

## $\Delta$ -ENDOMORPHISM NEAR-RINGS

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The concept of a distributively generated near-rings arise if we define addition and multiplication of endomorphisms of the group  $(G, +)$  in the usual manner. It is possible to consider the set of the mappings of  $(G, +)$  into itself which are similar to the endomorphisms of a group in such a way that their “linearity” is corrected by the elements from a normal subgroup  $\Delta$  of the group  $(G, +)$ . These mappings are called  $\Delta$ -endomorphisms of  $(G, +)$ . The set of  $\Delta$ -endomorphisms of  $G$  generate (additively) a near-ring  $\mathcal{E}_\Delta(G)$ , whose defect depends on the choice of the subgroup  $\Delta$ . Also,  $\Delta$ -endomorphisms for which is invariant every fully invariant subgroup of the group  $(G, +)$ , are investigated. In this case we obtain the subnearring  $E_\Delta(G)$  of the near-ring  $\mathcal{E}_\Delta(G)$ . Some known properties of the endomorphism near-rings were transferred to the  $\Delta$ -endomorphism near-rings.

Some elementary results relating to the  $E_\Delta$ -invariant subgroups of  $(G, +)$  are presented in Section 2. In Section 3 we consider the structure of ideals of the near-ring  $E_\Delta(G)$ , generalizing the results which were obtained by H. Johnson in [8] and [9] for the near-ring of endomorphisms. The result in Section 4 refers to the problem embedding of near-rings into some near-ring of  $\Delta$ -endomorphisms and generalizes the Theorem Heatherly and Malone in [7]. Also, a  $\mathcal{D}$ -direct sum of subnear-rings of the near-ring  $E_\Delta(G)$  is considered, where  $\mathcal{D}$  is a defect of  $E_\Delta(G)$ .

### 1. Preliminaries

Throughout this paper term “near-ring” shall mean “left near-ring”  $R$  satisfying  $ox = o$  for all  $x \in R$ . The necessary definitions concerning near-rings with a defect of distributivity are now given.

A set of generators of the near-ring  $R$  is a multiplicative subsemigroup  $S$  of  $R$  whose elements generate  $(R, +)$ . Let  $S$  be a set of generators of the near-ring  $R$  and let

$$D_S = \{d: d = -(xs + ys) + (x + y)s, \quad x, y \in R, \quad s \in S\}.$$

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The normal subgroup  $D$  of the group  $(R, +)$  which is generated by the set  $D_S$  is called the defect of distributivity of the near-ring  $R$ . Thus, for all  $x, y \in R$  and  $s \in S$  there exists  $d \in D$  such that

$$(x + y)s = xs + ys + d.$$

The near-ring  $R$  with the defect  $D$  will be denoted by  $(R, S)$  when we wish to stress the set of generators  $S$ . A near-ring  $R$  is called  $D$ -distributive if  $R = S$ , i.e. for each  $x, y, z \in R$  there exists  $d \in D$  such that

$$(x + y)z = xz + yz + d.$$

Let  $(R, S)$  be a near-ring with the defect  $D$  and  $A \subset R$ . The normal subgroup  $\bar{A}$  of  $(R, +)$  generated by the set  $A \cup AS$  has the elements of the form

$$\bar{a} = \sum_i (r_i \pm a_i s_i + m_i a_i' - r_i), \quad (r_i \in R, a_i, a_i' \in A, s_i \in S, m_i - \text{integers}).$$

For all  $r, r_i \in R, a_i, a_i' \in A$  and  $s, s_i \in S$  there exists  $d_1, d_2 \in D$  such that

$$\begin{aligned} (r + \bar{a})s &= rs + \bar{a}s + d_1 = rs + \left( \sum_i (r_i \pm a_i s_i + m_i a_i' - r_i) \right) s + d_1 \\ (r + \bar{a})s &= \sum_i (r_i s \pm a_i s_i s + m_i a_i' s - r_i s) + d_2 + d_1. \end{aligned}$$

The normal subgroup  $D_r$  of the group  $(R, +)$  generated by the elements  $d_2 + d_1 = d \in D$  which have been obtained in the previous manner, is called a relative defect of the subset  $A$  with respect to  $R$ . It is obvious that  $D_r \subseteq D$ .

LEMMA 1.1. ([4]. Lemma 3.2) *Let  $(R, S)$  be a near-ring with defect. The normal subgroup  $B$  of the group  $(R, +)$  is a right ideal of  $R$  if and only if  $B$  is an  $S$ -subgroup which contains the relative defect of the subset  $B$  with respect to  $R$ .*

PROPOSITION 1.2. ([5], Coroll. of Lemma 1.1) *Let  $(R, S)$  be a near-ring with defect and  $A \subset R$ . The normal subgroup  $\bar{A}$  of  $(R, +)$  generated by  $A \cup RA \cup AS$  is an ideal of  $R$  if and only if  $\bar{A}$  contains the relative defect of the subset  $A \cup RA$  with respect to  $R$ .*

PROPOSITION 1.3. ([4], Theorem 2.3 b) *Every direct sum of the near-rings  $R_i$  with the defect  $D_i$  respectively, is a near-ring  $R$  whose defect is a direct sum of the defects  $D_i$ .*

## 2. Elementary properties of $\Delta$ -endomorphisms

Let  $M_0(G)$  be a set of zero preserving mappings of the group  $(G, +)$  into itself.

DEFINITION. Let  $\Delta$  be a normal subgroup of the group  $(G, +)$ . The mapping  $f \in M_0(G)$  with  $(\Delta)f \subseteq \Delta$  is called  $\Delta$ -endomorphism of the group  $(G, +)$  if for all  $x, y \in G$  there exists  $\delta \in \Delta$  such that

$$(x + y)f = (x)f + (y)f + \delta.$$

It is easy to prove by induction that for each  $x_1, \dots, x_n \in G$  and some  $\Delta$ -endomorphism  $f$  there exists  $\delta \in \Delta$  such that

$$(x_1 + \dots + x_n)f = (x_1)f + \dots + (x_n)f + \delta.$$

In the case  $\Delta = (0)$  we obtain the endomorphisms of the group  $(G, +)$ . The set of all  $\Delta$ -endomorphisms of the group  $(G, +)$  will be denoted by  $\mathcal{E}nd_{\Delta}(G)$ . This set is a semigroup with respect to composition.

Let us denote by  $(G, \Delta)_0$  the set of all mappings  $h: G \rightarrow \Delta$  with  $(0)h = 0$ . It is clear that  $(G, \Delta)_0 \subseteq \mathcal{E}nd_{\Delta}(G)$ . Thus, for  $\Delta \neq (0)$  it follows that  $\mathcal{E}nd_{\Delta}(G) \neq \mathcal{E}nd(G)$ .

If  $(G, +)$  is non-commutative, then the set of all  $\Delta$ -endomorphisms of  $G$  will not be closed under pointwise addition. However, the set of all (finite) sums and differences of  $\Delta$ -endomorphisms of  $G$  forms a near-ring, which will be designated by  $\mathcal{E}_{\Delta}(G)$ . Namely, if  $f = \sum_i (\pm t_i)$  and  $h = \sum_j (\pm t_j')$ , ( $t_i, t_j' \in \mathcal{E}nd_{\Delta}(G)$ ), then for all  $x \in G$  we have

$$\begin{aligned} (x)fh &= \sum_j \pm \left( \sum_i ((\pm x)t_i) \right) t_j' \\ &= \sum_j \pm \left( \sum_i (\pm x)t_i t_j' + \delta_{ij} \right) \\ &= \sum_j \pm \left( \sum_i (\pm x)t_i t_j' \right) + \delta, \quad (\delta_{ij}, \delta \in \Delta). \end{aligned}$$

But, the element  $\delta \in \Delta$  depends on  $x$ . If we put  $\delta = (x)\alpha$ , then  $\alpha \in (G, \Delta)_0$  i.e.  $\alpha \in \mathcal{E}nd_{\Delta}(G)$ . Hence,

$$\begin{aligned} (x)fh &= (x) \left[ \left( \sum_j \left( \pm \sum_i t_i t_j' \right) \right) + \alpha \right], \text{ i.e.} \\ fh &= \sum_j \left( \sum_i (\pm t_{ij}) \right) + \alpha, \end{aligned}$$

where  $t_i t_j' = t_{ij} \in \mathcal{E}nd_{\Delta}(G)$  and  $\alpha \in \mathcal{E}nd_{\Delta}(G)$ .

The normal subgroup  $\mathcal{D}$  of the group  $(\mathcal{E}_{\Delta}(G), +)$  generated by

$$\{\delta: \delta = -(ht + ft) + (h + f)t, \quad h, f \in \mathcal{E}_{\Delta}(G), \quad t \in \mathcal{E}nd_{\Delta}(G)\}$$

is a defect of distributivity of the near-ring  $\mathcal{E}_{\Delta}(G)$ . It is clear that  $\mathcal{D} \subseteq (G, \Delta)_0$ . For example, the near-ring  $\mathcal{E}_{\Delta}(Z_4) = \{f_0, f_1, \dots, f_{15}\}$ , where  $\Delta = \{0, 2\}$ , has the defect  $\mathcal{D} = \{f_0, f_3, f_{12}, f_{13}\}$  (table 1).

If the commutator subgroup  $G'$  of  $(G, +)$  is a subset of  $\Delta$ , then  $\mathcal{E}_{\Delta}(G)$  is a  $\mathcal{D}$ -distributive near-ring, where  $\mathcal{D}$  is the defect of  $\mathcal{E}_{\Delta}(G)$ . Let  $G$  be a nilpotent group and  $\Delta$  its maximal subgroup. Then by Corollary 10.3.2 of [6] it follows that the near-ring  $\mathcal{E}_{\Delta}(G)$  is  $\mathcal{D}$ -distributive, where  $\mathcal{D}$  is the defect of  $\mathcal{E}_{\Delta}(G)$ .

Let  $(R, S)$  be a near-ring with the defect  $D$ . For all  $s \in S$  and  $x \in R$  there is a map  $f_s: x \rightarrow xs$  from  $R$  into  $R$ . These maps are  $D$ -endomorphisms. Let us denote by  $\mathcal{E}_D(R)$  the near-ring of "right multiplications" of the near-ring  $R$  with the defect  $D$ . The defect of distributivity of  $\mathcal{E}_D(R)$  is the set

$$\{f_d: (x)f_d = xd, \quad x \in R, \quad d \in D\}.$$

**PROPOSITION 2.1.** *If  $\Delta$  is a proper normal subgroup of the group  $(G, +)$ , then  $\mathcal{E}_{\Delta}(G) \subset M_0(G)$ .*

**PROOF.** Anyhow  $\mathcal{E}_{\Delta}(G) \subseteq M_0(G)$ . If  $(0) \neq \Delta \neq G$  and  $y \in G \setminus \Delta$ , then the map  $h \in M_0(G)$  can be defined as follows

$$x(h) = \begin{cases} y, & x \in \Delta, x \neq 0 \\ 0 & x = 0 \\ x, & x \notin \Delta \end{cases}$$

Since  $(\Delta) \mathcal{E}_{\Delta}(G) \subseteq \Delta$ , we have  $h \notin \mathcal{E}_{\Delta}(G)$ .

If  $B$  is a fully invariant subgroup of the group  $(G, +)$ , then  $B$  must not be invariant with respect to all  $\Delta$ -endomorphisms of  $(G, +)$ . For example, the subgroup  $B = \{0, 2, 4\}$  of  $(Z_6, +)$  is not invariant with respect to the  $\Delta$ -endomorphism  $f = \begin{pmatrix} 012345 \\ 003003 \end{pmatrix}$ , where  $\Delta = \{0, 3\}$ .

Let  $\Delta$  be a proper normal subgroup of the group  $(G, +)$ . There exist nontrivial  $\Delta$ -endomorphisms for which are invariant all subgroups of  $(G, +)$ . For instance, the mapping  $f \in M_0(G)$  with  $(x)f = x$  for all  $x \in \Delta$ , and  $(x)f = 0$  for all  $x \in G \setminus \Delta$  is such a  $\Delta$ -endomorphism. Let us denote by  $End_{\Delta}(G)$  the biggest subsemigroup of the semigroup  $\mathcal{E}nd_{\Delta}(G)$  for which are invariant all fully invariant subgroups of the group  $(G, +)$ . If we denote by  $E_{\Delta}(G)$  the additive group generated by  $\mathcal{E}nd_{\Delta}(G)$ , then  $E_{\Delta}(G)$  is a near-ring whose set of generators  $End_{\Delta}(G)$  is contained in a set of generators  $\mathcal{E}nd_{\Delta}(G)$  of the near-ring  $\mathcal{E}_{\Delta}(G)$ . Every fully invariant subgroup of  $(G, +)$  which is invariant with respect to  $End_{\Delta}(G)$ , is invariant with respect to

$E_\Delta(G)$  as well. For this reason we say that the subgroups of this kind are  $E_\Delta$ -invariant.

EXAMPLE 1. The group  $(Z_6, +)$  has 96  $\Delta$ -endomorphisms for which only the subgroup  $\Delta = \{0, 3\}$  is invariant. However, the set  $End_\Delta(Z_6) = \{f_0, f_1, \dots, f_{23}\}$  contains all  $\Delta$ -endomorphisms of  $(Z_6, +)$  for which both subgroups  $\Delta$  and  $B = \{0, 2, 4\}$  are invariant (table 2). If we take for  $\Delta$  the subgroup  $B$ , then there exist 486  $\Delta$ -endomorphisms. But by claiming that both subgroups of  $(Z_6, +)$  are invariant this number will be reduced to 54.

If  $\Delta$  is a fully invariant subgroup of  $(G, +)$ , then a near-ring  $E_\Delta(G)$  contains the endomorphism near-ring  $E(G)$ . A several following propositions are related to the elementary properties of  $E_\Delta$ -invariant subgroup and they generalize the corresponding results of M. Jonson in [8].

PROPOSITION 2.2. *Let  $\Delta$  be a fully invariant subgroup of  $(G, +)$  and let  $y \in G$ , ( $y \neq 0$ ). If  $\mathcal{H}$  is a right  $E_\Delta(G)$ -subgroup, then  $(y)\mathcal{H}$  is  $E_\Delta$ -invariant subgroup of  $(G, +)$ .*

The proof is quite analogous with that in ([8], Lemma 3.1).

COROLLARY. Let  $B$  be  $E_\Delta$ -invariant subgroup of  $(G, +)$  and let  $y \in B$ , ( $y \neq 0$ ). If  $\mathcal{H}$  is a right  $E_\Delta(G)$ -subgroup, then  $(y)\mathcal{H}$  is  $E_\Delta$ -invariant subgroup of  $(G, +)$ .

DEFINITION. Let  $B$  be a subgroup of the group  $(G, +)$  and  $\mathcal{H} \subseteq M_0(G)$ . If  $B$  is an invariant subgroup with respect to  $\mathcal{H}$ , then we say that  $\mathcal{H}$  acts transitively on  $B$  if for all  $x \in B$ , ( $x \neq 0$ ) we have  $(x)\mathcal{H} = B$ .

DEFINITION. The group  $(G, +)$  is called  $E_\Delta$ -simple if and only if  $(G, +)$  has not proper  $E_\Delta$ -invariant subgroups.

Using Corollary of Proposition 2.2 we obtain the following.

PROPOSITION 2.3. *Let  $B$  be an  $E_\Delta$ -invariant subgroup of the group  $(G, +)$ . Then  $B$  is a minimal  $E_\Delta$ -invariant subgroup of  $(G, +)$  if and only if  $E_\Delta(G)$  acts transitively on  $B$ .*

COROLLARY. Let  $\Delta$  be a fully invariant subgroup of  $(G, +)$ .  $E_\Delta(G)$  acts transitively on  $G$  if and only if  $G$  is  $E_\Delta$ -simple.

Let  $G$  be a group and  $B \subset G$ . Denote by  $\mathcal{A}(B)$  a right annihilator of  $B$  in  $E_\Delta(G)$ , that is,  $\mathcal{A}(B) = \{f \in E_\Delta(G) : (b)f = 0 \text{ for all } b \in B\}$ .

PROPOSITION 2.4. *Let  $B_i$  ( $i \in I$ ) be a collection of minimal  $E_\Delta$ -invariant subgroups of the group  $(G, +)$  and let  $\mathcal{N}$  be a right  $E_\Delta(G)$ -subgroup of  $E_\Delta(G)$  containing only nilpotent elements. Then  $\mathcal{N} \subseteq \cap_i \mathcal{A}(B_i)$ .*

PROOF. Let  $h \in \mathcal{N}$  and suppose that for some  $b \in B_p$  ( $p \in I$ ),  $(b)h \neq 0$ . By Proposition 2.2  $(b)hE_\Delta(G)$  is  $E_\Delta$ -invariant subgroup. Since  $B_p$  is a minimal  $E_\Delta$ -invariant subgroup of  $(G, +)$ , there exists  $f \in E_\Delta(G)$  such that  $(b)hf = b$ . Hence

$hf$  is not nilpotent. On the other hand,  $hf \in \mathcal{N}$  and this contradiction establishes the proposition.

The next proposition is easily verified.

**PROPOSITION 2.5.** *Let  $B_i$  ( $i \in I$ ) be a collection of  $E_\Delta$ -invariant subgroups of the group  $(G, +)$ . If  $\Delta \subseteq \sum_i B_i$  then  $\sum_i B_i$  is  $E_\Delta$ -invariant subgroup.*

### 3. The ideal structures of $E_\Delta(G)$

The results in this section refer to the ideal structures of the near-ring  $E_\Delta(G)$ . The results of M. Johnson ([8], Lemmas 6.1, 8.5, Thms 6.2, 6.11, 6.12, Propositions 8.9, 8.15) and ([9], Lemma 11, Thms 8 and 16) become a special case of these, when we take an endomorphism near-ring  $E(G)$  instead  $E_\Delta(G)$ .

If  $\mathcal{H}$  is a subset of  $E_\Delta(G)$ , we define

$$\mathfrak{S}(\mathcal{H}) = \{(x)h : x \in G, h \in \mathcal{H}\}.$$

Obviously,  $\mathfrak{S}(\mathcal{D}) \subseteq \Delta$ , where  $\mathcal{D}$  is the defect of the near-ring  $E_\Delta(G)$ .

**PROPOSITION 3.1.** *Let  $B$  be an  $E_\Delta$ -invariant subgroup of the group  $(G, +)$ . If  $\mathfrak{S}(\mathcal{D}_r) \subseteq B$ , where  $\mathcal{D}_r$  is the relative defect of the subset  $\mathcal{B} = \{f \in E_\Delta(G) : \mathfrak{S}(f) \subseteq B\}$  with respect to  $E_\Delta(G)$ , then  $\mathcal{B}$  is an ideal of  $E_\Delta(G)$ .*

**PROOF.** It is easy to show that  $\mathcal{B}$  is a normal subgroup of  $(E_\Delta(G), +)$  and  $E_\Delta(G)$ -subgroup of  $E_\Delta(G)$ . If  $\delta \in \mathcal{D}_r$  then  $\delta \in \mathcal{B}$  because  $\mathfrak{S}(\mathcal{D}_r) \subseteq B$ . Hence  $\mathcal{B}$  contains the relative defect of the subset  $\mathcal{B}$  with respect to  $E_\Delta(G)$ . Therefore, by Lemma 1.1 it follows that  $\mathcal{B}$  is a right ideal of  $E_\Delta(G)$ . Also,  $\mathcal{B}$  is a left  $E_\Delta(G)$ -subgroup. Thus  $\mathcal{B}$  is an ideal of  $E_\Delta(G)$ .

**PROPOSITION 3.2.** *Let  $\Delta \neq G$  be a nonzero fully invariant subgroup of the group  $(G, +)$ . Then  $E_\Delta(G)$  is not a simple near-ring.*

**PROOF.** Let  $\mathcal{D}_r$  be a relative defect of the subset

$$\mathcal{B} = \{f \in E_\Delta(G) : \mathfrak{S}(f) \subseteq \Delta\}$$

with respect to  $E_\Delta(G)$ . Because  $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G, \Delta)_0$ , we have  $\mathfrak{S}(\mathcal{D}_r) \subseteq \Delta$ . By Proposition 3.1,  $\mathcal{B}$  is an ideal of  $E_\Delta(G)$ . Since  $\Delta \neq G$  it follows that the identity map is not in  $\mathcal{B}$ , i.e.  $\mathcal{B} \neq E_\Delta(G)$ . Let us define the map  $h \in (G, \Delta)_0$  as follows

$$(x)h = \begin{cases} x, & x \in \Delta \\ 0, & x \notin \Delta \end{cases}$$

This map is a nonzero  $\Delta$ -endomorphism and  $\mathfrak{S}(h) \subseteq \Delta$ , i.e.  $h \in \mathcal{B}$ . Hence,  $\mathcal{B}$  is a proper ideal of  $E_\Delta(G)$ .

**PROPOSITION 3.3.** *Let  $\Delta$  be a fully invariant subgroup of the group  $(G, +)$ .  $E_\Delta(G)$  is simple if and only if  $G$  is  $E_\Delta$ -simple.*

PROOF. If  $G$  is a nonzero  $E_\Delta$ -simple group it must be either  $\Delta = (0)$  or  $\Delta = G$ . For  $\Delta = (0)$  the results follows from ([8], Th. 6.12) and for  $\Delta = G$  it follows from ([2], Lemma 4).

Conversely, let now  $E_\Delta(G)$  be a simple near-ring. If  $\Delta = (0)$  the result follows from ([8], Th. 6.12). If  $\Delta \neq (0)$  then it is not a proper subgroup of  $G$ . Namely, if  $\Delta \neq G$  then by Proposition 3.2  $E_\Delta(G)$  is not a simple near-ring. Thus, let  $\Delta = G$ , i.e.  $E_\Delta(G) = H_0(G)$ . If  $B$  is a proper subgroup of  $(G, +)$ , then there always exists  $f \in M_0(G)$  for that  $B$  is not invariant. Therefore,  $G$  is an  $E_\Delta$ -simple group.

**THEOREM 3.4.** *If  $B$  is a sum of all minimal nonzero  $E_\Delta$ -invariant subgroups of a finite group  $(G, +)$  and  $\Delta \subseteq B$  is fully invariant subgroup of  $(G, +)$ , then  $\mathfrak{B} = \{h \in E_\Delta(G) : \mathfrak{S}(h) \subseteq B\}$  is a proper nonzero ideal of  $E_\Delta(G)$ .*

PROOF. By Proposition 2.5 it follows that  $B$  is  $E_\Delta$ -invariant subgroup. If  $\mathcal{D}_r$  is a relative defect of the subset  $\mathcal{B}$  with respect to  $E_\Delta(G)$ , then  $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G, \Delta)_0$ . Since,  $\Delta \subseteq B$  we have  $\mathfrak{S}(\mathcal{D}_r) \subseteq B$ . Thus, by Proposition 3.1  $\mathcal{B}$  is an ideal of  $E_\Delta(G)$ . Clearly,  $\mathcal{B} \neq E_\Delta(G)$ . Let  $\{x_1, \dots, x_n\} = G$ . By Proposition 2.2  $(x_p)E_\Delta(G)$  ( $p = 1, \dots, n$ ) is  $E_\Delta$ -invariant subgroup of  $(G, +)$ . Thus,  $(x_p)E_\Delta(G) \cap B \neq (0)$  for all  $p = 1, \dots, n$ . Now the proof is similar to the proof of the Theorem 6.2 in [8].

**PROPOSITION 3.5.** *Let  $B$  be a sum of all minimal nonzero  $E_\Delta$ -invariant subgroups of a finite group  $(G, +)$  and let  $\Delta \subseteq B$  be a fully invariant subgroup of  $(G, +)$ . If  $\mathcal{H}$  is a minimal right  $E_\Delta(G)$ -subgroup of  $E_\Delta(G)$  then  $\mathfrak{S}(\mathcal{H}) \subseteq B$ .*

*The proof is the same as that in ([9], Proposition 6.)*

**THEOREM 3.6.** *Let  $B$  a minimal nonzero  $E_\Delta$ -invariant subgroup of the group  $(G, +)$ . If  $b \in B$  ( $b \neq 0$ ), then  $\mathcal{A}(b)$  is a maximal right ideal of  $E_\Delta(G)$ .*

PROOF. If  $\Delta = G$  then  $E_\Delta(G) = M_0(G)$ . In this case the result follows from ([10], Th. 3). If  $\Delta = (0)$  then result follows by Lemma 8.5 of [8]. Let now  $\Delta \neq (0)$  and  $\Delta \neq G$ . Since  $e \notin \mathcal{A}(b)$  ( $e$  is the identity map), we have that  $\mathcal{A}(b) \neq E_\Delta(G)$ . Let us suppose that there is a right ideal  $\mathcal{P}$  of  $E_\Delta(G)$  such that  $\mathcal{A}(b)$  is a proper subset of  $\mathcal{P}$ . By Corollary of Proposition 2.2 it follows that  $(b)\mathcal{P}$  is an  $E_\Delta$ -invariant subgroup of  $(G, +)$ . Thus, either  $(b)\mathcal{P} = B$  or  $(b)\mathcal{P} = (0)$ , because  $B$  is a minimal  $E_\Delta$ -invariant subgroup. Since  $\mathcal{A}(b) \subset \mathcal{P}$  we have  $(b)\mathcal{P} = B$ . Consequently, there exists  $f \in \mathcal{P}$  such that  $(b)f = b$ . Let  $h = -f + e$ , where  $e$  is the identity map of  $G$  itself. Clearly  $h \in \mathcal{A}(b)$ . Thus,  $e = h + f \in \mathcal{P}$  and  $\mathcal{P} = E_\Delta(G)$ . Therefore,  $\mathcal{A}(b)$  is a maximal ideal of  $E_\Delta(G)$ .

**THEOREM 3.7.** *Let  $B$  be a minimal nonzero  $E_\Delta$ -invariant subgroup of the group  $(G, +)$ . Then  $\mathcal{A}(B)$  is a maximal ideal of  $E_\Delta(G)$ .*

The proof is similar to the proof of the Proposition 8.15 in [8].

**EXAMPLE 2.** Let  $E_\Delta(Z_6)$  be a near-ring of  $\Delta$ -endomorphisms of the group  $(Z_6, +)$  (table 2). The subgroups  $B_1 = \Delta = \{0, 3\}$  and  $B_2 = \{0, 2, 4\}$  of  $(Z_6, +)$

are minimal  $E_\Delta$ -invariant subgroups. The annihilator ideals

$$\mathcal{A}(B_1) = \{f_0, f_2, f_4, f_6, f_7, f_9, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$$

and

$$\mathcal{A}(B_2) = \{f_0, f_3, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{21}\}$$

are maximal ideals of  $E_\Delta(Z_6)$ .

The following theorem gives another type of a maximal right ideal of  $E_\Delta(G)$  and generalizes the Proposition 8.9 in [8].

**THEOREM 3.8.** *Let  $B$  be a maximal  $E_\Delta$ -invariant subgroup of a finite group  $(G, +)$  and let  $\Delta \subseteq B$  be a fully invariant subgroup of  $(G, +)$ . If  $x \in G \setminus B$  then  $\mathcal{B} = \{\beta \in E_\Delta(G) : (x)\beta \in B\}$  is a maximal right ideal of  $E_\Delta(G)$ .*

**PROOF.** It is easy to show that  $\mathcal{B}$  is a normal  $E_\Delta(G)$ -subgroup. Let  $\mathcal{D}_r$  be a relative defect of the subset  $\mathcal{B}$  with respect to  $E_\Delta(G)$ . Since  $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G, \Delta)_0$  we have  $\mathcal{D}_r \subseteq \mathcal{B}$ . Thus, by Lemma 1.1 it follows that  $\mathcal{B}$  is a right ideal of  $E_\Delta(G)$ . Moreover,  $\mathcal{B} \neq E_\Delta(G)$ , because  $\mathcal{B}$  contains no the identity map  $e$  of  $G$  into itself.

We will prove that  $\mathcal{B}$  is a maximal right ideal of  $E_\Delta(G)$ . Let  $\mathcal{P}$  be a right ideal of  $E_\Delta(G)$  such that  $\mathcal{B} \subset \mathcal{P}$ . Assume that  $\alpha \in \mathcal{P}$  and  $\alpha \notin \mathcal{B}$  i.e.  $(x)\alpha \notin B$ . The normal subgroup  $(x)\alpha E_\Delta(G) + B$  is  $E_\Delta$ -invariant. Namely, for all  $f \in E_\Delta(G)$  and  $t \in \text{End}_\Delta(G)$  we have

$$((x)\alpha f + b)t = (x)\alpha ft + (b)t + \delta \in (x)\alpha E_\Delta(G) + B,$$

because  $\delta \in \Delta \subseteq B$  and  $b, (b)t \in B$ . Since  $B$  is a maximal  $E_\Delta$ -invariant subgroup of  $(G, +)$ , then  $(x)\alpha E_\Delta(G) + B = G$ . Thus, there exist  $f \in E_\Delta(G)$  and  $b \in B$  such that  $(x)\alpha f + b = x$ . The map  $h: G \rightarrow G$  with  $h = -\alpha f + e$  belongs to  $E_\Delta(G)$ . Since  $(x)h = -(x)\alpha f + x = b - x + x = b \in B$  we have  $h \in \mathcal{B}$ , i.e.  $h \in \mathcal{P}$ . Also,  $\alpha f \in \mathcal{P}$ . Hence  $e = (\alpha f + h) \in \mathcal{P}$  and  $\mathcal{P} = E_\Delta(G)$ . Therefore,  $\mathcal{B}$  is a maximal right ideal of  $E_\Delta(G)$ .

**EXAMPLE 3.** Let  $E_\Delta(Z_4)$  be a near-ring of  $\Delta$ -endomorphisms of the group  $(Z_4, +)$  (table 1). The subgroup  $\Delta = \{0, 2\}$  is a maximal  $E_\Delta$ -invariant subgroup of  $(Z_4, +)$ . For  $x = 3 \notin \Delta$  the set

$$\mathcal{B} = \{f \in E_\Delta(Z_4) : (3)f \in \Delta\} = \{f_0, f_3, f_7, f_8, f_{12}, f_{13}, f_{14}, f_{15}\}$$

is a maximal right ideal of  $E_\Delta(Z_4)$ .

**THEOREM 3.9** *Let  $B \neq G$  be a sum of all minimal nonzero  $E_\Delta$ -invariant subgroups of a finite group  $(G, +)$ . If  $\Delta \subseteq B$  is a fully invariant subgroup of  $(G, +)$  then the nil radical of  $E_\Delta(G)$  is nonzero.*

**PROOF.** Let  $B_i$  ( $i \in I$ ) be a collection of all minimal nonzero  $E_\Delta$ -invariant subgroups of  $(G, +)$  and let  $\mathcal{A}(B_i)$  be annihilator ideals of the subgroups  $B_i$  ( $i \in I$ ).



We prove first that  $\cap_i \mathcal{A}(B_i)$  is nonzero. Suppose, if possible  $\cap_i \mathcal{A}(B_i) = (0)$ . By using the Proposition 2.4 it follows that  $E_\Delta(G)$  contains no nonzero right  $E_\Delta(G)$ -subgroup consisting of nilpotent elements. Thus, by Theorem 3 of [3]  $E_\Delta(G)$  is a direct sum of minimal nonzero  $E_\Delta(G)$ -subgroups. Hence, by Proposition 3.5 we obtain  $\mathfrak{S}(E_\Delta(G)) \subseteq B$ . In particular, for identity map  $e \in E_\Delta(G)$  we have  $G = (G)e \subseteq B$ , i.e.  $G = B$ . But this is contradictory to the supposition that  $G \neq B$ . Therefore  $\cap_i \mathcal{A}(B_i) \neq (0)$ . Since the nil radical is the sum of all nil ideals and  $\cap_i \mathcal{A}(B_i)$  is nonzero nil ideal, it follows that the nil radical of  $E_\Delta(G)$  is nonzero.

**PROPOSITION 3.10.** *Let  $\Delta$  be a minimal fully invariant subgroup of a finite group  $(G, +)$  and let  $\mathcal{N}$  be any nilpotent  $E_\Delta(G)$ -subgroup of  $E_\Delta(G)$ . If the normal subgroup  $\mathcal{W}$  of the group  $(E_\Delta(G), +)$ , generated by the set  $E_\Delta(G)\mathcal{N}$ , contains the relative defect of the subset  $E_\Delta(G)\mathcal{N}$  with respect to  $E_\Delta(G)$ , then  $\mathcal{W}$  is a nilpotent ideal of  $E_\Delta(G)$ .*

**PROOF.** By Proposition 1.2  $\mathcal{W}$  is an ideal of  $E_\Delta(G)$ . Since  $\mathcal{N}$  is a right  $E_\Delta(G)$ -subgroup of  $E_\Delta(G)$  and  $E_\Delta(G)$  has identity, the elements of  $\mathcal{W}$  have the form  $w = \sum_i (f_i \pm h_i n_i - f_i)$ , ( $f_i, h_i \in E_\Delta(G)$ ,  $n_i \in \mathcal{N}$ ). If  $x \in G$ ,  $x \neq 0$ , and  $n \in \mathcal{N}$ , then  $E_\Delta$ -invariant subgroup of  $(G, +)$  generated by  $(x)n$  is properly contained in the  $E_\Delta$ -invariant subgroup generated by  $x$ . Indeed, let  $X$  be  $E_\Delta$ -invariant subgroup generated by  $x$  and let  $Y$  be  $E_\Delta$ -invariant subgroup generated by  $(x)n$ . Clearly  $Y \subseteq X$ . Let us suppose that  $Y = X$ . Then there exists  $f \in E_\Delta(G)$  such that  $(x)nf = x$  and, we have a contradiction, because  $nf \in \mathcal{N}$  and  $\mathcal{N}$  is a nilpotent  $E_\Delta(G)$ -subgroup. Thus  $Y \subset X$ .

Let  $B = \sum_k B_k$  be a sum of all minimal  $E_\Delta$ -invariant subgroups of  $(G, +)$  and let  $w = \sum_i (f_i \pm h_i n_i - f_i) \in \mathcal{W}$ , ( $f_i, h_i \in E_\Delta(G)$ ,  $n_i \in \mathcal{N}$ ). Then there exists a positive integer  $p$  such that  $(x)w^p \in B$ , because every fully invariant subgroup generated by  $(x)h_i n_i$  is properly contained in the fully invariant subgroup generated by  $(x)h_i$ . Thus,

$$\begin{aligned} (x)w^{p+1} &= ((x)w^p)w = \left( \sum_k b_k \right) w \\ &= \sum_i \left[ \left( \sum_k b_k \right) f_i \pm \left( \sum_k b_k \right) h_i n_i - \left( \sum_k b_k \right) f_i \right]. \end{aligned}$$

By Proposition 2.5  $B$  is  $E_\Delta$ -invariant subgroup, i.e.

$$\left( \sum_k b_k \right) h_i n_i = \left( \sum_k b_k' \right) n_i, \quad (b_k, b_k' \in B_k).$$

Let  $n_i = \sum_j (\pm t_{ij})$ , ( $t_{ij} \in \text{End}_\Delta(G)$ ), then

$$\begin{aligned} \left( \sum_k b_k \right) h_i n_i &= \left( \sum_k b_k' \right) n_i = \left( \sum_k b_k' \right) \sum_j (\pm t_{ij}) = \\ &= \sum_j \pm \left( \sum_k b_k' \right) t_{ij} = \sum_j \pm \left( \sum_k (b_k') t_{ij} \right) + \delta, \quad (\delta \in \Delta). \end{aligned}$$

The elements of different minimal  $E_\Delta$ -invariant subgroups  $B_k$  commute element-wise. Thus

$$\left(\sum_k b_k\right) h_i n_i = \sum_k \left[ (b_k') \sum_j (\pm t_{ij}) \right] + \delta = \sum_k (b_k') n_i + \delta.$$

Therefore

$$(x)w^{p+1} = \sum_i \left[ \left(\sum_k b_k\right) f_i \pm \left(\sum_k (b_k') n_i + \delta\right) - \left(\sum_k b_k\right) f_i \right].$$

By Proposition 2.4,  $n_i \in \mathcal{A}(B_k)$  for all  $k$  and hence  $(x)w^{p+1} \in \Delta$ . Thus, there exist  $\delta', \delta'' \in \Delta$  such that

$$\begin{aligned} (x)w^{p+2} &= ((x)w^{p+1})w = (\delta')w = (\delta') \sum_i (f_i \pm h_i n_i - f_i) = \\ &= \sum_i [(\delta') f_i \pm (\delta') h_i n_i - (\delta') f_i] = \\ &= \sum_i [(\delta') f_i \pm (\delta'') n_i - (\delta') f_i] = 0. \end{aligned}$$

Thus, every element  $w \in \mathcal{W}$  is nilpotent. Because  $G$  is finite it follows that  $\mathcal{W}$  is nilpotent.

**THEOREM 3.11.** *Let  $\Delta$  be a minimal fully invariant subgroup of a finite group  $(G, +)$  and let  $\mathcal{N}$  be any nilpotent  $E_\Delta(G)$ -subgroup of  $E_\Delta(G)$ . If the normal subgroup  $\omega$  of the group  $(E_\Delta(G), +)$  generated by the set  $E_\Delta(G)\mathcal{N}$  contains the relative defect of the subset  $E_\Delta(G)\mathcal{N}$  with respect to  $E_\Delta(G)$ , then the nil radical  $\eta(E_\Delta(G))$  coincides with the radical  $J_2(E_\Delta(G))$ .*

**PROOF.** By Proposition 3.10  $\mathcal{N} \subseteq \eta(E_\Delta(G))$ , because the nil radical  $\eta(E_\Delta(G))$  is the sum of all nil ideals. Thus,  $E_\Delta(G)/\eta(E_\Delta(G))$  contains no nonzero nilpotent right  $E_\Delta(G)$ -subgroups. By using two theorems of Blackett ([3], Thms 1 and 2) it follows that every minimal right ideal of  $E_\Delta(G)/\eta(E_\Delta(G))$  contains an idempotent element. By Beidleman [1], a proper ideal  $B$  of a near-ring  $R$  is called a strong radical-ideal of  $R$  if and only if every nonzero right ideal  $R/B$  contains a minimal right ideal which contains an idempotent element. Hence,  $\eta(E_\Delta(G))$  is a strong radical-ideal of  $E_\Delta(G)$ . The following step in the proof is the same as that of ([1], Th. 8).

If the group  $(G, +)$  is equal to the sum of its minimal fully invariant subgroups, then as an immediate consequence of Proposition 3 of [1],  $J_2(E(G)) = (0)$ , where  $E(G)$  is an endomorphism near-ring. However, this is not true for near-ring  $E_\Delta(G)$  if  $(G, +)$  is equal to the sum its minimal  $E_\Delta$ -invariant subgroups, where  $\Delta$  is a proper minimal  $E_\Delta$ -invariant subgroup of  $(G, +)$ . For example, the group  $(Z_6, +)$

is a direct sum of a minimal  $E_\Delta$ -invariant subgroups  $B_1 = \Delta = \{0, 3\}$  and  $B_2 = \{0, 2, 4\}$ , but the radical

$$J_2(E_\Delta(Z_6)) = \mathcal{D} = \{f_0, f_9, F_{12}, f_{14}\} \neq (0),$$

where  $\mathcal{D}$  is the defect of the near-ring  $E_\Delta(Z_6)$  (table 2). In general, let  $(G, +)$  be a direct sum of minimal  $E_\Delta$ -invariant subgroups, where  $\Delta$  is a proper  $E_\Delta$ -invariant subgroup and let  $\mathcal{D}$  be the defect of the near-ring  $E_\Delta(G)$ . Is it  $J_2(E_\Delta(G)) = \mathcal{D}$ ? The answer is connected to the possibility that every  $\Delta$ -endomorphism  $f$  of  $(G, +)$  can be uniquely expressed in the form  $f = h + \delta$ , where  $h \in E(G)$  and  $\delta \in \mathcal{D}$ .

#### 4. Embeddings of near-ring with defect into some $E_\Delta(G)$

The problem of embedding the near-rings with the defect of distributivity is not easy. The following results refer to the particular case and generalize corresponding results for distributively generated near-ring (see [7]).

By using the technique of “right multiplier” we have.

**PROPOSITION 4.1.** *Let  $(R, S)$  be a near-ring with the defect  $D$ . If  $A(R) = (0)$ , then  $R$  embeds in  $E_D(R)$ .*

**PROPOSITION 4.2.** *Let  $R$  be a near-ring such that  $R = A(R) \oplus B$ , where  $B$  is an ideal of  $R$ . Let  $D \neq R$  be the defect of distributivity of  $R$ . Then  $D$  is the defect of the near-ring  $B$ .*

**PROOF.** Since  $B \simeq R/A(R)$  it follows that  $B$  is a near-ring with the defect  $D'$ . On the other hand  $A(R) = \{a \in R: ra = 0, \text{ for all } r \in R\}$ , i.e.  $A(R)$  is a near-ring with the defect  $D'' = (0)$ . By Proposition 1.3  $R$  is a near-ring with the defect  $D = D' \oplus D'' = D'$ .

**THEOREM 4.3.** *Let  $(R, S)$  be a near-ring with the defect  $D \neq R$  and let  $R$  be a direct sum of ideals which include  $A(R)$ , where  $A(R)$  is finite. Then there exist the group  $(G, +)$  and its normal subgroup  $\Delta$  such that  $R$  embeds in  $E_\Delta(G)$ .*

**PROOF.** Let  $R = A(R) \oplus B$ . By Proposition 3 of [7],  $A(R)$  embeds in some  $E(G_1)$ . By Lemma 2 of [7],  $A(B) = (0)$ . Since  $D$  is a defect of  $B$  (Proposition 4.2), it follows that  $B$  embeds in  $E_D(B)$  (Proposition 4.1). Thus,  $R$  embeds in  $\mathcal{R} = E(G_1) \oplus E_D(B)$ , whereby multiplication on  $\mathcal{R}$  is componentwise. Let  $\mathcal{D}$  be a defect of the near-ring  $E_D(B)$ . Then, by Proposition 1.3 it follows that  $\mathcal{R}$  is a near-ring with defect  $\mathcal{D} \neq \mathcal{R}$ , because the defect of  $E(G_1)$  is zero. The nearring  $\mathcal{R}$  contains identity  $e = (e_1, e_2)$ , where  $e_1 \in E(G_1)$  and  $e_2 \in E_D(B)$  are identity mappings, thus  $A(\mathcal{R}) = (0)$ . Hence by Proposition 4.1  $\mathcal{R}$  embeds in  $E_D(\mathcal{R})$ . Consequently, there exist the group  $(G, +)$  and its normal subgroup  $\Delta$  such that  $R$  embeds in  $E_\Delta(G)$ .

**DEFINITION.** Let  $(R, +)$  be a direct sum of the subgroups  $(A, +)$  and  $(B, +)$ . Let  $(A, +, \cdot)$  and  $(B, +, \cdot)$  be two subnear-rings of the nearring

TABLE 1.

The  $\Delta$ -endomorphisms of  $(Z_4, +)$  for  $\Delta = \{0, 2\}$ .The group  $(E_\Delta(Z_4), +)$  and the semigroup  $(E_\Delta(Z_4), \circ)$   $\Delta$ -endomorphisms

| 0123            | +       | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
|-----------------|---------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $f_0 = 0000$    | 0       | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
| $f_1 = 0123$    | 1       | 1  | 3  | 0  | 2  | 7  | 12 | 13 | 9  | 11 | 14 | 8  | 15 | 6  | 5  | 4  | 10 |
| $f_2 = 0321$    | 2       | 2  | 0  | 3  | 1  | 14 | 13 | 12 | 4  | 10 | 7  | 15 | 8  | 5  | 6  | 9  | 11 |
| $f_3 = 0202$    | 3       | 3  | 2  | 1  | 0  | 9  | 6  | 5  | 14 | 15 | 4  | 11 | 10 | 13 | 12 | 7  | 8  |
| $f_4 = 0103$    | 4       | 4  | 7  | 14 | 9  | 3  | 8  | 15 | 2  | 6  | 0  | 12 | 13 | 11 | 10 | 1  | 5  |
| $f_5 = 0121$    | 5       | 5  | 12 | 13 | 6  | 8  | 3  | 0  | 11 | 9  | 13 | 7  | 14 | 2  | 1  | 10 | 4  |
| $f_6 = 0323$    | 6       | 6  | 13 | 12 | 5  | 15 | 0  | 3  | 10 | 4  | 8  | 14 | 7  | 1  | 2  | 11 | 9  |
| $f_7 = 0222$    | 7       | 7  | 9  | 4  | 14 | 2  | 11 | 10 | 0  | 13 | 1  | 6  | 5  | 15 | 8  | 3  | 12 |
| $f_8 = 0220$    | 8       | 8  | 11 | 10 | 15 | 6  | 9  | 4  | 13 | 0  | 5  | 2  | 1  | 14 | 7  | 12 | 3  |
| $f_9 = 0301$    | 9       | 9  | 14 | 7  | 4  | 0  | 15 | 8  | 1  | 5  | 3  | 13 | 12 | 10 | 11 | 2  | 6  |
| $f_{10} = 0101$ | 10      | 10 | 8  | 15 | 11 | 12 | 7  | 14 | 6  | 2  | 13 | 3  | 0  | 9  | 4  | 5  | 1  |
| $f_{11} = 0303$ | 11      | 11 | 15 | 8  | 10 | 13 | 14 | 7  | 5  | 1  | 12 | 0  | 3  | 4  | 9  | 6  | 2  |
| $f_{12} = 0200$ | 12      | 12 | 6  | 5  | 13 | 11 | 2  | 1  | 15 | 14 | 10 | 9  | 4  | 0  | 3  | 8  | 7  |
| $f_{13} = 0002$ | 13      | 13 | 5  | 6  | 12 | 10 | 1  | 2  | 8  | 7  | 11 | 4  | 9  | 3  | 0  | 15 | 14 |
| $f_{14} = 0020$ | 14      | 14 | 4  | 9  | 7  | 1  | 10 | 11 | 3  | 12 | 2  | 5  | 6  | 8  | 15 | 0  | 13 |
| $f_{15} = 0022$ | 15      | 15 | 10 | 11 | 8  | 5  | 4  | 9  | 12 | 3  | 6  | 1  | 2  | 7  | 14 | 13 | 0  |
|                 | $\circ$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |
| 0               | 0       | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1               | 0       | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 |    |
| 2               | 0       | 2  | 1  | 3  | 9  | 5  | 6  | 7  | 15 | 4  | 10 | 11 | 13 | 12 | 14 | 8  |    |
| 3               | 0       | 3  | 3  | 0  | 0  | 3  | 3  | 3  | 3  | 0  | 0  | 0  | 0  | 0  | 3  | 3  |    |
| 4               | 0       | 4  | 9  | 3  | 4  | 10 | 11 | 3  | 12 | 9  | 10 | 11 | 12 | 13 | 0  | 13 |    |
| 5               | 0       | 5  | 6  | 3  | 10 | 5  | 6  | 7  | 7  | 11 | 10 | 11 | 3  | 0  | 14 | 14 |    |
| 6               | 0       | 6  | 5  | 3  | 11 | 5  | 6  | 7  | 14 | 10 | 10 | 11 | 0  | 3  | 14 | 7  |    |
| 7               | 0       | 7  | 7  | 0  | 0  | 7  | 7  | 7  | 7  | 0  | 0  | 0  | 0  | 0  | 7  | 7  |    |
| 8               | 0       | 8  | 8  | 0  | 0  | 8  | 8  | 8  | 8  | 0  | 0  | 0  | 0  | 0  | 8  | 8  |    |
| 9               | 0       | 9  | 4  | 3  | 9  | 10 | 11 | 3  | 13 | 4  | 10 | 11 | 13 | 12 | 0  | 12 |    |
| 10              | 0       | 10 | 11 | 3  | 10 | 10 | 11 | 3  | 3  | 11 | 10 | 11 | 3  | 0  | 0  | 0  |    |
| 11              | 0       | 11 | 10 | 3  | 11 | 10 | 11 | 3  | 0  | 10 | 10 | 11 | 0  | 3  | 0  | 3  |    |
| 12              | 0       | 12 | 12 | 0  | 0  | 12 | 12 | 12 | 12 | 0  | 0  | 0  | 0  | 0  | 12 | 12 |    |
| 13              | 0       | 13 | 13 | 0  | 0  | 13 | 13 | 13 | 13 | 0  | 0  | 0  | 0  | 0  | 13 | 13 |    |
| 14              | 0       | 14 | 14 | 0  | 0  | 14 | 14 | 14 | 14 | 0  | 0  | 0  | 0  | 0  | 14 | 14 |    |
| 15              | 0       | 15 | 15 | 0  | 0  | 15 | 15 | 15 | 15 | 0  | 0  | 0  | 0  | 0  | 15 | 15 |    |

The near-ring  $E_\Delta(Z_4)$  has the defect  $\mathcal{D} = \{f_0, f_3, f_{12}, f_{13}\}$ .

TABLE 2.  
The  $\Delta$ -endomorphisms of  $(Z_6, +)$  for which the subgroups  $B_1 = \Delta = \{0, 3\}$  and  $B_2 = \{0, 2, 4\}$  are invariant. The group  $(E_\Delta(Z_6), +)$ .

| $\Delta$ -endomorphisms | +  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|-------------------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $0 = 000000$            | 0  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| $f_1 = 012345$          | 1  | 1  | 2  | 3  | 4  | 5  | 0  | 10 | 11 | 7  | 8  | 9  | 6  | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 14 | 13 | 22 | 23 | 12 |
| $f_2 = 024024$          | 2  | 2  | 3  | 4  | 5  | 0  | 1  | 9  | 6  | 11 | 7  | 8  | 10 | 18 | 19 | 20 | 21 | 14 | 13 | 22 | 17 | 16 | 23 | 12 | 15 |
| $f_3 = 030303$          | 3  | 3  | 4  | 5  | 0  | 1  | 2  | 8  | 10 | 6  | 11 | 7  | 9  | 21 | 14 | 13 | 22 | 17 | 16 | 23 | 20 | 19 | 12 | 15 | 18 |
| $f_4 = 042042$          | 4  | 4  | 5  | 0  | 1  | 2  | 3  | 7  | 9  | 10 | 6  | 11 | 8  | 22 | 17 | 16 | 23 | 20 | 19 | 12 | 13 | 14 | 15 | 18 | 21 |
| $f_5 = 054321$          | 5  | 5  | 0  | 1  | 2  | 3  | 4  | 11 | 8  | 9  | 10 | 6  | 7  | 23 | 20 | 19 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 21 | 22 |
| $f_6 = 012045$          | 6  | 6  | 10 | 9  | 8  | 7  | 11 | 2  | 0  | 5  | 4  | 3  | 1  | 16 | 15 | 22 | 19 | 18 | 23 | 14 | 21 | 12 | 17 | 20 | 13 |
| $f_7 = 054021$          | 7  | 7  | 11 | 6  | 10 | 9  | 8  | 0  | 4  | 3  | 2  | 1  | 5  | 20 | 23 | 18 | 13 | 12 | 21 | 16 | 15 | 22 | 19 | 14 | 17 |
| $f_8 = 042342$          | 8  | 8  | 7  | 11 | 6  | 10 | 9  | 5  | 3  | 2  | 1  | 0  | 4  | 17 | 22 | 15 | 20 | 23 | 18 | 13 | 12 | 21 | 16 | 19 | 14 |
| $f_9 = 030003$          | 9  | 9  | 8  | 7  | 11 | 6  | 10 | 4  | 2  | 1  | 0  | 5  | 3  | 14 | 21 | 12 | 17 | 22 | 15 | 20 | 23 | 18 | 13 | 16 | 19 |
| $f_{10} = 024324$       | 10 | 10 | 9  | 8  | 7  | 11 | 6  | 3  | 1  | 0  | 5  | 4  | 2  | 19 | 18 | 23 | 14 | 21 | 12 | 17 | 22 | 15 | 20 | 13 | 16 |
| $f_{11} = 000300$       | 11 | 11 | 6  | 10 | 9  | 8  | 7  | 1  | 5  | 4  | 3  | 2  | 0  | 13 | 12 | 21 | 16 | 15 | 22 | 19 | 18 | 23 | 14 | 17 | 20 |
| $f_{12} = 000003$       | 12 | 12 | 15 | 18 | 21 | 22 | 23 | 16 | 20 | 17 | 14 | 19 | 13 | 0  | 11 | 9  | 1  | 6  | 8  | 2  | 10 | 7  | 3  | 4  | 5  |
| $f_{13} = 000303$       | 13 | 13 | 16 | 19 | 14 | 17 | 20 | 15 | 23 | 22 | 21 | 18 | 12 | 11 | 0  | 3  | 6  | 1  | 4  | 10 | 2  | 5  | 9  | 8  | 7  |
| $f_{14} = 030000$       | 14 | 14 | 17 | 20 | 13 | 16 | 19 | 22 | 18 | 15 | 12 | 23 | 21 | 9  | 3  | 0  | 8  | 4  | 1  | 7  | 5  | 2  | 11 | 6  | 10 |
| $f_{15} = 012342$       | 15 | 15 | 18 | 21 | 22 | 23 | 12 | 19 | 13 | 20 | 17 | 14 | 16 | 1  | 6  | 8  | 2  | 10 | 7  | 3  | 9  | 11 | 4  | 5  | 0  |
| $f_{16} = 012042$       | 16 | 16 | 19 | 14 | 17 | 20 | 13 | 18 | 12 | 23 | 22 | 21 | 15 | 6  | 1  | 4  | 10 | 2  | 5  | 9  | 3  | 0  | 8  | 7  | 11 |
| $f_{17} = 042345$       | 17 | 17 | 20 | 13 | 16 | 19 | 14 | 23 | 21 | 18 | 15 | 12 | 22 | 8  | 4  | 1  | 7  | 5  | 2  | 11 | 0  | 3  | 6  | 10 | 9  |
| $f_{18} = 024021$       | 18 | 18 | 21 | 22 | 23 | 12 | 15 | 14 | 16 | 13 | 20 | 17 | 29 | 2  | 10 | 7  | 3  | 9  | 11 | 4  | 8  | 6  | 5  | 0  | 1  |
| $f_{19} = 024321$       | 19 | 10 | 14 | 17 | 20 | 13 | 16 | 21 | 15 | 12 | 23 | 22 | 18 | 10 | 2  | 5  | 9  | 3  | 0  | 8  | 4  | 1  | 7  | 11 | 6  |
| $f_{20} = 054024$       | 20 | 29 | 13 | 16 | 19 | 14 | 17 | 12 | 22 | 21 | 18 | 15 | 23 | 7  | 5  | 2  | 11 | 0  | 3  | 6  | 1  | 4  | 10 | 9  | 8  |
| $f_{21} = 030300$       | 21 | 21 | 22 | 23 | 12 | 15 | 18 | 17 | 19 | 16 | 13 | 20 | 14 | 3  | 9  | 11 | 4  | 8  | 6  | 5  | 7  | 10 | 0  | 1  | 2  |
| $f_{22} = 042045$       | 22 | 22 | 23 | 12 | 15 | 18 | 21 | 20 | 14 | 19 | 16 | 13 | 17 | 4  | 8  | 6  | 5  | 7  | 10 | 0  | 11 | 9  | 1  | 2  | 3  |
| $f_{23} = 054324$       | 23 | 23 | 12 | 15 | 18 | 21 | 22 | 13 | 17 | 14 | 19 | 16 | 20 | 5  | 7  | 10 | 0  | 11 | 9  | 1  | 6  | 8  | 2  | 3  | 4  |

The semigroup  $(E_{\Delta}(Z_0), \circ)$ . The near-ring  $E_{\Delta}(Z_0)$  has the defect

$$\mathcal{D} = \{f_0, f_9, f_{12}, f_{14}\}.$$

| $\circ$ | 0 | 1  | 2 | 3  | 4 | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---------|---|----|---|----|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0       | 0 | 0  | 0 | 0  | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |
| 1       | 0 | 1  | 2 | 3  | 4 | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 2       | 0 | 2  | 4 | 0  | 2 | 4  | 2  | 4  | 2  | 0  | 4  | 0  | 0  | 0  | 0  | 2  | 2  | 2  | 4  | 4  | 4  | 0  | 2  | 4  |
| 3       | 0 | 3  | 0 | 3  | 0 | 3  | 0  | 0  | 3  | 0  | 3  | 3  | 0  | 3  | 0  | 3  | 0  | 3  | 0  | 3  | 0  | 3  | 0  | 3  |
| 4       | 0 | 4  | 2 | 0  | 4 | 2  | 4  | 2  | 4  | 0  | 2  | 0  | 0  | 0  | 4  | 4  | 4  | 4  | 2  | 2  | 2  | 0  | 4  | 2  |
| 5       | 0 | 5  | 4 | 3  | 2 | 1  | 7  | 6  | 10 | 9  | 8  | 11 | 14 | 21 | 12 | 19 | 18 | 23 | 16 | 15 | 22 | 13 | 20 | 17 |
| 6       | 0 | 6  | 2 | 9  | 4 | 7  | 6  | 7  | 4  | 9  | 2  | 0  | 0  | 12 | 14 | 16 | 16 | 22 | 18 | 18 | 20 | 14 | 22 | 20 |
| 7       | 0 | 7  | 4 | 9  | 2 | 6  | 7  | 6  | 2  | 9  | 4  | 0  | 14 | 14 | 12 | 18 | 18 | 20 | 16 | 16 | 22 | 12 | 20 | 22 |
| 8       | 0 | 8  | 2 | 11 | 4 | 10 | 4  | 2  | 8  | 0  | 10 | 11 | 0  | 11 | 0  | 8  | 4  | 8  | 2  | 10 | 2  | 11 | 4  | 10 |
| 9       | 0 | 9  | 0 | 9  | 0 | 9  | 0  | 0  | 9  | 0  | 9  | 9  | 0  | 9  | 0  | 9  | 0  | 9  | 0  | 9  | 0  | 9  | 0  | 9  |
| 10      | 0 | 10 | 4 | 11 | 2 | 8  | 2  | 4  | 10 | 0  | 8  | 11 | 0  | 11 | 0  | 10 | 2  | 10 | 4  | 8  | 4  | 11 | 2  | 8  |
| 11      | 0 | 11 | 0 | 11 | 0 | 11 | 0  | 0  | 11 | 0  | 11 | 11 | 0  | 11 | 0  | 11 | 0  | 11 | 0  | 11 | 0  | 11 | 0  | 11 |
| 12      | 0 | 12 | 0 | 0  | 0 | 12 | 0  | 12 | 12 | 0  | 12 | 12 | 0  | 12 | 0  | 12 | 0  | 12 | 0  | 12 | 0  | 12 | 0  | 12 |
| 13      | 0 | 13 | 0 | 13 | 0 | 13 | 0  | 0  | 13 | 0  | 13 | 13 | 0  | 13 | 0  | 13 | 0  | 13 | 0  | 13 | 0  | 13 | 0  | 13 |
| 14      | 0 | 14 | 0 | 14 | 0 | 14 | 0  | 0  | 14 | 0  | 14 | 14 | 0  | 14 | 0  | 14 | 0  | 14 | 0  | 14 | 0  | 14 | 0  | 14 |
| 15      | 0 | 15 | 2 | 3  | 4 | 23 | 16 | 20 | 8  | 14 | 10 | 11 | 0  | 11 | 14 | 15 | 16 | 18 | 2  | 10 | 20 | 21 | 4  | 23 |
| 16      | 0 | 16 | 2 | 14 | 4 | 20 | 16 | 20 | 4  | 14 | 2  | 0  | 0  | 0  | 14 | 16 | 16 | 4  | 2  | 2  | 20 | 14 | 4  | 20 |
| 17      | 0 | 17 | 2 | 13 | 4 | 19 | 22 | 18 | 8  | 12 | 10 | 11 | 12 | 13 | 0  | 8  | 4  | 17 | 18 | 19 | 2  | 11 | 22 | 10 |
| 18      | 0 | 18 | 4 | 12 | 2 | 22 | 18 | 22 | 2  | 12 | 4  | 0  | 0  | 0  | 12 | 18 | 18 | 4  | 4  | 22 | 12 | 2  | 22 | 10 |
| 19      | 0 | 19 | 4 | 13 | 2 | 17 | 18 | 22 | 10 | 12 | 8  | 11 | 0  | 11 | 12 | 19 | 18 | 10 | 4  | 8  | 22 | 13 | 2  | 17 |
| 20      | 0 | 20 | 4 | 14 | 2 | 16 | 20 | 16 | 2  | 14 | 4  | 0  | 14 | 14 | 0  | 2  | 2  | 20 | 16 | 4  | 0  | 20 | 4  | 17 |
| 21      | 0 | 21 | 0 | 21 | 0 | 21 | 0  | 0  | 21 | 0  | 21 | 21 | 0  | 21 | 0  | 21 | 0  | 21 | 0  | 21 | 0  | 21 | 0  | 21 |
| 22      | 0 | 22 | 2 | 12 | 4 | 18 | 22 | 18 | 4  | 12 | 2  | 0  | 12 | 12 | 0  | 4  | 4  | 22 | 18 | 18 | 2  | 0  | 22 | 2  |
| 23      | 0 | 23 | 4 | 21 | 2 | 15 | 20 | 16 | 10 | 14 | 8  | 11 | 14 | 21 | 0  | 10 | 2  | 23 | 16 | 15 | 4  | 11 | 20 | 8  |

$(R, +, \cdot)$  with the defect  $D$ . A multiplication on  $R$  is  $D$ -componentwise if for all  $a, a' \in A$  and  $b, b' \in B$  there exists  $d \in D$  such that  $(a + b)(a' + b') = aa' + bb + d$ . We say that  $R$  is a  $D$ -direct sum of three subnear-rings  $A$  and  $B$ .

Let  $E_\Delta(G)$  be a  $\Delta$ -endomorphism near-ring with the defect  $\mathcal{D}$ . For some idempotent  $e \in E_\Delta(G)$  let  $\mathcal{A}$  be the subgroup of  $(E_\Delta(G), +)$  generated by  $\{s - es : s \in \text{End}_\Delta(G)\}$  and  $\mathcal{M}$  be the subgroup of  $(E_\Delta(G), +)$  generated by  $\{es : s \in \text{End}_\Delta(G)\}$ .

**THEOREM 4.4.** *Let  $G = B \oplus C$  be a direct sum of  $E_\Delta$ -invariant subgroups  $B$  and  $C$ , where  $B$  is summand and  $\Delta$  is a subset of one of the summands. If  $e$  is the projection map  $e: G \rightarrow B$  and  $\mathcal{AM} \subseteq \mathcal{D}$ , then  $E_\Delta(G)$  is the  $\mathcal{D}$ -direct sum of the subnear-rings  $\mathcal{A}$  and  $\mathcal{M}$ , where  $\mathcal{D}$  is the defect of  $E_\Delta(G)$ .*

**PROOF.** The projection map  $e: G \rightarrow B$  is an endomorphism of  $(G, +)$ . The idempotent  $e \in \text{End}(G)$  is a right identity for  $\mathcal{M}$ . Hence,  $\mathcal{M}$  is a subnear-ring of  $E_\Delta(G)$ . Also, by Corollary 2.3 of [11] it follows that  $\mathcal{A}$  is an ideal of  $E_\Delta(G)$ . Because  $B$  and  $C$  commute elementwise and  $B$  is  $E_\Delta$ -invariant abelian summand, it follows that the decomposition  $E_\Delta(G) = \mathcal{A} + \mathcal{M}$  has  $\mathcal{M}$  in the additive center of  $E_\Delta(G)$ , i.e. semidirect sum  $\mathcal{A} + \mathcal{M}$  is direct.

We shall now prove that the multiplication on  $E_\Delta(G)$  is  $\mathcal{D}$ -componentwise. Let  $a, a' \in \mathcal{A}$  and  $m, m' \in \mathcal{M}$ , where  $a' = s' - es'$ ,  $m = et$ ,  $m' = et'(s', t, t' \in \text{End}_\Delta(G))$ . Then

$$\begin{aligned} (a + m)(a' + m') &= (a + m)(s' - es') + (a + m)et' \\ &= (a + m)s' - (a + m)es' + (a + m)et' = \\ &= as' + ms' + \delta_1 - (aes' + mes' + \delta_2) + aet' + met' + \delta_3 = \\ &= as' - aes' + aet' + ms' - mes' + met' + \delta = \\ &= aa' + am' + ma' + mm' + \delta \\ &= aa' + mm' + \delta', \quad (\delta_1, \delta_2, \delta_3, \delta, \delta' \in \mathcal{D}) \end{aligned}$$

because  $ma' = et(s' - es') = ets' - etes' = 0$  and  $\mathcal{AM} \subseteq \mathcal{D}$ .

For example, if for an idempotent of the near-ring  $E_\Delta(Z_6)$  with the defect  $\mathcal{D} = \{f_0, f_9, f_{12}, f_{14}\}$  (table 2) we take the map  $e = f_3: G \rightarrow B = \{0, 3\}$  then,  $E_\Delta(Z_6)$  is a  $\mathcal{D}$ -direct sum of the subnear-rings

$$\mathcal{A} = \{f_0, f_2, f_4, f_6, f_7, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$$

and  $\mathcal{M} = \{f_0, f_3\}$

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