

SOME FIXED POINT THEOREMS IN BOOLEAN ALGEBRA

Koriolan Gilezan

A Boolean transformation is a mapping

$$F: D \rightarrow B^n$$

where $D \subseteq B^n$ is a set of solutions of Boolean equations $d_j(X) = 0$ $j = 1, \dots, k$.
 F is defined as follows

$$F(X) = (f_1(X), \dots, f_n(X)), \quad X = (x_1, \dots, x_n),$$

each f_i is a Boolean function $f_i: B^n \rightarrow B$, $i = 1, \dots, n$, $B = \{0, 1\}$.

DEFINITION 1. X is a fixed point of a Boolean transformation F with restriction $d_j(X) = 0$, $j = 1, \dots, k$, if and only if

$$(1) \quad F(X) = X \text{ and } d_j(X) = 0, \quad (j = 1, \dots, k).$$

X is a fixed point of a Boolean transformation F with restrictions d_j ($j = 1, \dots, k$) if X is a solution of the system of the Boolean equations:

$$(1') \quad f_i(X) = x_i, \quad i = 1, \dots, n; \quad d_j(X) = 0, \quad j = 1, \dots, k,$$

which follows from the definitions.

LEMMA 1. *The system of Boolean equations (1') is equivalent to the Boolean equation:*

$$(2) \quad \bigcup_{j=1}^k d_j(X) \cup \bigcup_{i=1}^n (x_i \bar{f}_i(X) \cup \bar{x}_i f_i(X = 0))$$

PROOF. It is known that Boolean equations $f = g$ and $\bar{f}g \cup \bar{f}g = 0$ are equivalent. If this property is applied to the system (1) it will become an equivalent system of Boolean equations:

$$(3) \quad \begin{aligned} x_i \bar{f}_i(X) \cup \bar{x}_i f_i(X) &= 0, & i = 1, \dots, n \\ d_j(X) &= 0 & j = 1, \dots, k. \end{aligned}$$

Furthermore, we know that a system of Boolean equations $g_i = 0$, $i = 1, \dots, m$ is equivalent to a Boolean equation $\bigcup_{i=1}^m g_i = 0$, if this property is applied to the system (3) it will become the Boolean equation (2) which is equivalent with the system (3).

LEMMA 2. *The system of Boolean equations (2) has a solution if and only if*

$$(4) \quad \prod_{\alpha \in B^n} \left[\bigcup_{j=1}^n d_j(\alpha) \cup \bigcup_{i=1}^n (\alpha_i \bar{f}_i(\alpha) \cup \bar{\alpha}_i f_i(\alpha)) \right] = 0,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$.

PROOF. Let $f(x)$ be the left side of the equation (2). It is known that every Boolean function can be written in a disjunctive canonical form. Therefore, the equation (2) can be written in the following form:

$$(5) \quad f(x_1, \dots, x_n) = \bigcup_{(\alpha_1, \dots, \alpha_n) \in B^n} f(\alpha_1, \dots, \alpha_n) x_1^{\alpha_1} \dots x_n^{\alpha_n} = 0,$$

where $x^0 = \bar{x}$, $x^1 = x$.

The equation (5) has a solution if and only if

$$(6) \quad \bigcap_{\alpha \in B^n} f(\alpha) = 0.$$

The lemma is proved.

Directly from the lemmas 1 and 2 we have the following theorem.

THEOREM 1. *A Boolean transformation $f(x) = x$ with the restrictions $d_j(x) = 0$ ($j = 1, \dots, k$) has a solution if and only if the condition (4) is fulfilled.*

The matrix $A = [a_{ij}]$, $i, j = 1, \dots, n$, is a Boolean matrix when the elements a_{ij} are Boolean terms from the Boolean algebra $(\{0, 1\}, \cup, \cdot, -)$. The linear Boolean transformation is a linear operator

$$(7) \quad F(X) = AX$$

where $X = [x_{ij}]$

$$A = [a_{ij}], \quad i, j = 1, \dots, n.$$

The matrix product is

$$AX = \left[\bigcup_{k=1}^n a_{ik}x_{kj} \right].$$

Let A be a given matrix. The problem is to find a matrix X , such that

$$(8) \quad X = AX.$$

The Boolean matrix equation (8) is equivalent to the system of Boolean equations

$$(9) \quad \bigcup_{k=1}^n a_{ik}x_{kj} = x_{kj}, \quad i, j = 1, \dots, n.$$

If we replace (9) in (2) follows:

CONSEQUENCE 1. The system of Boolean equations (9) is equivalent to the Boolean equations:

$$(10) \quad \bigcup_{i,j=1}^n \left(\bar{x}_{ij} \bigcup_{k=1}^n a_{ik}x_{kj} \cup x_{ij} \prod_{k=1}^n (\bar{a}_{ik} \cup \bar{x}_{kj}) \right) = 0$$

The replacement of (10) in (4) gives:

CONSEQUENCE 2. The system of Boolean equations (9) has a solution if and only if

$$(11) \quad \prod_{\alpha \in B^{n^2}} \left[\bigcup_{i,j=1}^n \left(\bar{\alpha}_{ij} \bigcup_{k=1}^n a_{ik}\alpha_{kj} \cup \alpha_{ij} \prod_{k=1}^n (\bar{\alpha}_{ik} \cup \bar{\alpha}_{kj}) \right) \right] = 0,$$

where

$$\alpha = (\alpha_{11}, \dots, \alpha_{1n}, \alpha_{21}, \dots, \alpha_{2n}, \dots, \alpha_{n1}, \dots, \alpha_{nn}).$$

Directly from the Consequence 2. the statement follows:

THEOREM 2. Let A be a given matrix. The matrix equation $X = AX$ has a solution if and only if the condition (11) is fulfilled.

LEMMA 3. The Boolean matrix equation $AX = X$ always has at least one solution. That is

$$X = 0.$$

THEOREM 3. The Boolean matrix equation $AX = X$ has a singular solution $x = 0$ if and only if the matrix A is of the form:

$$(12) \quad \left(\begin{array}{ccc|ccc} 0 & \dots & 0 & a_{1k} & & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{k-1k} & & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{k+1k} & & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & a_{nk} & & 0 & \dots & 0 \end{array} \right)$$

where $a_{1k}, \dots, a_{k-k}, a_{k+1k}, \dots, a_{nk}$ are the elements of the set $\{0, 1\}$.

PROOF. Let A be a matrix of the form (12). If we replace it in the Boolean matrix equation $AX = X$ we get a system of Boolean equations:

$$(13) \quad \begin{array}{l} \text{for } i \neq k, \quad a_{ik}x_{kj} = x_{ij}, \quad i, j = 1, \dots, n \\ \text{for } i = k \quad \quad 0 = x_{ij} \quad j = 1, \dots, n. \end{array}$$

From (13) it follows that $x_{ij} = 0, i, j = 1, \dots, n$, i.e. $X = 0$.

Suppose that the matrix A^* is not of the form (12) and $X = 0$ is a singular solution, i.e. there exist some elements $a_{i,ji} \neq 0$ which are not in the k -th column.

Let us replace the matrix A^* in the Boolean matrix equation $AX = X$. We get a system of Boolean equations:

$$(14) \quad \begin{array}{l} \text{for } i \neq k, \quad i \neq i_1, \quad a_{ik}x_{ki} = x_{ij}, \quad i, j = 1, \dots, n \\ \text{for } i \neq k, \quad i = i_1, \quad a_{i_1j}x_{i,j} \cup a_{i_1j}x_{ij}x_{i_1,j}, \quad j = 1, \dots, n \\ \text{for } i = k \quad \quad \quad 0 = x_{ij}, \quad j = 1, \dots, n. \end{array}$$

The system of Boolean equations (14) is equivalent to the system of Boolean equations

$$(15) \quad \begin{array}{l} \text{for } i \neq k \quad i = i_1, \quad x_{ij} = 0, \quad i, j = 1, \dots, n \\ \text{for } i \neq k, \quad i \neq i_1, \quad a_{i_1j}x_{i_1,j} = x_{i_1,j}, \quad j = 1, \dots, n \\ \text{for } i = k \quad \quad \quad 0 = x_{ij}, \quad j = 1, \dots, n. \end{array}$$

Zero matrix is not the only solution of the system (15). It is contradiction.

So the theorem is proved.

LEMMA 4. $A \cup X = X$ (where $A \cup X = [a_{ij} \cup x_{ij}] i, j = 1, \dots, n$) if and only if $X = A \cup P$, P is an arbitrary Boolean $n \times n$ matrix.

PROOF. The Boolean matrix equation $A \cup X = X$ is equivalent to a system of Boolean matrices:

$$(16) \quad a_{ij} \cup x_{ij} = x_{ij}, \quad i, j = 1, \dots, n.$$

The system (16) is equivalent to the system of Boolean equations:

$$(17) \quad a_{ij}\bar{x}_{ij} = 0, \quad i, j = 1, \dots, n.$$

It is known that the equations $AX \cup B\bar{X} = 0, X = \bar{A}P \cup B\bar{P}, P \in \{0, 1\}$ are equivalent. If we use this property in (16) we get the system of Boolean equations.

$$(18) \quad x_{ij} = a_{ij} \cup p_{ij}, \quad p_{ij} \in \{0, 1\}, \quad i, j = 1, 2, \dots, n.$$

This system is equivalent to the system (16) therefore, from (18) there follows that

$$X = A \cup P, \text{ where } P \text{ is an arbitrary Boolean matrix.}$$

LEMMA 5. $A \odot X = X$ ($A \odot X = [a_{ij}, x_{ij}]$, $i, j = 1, \dots, n$) if and only if $X = A \odot P$, P is an arbitrary Boolean $n \times n$ matrix.

PROOF. The Boolean matrix $A \odot A = X$ is equivalent to the system of Boolean equations

$$(19) \quad a_{ij}x_{ij} = x_{ij}, \quad i, j = 1, \dots, n.$$

The system (19) is equivalent to the system of Boolean equations

$$(20) \quad \bar{a}_{ij}x_{ij} = 0, \quad i, j = 1, \dots, n.$$

Let us apply to (20) the property that the equations $AX \cup B\bar{X} = 0$, $X = \bar{A}P \cup B\bar{P}$, $P \in \{0, 1\}$ are equivalent. We get the system of Boolean equations:

$$(21) \quad x_{ij} = a_{ij}p_{ij} \quad p_{ij} \in \{0, 1\}, \quad i, j = 1, \dots, n.$$

Therefore, the system (21) is equivalent to the system (19). From (21) there follows that:

$$X = A \odot P, \quad P \text{ is an arbitrary Boolean matrix.}$$

LEMMA 6. $A * X = X$ (where $A * P = [\bar{a}_{ij}x_{ij} \cup a_{ij}\bar{x}_{ij}]$ $i, j = 1, \dots, n$) if and only if $X = A * P$, P is an arbitrary Boolean matrix.

PROOF. The Boolean matrix $A * P = X$ is equivalent to the system of Boolean equations

$$(22) \quad \bar{a}_{ij}x_{ij} \cup a_{ij}\bar{x}_{ij} = x_{ij}, \quad i, j = 1, \dots, n.$$

This system is equivalent to the system of Boolean equations

$$(23) \quad a_{ij}x_{ij} \cup a_{ij}\bar{x}_{ij} = 0 \quad i, j = 1, \dots, n.$$

The system (23) is equivalent to the system of Boolean equations:

$$x_{ij} = \bar{a}_{ij}p_{ij} \cup a_{ij}\bar{p}_{ij}, \quad i, j = 1, \dots, n, \quad p_{ij} \in \{0, 1\},$$

i.e. $X = A * P$, P is an arbitrary Boolean matrix.

THEOREM 4. The Boolean matrix equation $A \oplus X = X$ (where

$$A \oplus X = \left[\bigcup_{k=1}^n e_{ik}\bar{x}_{kj} \cup \bar{a}_{ik}x_{kj} \right] \quad i, j = 1, \dots, n)$$

for a given matrix A has a solution if and only if

$$\prod_{\alpha \in B^{n^2}} \left(\bigcup_{i,j=1}^n \left(\alpha_{ij} \bigcup_{k=1}^n (\bar{a}_{ik} \alpha_{kj} \cup a_{ik} \bar{\alpha}_{kj}) \cup \alpha_{ij} \prod_{k=1}^n (a_{ik} \cup \bar{a}_{kj}) (\bar{a}_{ik} \cup \alpha_{kj}) \right) \right) = 0$$

where $\alpha = (\alpha_{11}, \dots, \alpha_{1n}, \alpha_{21}, \dots, \alpha_{2k}, \dots, \alpha_{n1}, \dots, \alpha_{nn})$.

THEOREM 5. *The Boolean matrix equations $A \circ X = X$ (where*

$$A \circ X = \left[\bigcup_{k=1}^n (a_{ik} \cup x_{kj}) \right]. \quad i, j = 1, \dots, n)$$

for a given matrix A has a solution if and only if

$$\prod_{\alpha \in B^{n^2}} \left[\bigcup_{i,j=1}^n \left(\bar{\alpha}_{ij} \bigcup_{k=1}^n (a_{ik} \cup \alpha_{kj}) \cup \alpha_{ij} \prod_{k=1}^n \bar{a}_{ik} \bar{\alpha}_{kj} \right) \right] = 0$$

where $\alpha = (\alpha_{11}, \dots, \alpha_{1n}, \alpha_{21}, \dots, \alpha_{2n}, \dots, \alpha_{n1}, \dots, \alpha_{nn})$.

The proof of Theorem 4. and Theorem 5. directly follows from Lemma 2.

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