad \omega(2d+1) = 3d+1, \\ 2d+1 = \omega(3d+1), \dots .$$

Generally, we obtain

$$\omega(i) = i + (-1)^{[(i-1)/d]} d.$$

If now $\omega_1, \ \omega_2 \in \Gamma(G)$ are the automorphisms with the corresponding values $d_1 = d_2$, by the last relation we immediately find $\omega_1 = \omega_2$.

Next, let $\omega_1, \ \omega_2$ be the different automorphisms with $d_1 < d_2$. Then we get

$$\omega_2 \omega_1(1) = \omega_2(1 + d_1) = 1 + d_1 + d_2,$$

and also

$$\omega_2 \omega_1(d_1 + 1) = \omega_2(1) = 1 + d_2,$$

whence $d_1 = d_2 = d_2 - d_1$; thus $d_1 = 0, \ \omega_1 = \mathrm{id}$, *q*.e.d.

Hence, $\Gamma(G)$ is always either trivial or contains at most one non-trivial element (which is then—involution). □

So, the most important question concerning $\Gamma(G)$ is when it is non-trivial. In the next main theorem we completely describe all the infinite graphs which have a non-trivial automorphism group. It appears that the considered property depends only on the structure of the graph, and on the way of labelling of its vertex set.

First, let G be any bipartite graph. Its characteristic parts are denoted by N_1 and N_2, assuming always that the minimal element is in N_1. Note that N_1, N_2 are not the cardinals, but the corresponding sets of indices.

Next, we need the notion of *symmetric bipartite graphs* (briefly, SBGs). We call an infinite bipartite graph with the characteristic parts N_1, N_2–*symmetric*, if there is a bijection $\pi : N_1 \to N_2$ such that two vertices $a \in N_1, \ \pi(b) \in N_2$ are adjacent if and only if the vertices $b \in N_1, \ \pi(a) \in N_2$ are.

If G is a SBG, then obviously N_1, N_2 are infinite.

If, additionally, we have that $\pi(a) - a = d = \pi(1) - 1$ for each $a \in N_1$, we say that $N_1, \ N_2$ are *good*. In this case, we obtain that

$$N_1 = \{(2s-2)d + r \mid r \leq d, \ \ s \in N\}, \ \ N_2 = \{(2s-1)d + r \mid r \leq d, \ \ s \in N\}.$$

Theorem 2. *Graph G has a non-trivial automorphism group if and only if it is a SBG with the good characteristic parts N_1 and N_2.*

Proof. Let $\Gamma(G)$ be non-trivial, and let $\omega \in \Gamma(G)$ be the unique non-trivial automorphism (involution) of G. Then, by Lemma 1(iii), G can not have any odd cycle as an induced subgraph, thus it must be bipartite.

Let, next, the characteristic parts of G be N_1, N_2 with the minimal element in N_1. Then, by the odd-path and the even-path characterizations of N_1, N_2, we easily find that

$$\omega(a) = a + d(a \in N_1), \quad \omega(b) = b - d \quad (b \in N_2),$$

where $d = d(\omega) = \omega(1) - 1 > 0$.

Hence, N_1, N_2 are good, and ω is a needed bijection between N_1 and N_2.

Since the converse statement is immediate, this completes the proof. □

As examples, we consider the infinite graphs with a finite number $p (p \geq 2)$ of non-zero eigenvalues.

Proposition 1. *Let G have an odd number of non-zero eigenvalues. Then its automorphism group is always trivial.*

Proof. As is known ([**4**]), each bipartite infinite graph, for every $a \in (0,1)$, has the spectrum symmetric about the zero. Hence, if G has an odd number of non-zero eigenvalues, it cannot be bipartite, whence $\Gamma(G)$ is trivial. □

Next, consider the infinite graphs with $p = 2$ or 4 non-zero eigenvalues.

We need the notion of characteristic subsets of G. The characteristic subsets $N_1, N_2, \dots$ of an infinite graph are the equivalence classes related to the equivalence relation on the vertex set $N : x \sim y$ if and only if x, y are not adjacent and they have the same neighbors. Their number is finite or infinite and always greater than 1. If it is finite, G is said to be of finite type (type p, if this number is p) [**5**]. The corresponding quotient graph is denoted by g, and often called the canonical graph of G. If, for example, G is the complete m- partitive graph $K(N_1, \dots, N_m) (m \geq 2)$, then its characteristic subsets will be $N_1, \dots, N_m$, and its canonical graph is K_m.

Lemma 2. *(i) If $\omega(x) \in N_i$ for an $x \in N_i$, then $\omega =$*id.

(ii) *If G is of finite type p and $\Gamma(G)$ is non-trivial, then p is even.*

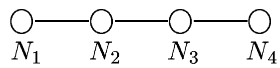
Proof. (i) Assume, on the contrary, $\omega \neq$id, and denote by M_1, M_2 the characteristic parts of G. Since if follows easily that each N_i is contained either in M_1 or in M_2, we get the statement.

(ii) Let ω be the non-trivial automorphism of $\Gamma(G)$. Since by (i), ω is an involution on the set $\{N_1, \dots, N_p\}$, without fixed elements, we have that p must be even. □

PROPOSITION 2. *Let G have exactly two-non-zero eigenvalues. Then $\Gamma(G)$ is non-trivial iff G is a complete bipartite graph with good characteristic parts.*

Proof. In [**4**], we proved that G has exactly two non-zero eigenvalues if and only if it is a complete bipartite graph. Hence, $\Gamma(G)$ is non-trivial iff N_1, N_2 are good (and consequently-infinite). □

PROPOSITION 3. *The following graph*



where $N_4 = N_1 + d$, $N_3 = N_2 - d$ $(d \neq 0)$ *is the unique connected infinite graph with four non-zero eigenvalues and a non-trivial automorphism group.*

Proof. In [**5**] we proved that G has exactly four non-zero eigenvalues if and only if its canonical graph is one of the eight particular graphs with 4,5 or 6 vertices. Since six of them have a triangle as a subgraph, their automorphism groups must be trivial. Since next, the seventh of them is P_5 with 5 characteristic subsets, by Lemma 2 (ii), its automorphism group is trivial, too. Hence, only P_4 remains, and the remaining proof is easy. □

The general problem for eny even number p of non-zero eigenvalues $(p \geq 6)$ is obviously equivalent to the determination of all finite connected canonical symmetric bipartite graphs with exactly p non-zero eigenvalues. The present author thinks it can be solved at least for $p = 6$, and may be for $p = 8$.

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