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ON AXIOMATIZABILITY AND PRESERVATION IN KRIPKE MODELS

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Section 2. of Chapter 3. of Chang and Keisler's Model Theory [**1**] begins with an inconspicuous looking little lemma (due to Keisler) which, at the first sight, might seem somewhat artifical, without much intuitive appeal. It turns out, however, that it occurs as an essential part of the proof of practically every preservation theorem. It appears that in it a crucial idea, common to preservation theorems, has been isolated. Clearly, it would be useful to have a similar result for Kripke model theory of intuitionistic first order logic. The orginal proof, however, applies essentialy the law of double negation $(\neg\neg\,\varphi \leftrightarrow \varphi)$ and the De Morgan law $(\neg(\varphi\wedge\psi) \leftrightarrow \neg\varphi\vee\neg\psi)$, principles which are not valid in Heyting's predicate calculus (HPC). It is shown here that, nevertheless, the same theorem can be obtained. Appeal to unacceptable principles is avoided by worging with pairs of theories. Pairs of intuitionistic theories were used by Gabbay in [**3**] and [**4**]. Essetially the same technique is the method of sets of signed formulas used by Fitting in [**2**] which originates in Smullyan's modification of Beth's tableaus.

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The notation and terminology are, more or less, standard so we shall just briefly review the basic definitions. By a theory (intuitionistic), in the language $\mathcal{L}$ we mean the same thing as in classical case, except that all the logical connectives $(\Box, \wedge, \vee, \rightarrow, \exists, \forall)$ occur as primitive symbols. A Kripke structure for the language $\mathcal{L}$ is a structure $\mathfrak{M} = \langle \mathcal{I}; \mathfrak{U}_t : t \in T\rangle$, where $\mathcal{I} = \langle T, 0, \leq\rangle$ is a p.o. set with the least element 0, and $\mathfrak{U}_t (t \in T)$ are classical structures for the language $\mathcal{L}$ satisfying the condition: $s \leq t$ implies $\mathfrak{U}_s \subseteq +\mathfrak{U}_t$ (i.e., $\mathfrak{U}_s$ is a positive submodel of $\mathfrak{U}_t$, i.e., $A_s \subseteq A_t$ and for any atomie formula $\varphi(x_1, \dots , x_n)$ and any $a_1, \dots , a_n \in A_s$, if $\mathfrak{U}_s \models \varphi[a_1, \dots , a_n]$ then $\mathfrak{U}_t \models \varphi[a_1, \dots , a_n]$. Forcing, by a node $t \in T$, of a sentence φ in the language $\mathcal{L} \cup \{c_a : a \in A_t\}$ is defined as usual: inductively, starting by identifying, for atomic sentence φ, forcing $(t \Vdash \varphi)$ with (classical) satisfiability $(\mathfrak{U}_t \models \varphi)$. We say that a sentence φ is valid in a Kripke structure $\mathfrak{M}(\mathfrak{M}$ is a model of φ, $\mathfrak{M} \models \varphi$) iff $0 \Vdash \varphi$) (or equivalently: for any $t \in T$, $t \Vdash \varphi$). For mode details on Kripke models cf. [**5, 2,**] or [**6**].

If Γ and Δ are sets of sentences in the language $\mathcal{L}$, we say that the pair (Γ, Δ) is consistent iff for any $\gamma_1, \dots, \gamma_k \in \Gamma$ and $\delta_1, \dots, \delta_n \in \Delta$

$$\nvdash (\gamma_1 \wedge \cdots \wedge \gamma_k) \to (\delta_1 \vee \cdots \vee \delta_n)$$

(where "$\vdash$" denotes derivability in HPC). A Kripke structure $\mathfrak{M}$ is said to be a model of the pair (Γ, Δ) iff for all $\gamma \in \Gamma, \gamma$ is valid in $\mathfrak{M}$ and *no* $\delta \in \Delta$ is valid in $\mathfrak{M}$.

We shall need the following result (Gabbay [**3**], Fitting [**2**]) which in effect is a form of strong completeness.

LEMMA: *If* (Γ, Δ) *is a consistent pair of intuitionistic theories, then it has a (Kripke) model.*

The statement of the theorem is practically the same as in [**1**].

THEOREM: *Let T be a consistent theory in a language $\mathcal{L}$ and let Δ be a set of sentences of $\mathcal{L}$ closed under disjunctions. Then the following are equivalent:*

(i) *T has a set of axioms from Δ.*

(II) *If $\mathfrak{M} \models T$ and for every $\delta \in \Delta$, ($\mathfrak{M} \models \delta$ implies $\mathfrak{N} \models \delta$) then $\mathfrak{N} \models T$.*

Proof 2. By strong completeness of Kripke models for HPC, (i) implies (ii). For the converse, assume (ii) and let $\Gamma = \{\delta \in \Delta : T \models \delta\}$. Again by strong completeness, it is enough to show $\Gamma \models T$. Since T is consistent, Γ must also be consistent, so suppose that $\mathfrak{N}$ is one model of Γ and let $\Sigma = \{\delta \in \Delta : \mathfrak{N}| \not\models \delta\}$. We claim that (T, Σ) is a consistent pair. For suppose that it is not. Then for some sentences $\varphi_1, \dots, \varphi_k \in T$ nad $\sigma_1, \dots, \sigma_n \in \Sigma$ we would have that the sentence $(\varphi_1 \wedge \cdots \wedge \varphi_k) \to (\sigma_1 \vee \cdots \vee \sigma_n)$ is a theorem of HPC. Hence $T \models \sigma_1 \vee \cdots \vee \sigma_n$. But $\sigma_1 \vee \cdots \vee \sigma_n \in \Delta$ (since Δ is closed under finite disjunctions), so we would have $\sigma_1 \vee \cdots \vee \sigma_n \in \Gamma$. Therefore $\mathfrak{N} \models \sigma_1 \vee \cdots \vee \sigma_n$ so, for some $i \in \{1, \dots, n\}$, $\mathfrak{N} \models \sigma_i$ which contradicts $\sigma_i \in \Sigma$. Therefore, the pair (T, Σ) is consistent and by the Lemma, it has a model $\mathfrak{M}$ in which $\mathfrak{M} \models \varphi$ for every sentence $\varphi \in T$ and $\mathfrak{M}| \not\models \sigma$ for every sentence $\sigma \in \Sigma$. Now models $\mathfrak{M}$ and $\mathfrak{N}$ satisfy the conditions of (ii), i.e., for any sentence $\delta \in \Delta, \mathfrak{M} \models \delta$ implies $\mathfrak{N} \models \delta$ (because $\mathfrak{N}| \not\models \delta$ implies $\delta \in \Sigma$ and consequently $\mathfrak{M}| \not\models \delta$). Therefore, by (ii), $\mathfrak{N}$ is a model of T.

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