

## M-CONVEXITY AND BEST APPROXIMATION

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**Abstract.** The notion of M-convexity is introduced in Metric Spaces. The relations between M-convex, strictly M-convex and uniformly M-convex metric spaces are studied. The Best approximation properties for M-convex subsets of metric spaces are considered and many new results derived.

**Introduction.** The problem of best approximation has been extensively studied in normed linear spaces. The study of similar problems in metric linear spaces was initiated by G. Albinus [1, 2, 3] and I. Singer [12]. In metric linear spaces the proximality of finite dimensional linear subspaces has been studied by K. Iseki [7] and in Frechet spaces, similar problems have been considered by V.N. Nikolski [10]. The consideration of best approximation problems in normed linear spaces was made by B.A. Hirschfeld [6] and A.M. Flomin [5]. In metric spaces many of the results of metric linear spaces were extended by I. Singer [12] and other. In this paper we have studied some such problems in a new kind of metric space which admits extensions of many results true in metric linear spaces and normed linear spaces.

In Section 1, we have defined M-convexity for metric spaces. The idea is essentially due to K. Menger who has survived in the prefix of the concept. Strict M-convexity and uniform M-convexity have been defined then and the relations among these spaces have been studied. In Section 2 M-convexity is defined for subsets and Chebyshev and proximal properties are studied there.

**DEFINITION 1.1.** A metric space  $(X, d)$  is said to be M-convex if for every  $x, y$  in  $X$ ,  $x \neq y$ , there exists a  $z$  in  $X$  different from  $x$  and  $y$  such that

$$d(x, y) = d(x, z) + d(z, y).$$

One can immediately see that

**PROPOSITION 1.2.** Every normed linear space is an M-convex metric space.

We now give two examples to illustrate the fact that not every metric space is M-convex and also the fact not every M-convex metric space is a normed linear space.

EXAMPLE 1.3. Let  $K$  be a non-convex closed subset of  $R^n$  equipped with the relative topology. Then it is easy to see that  $K$  is a metric space which is not M-convex.

EXAMPLE 1.4. Let  $U$  denote the unit ball of  $R^2$  i.e.

$$U = \{(x, y) \in R^2; x^2 + y^2 \leq 1\}$$

Then, if  $U$  is equipped with the usual Euclidean metric, then  $U$  becomes an M-convex metric space which is not a normed linear space.

DEFINITION 1.5. A metric space  $(X, d)$  is said to be strictly M-convex if for every  $x, y, t$  in  $X$ , all different and  $r > 0$ , there exists a  $z$  in  $X$  different from  $x, y$  and  $t$  such that

- (1)  $d(x, y) = d(x, z) + d(z, y)$
- (2)  $d(x, t) \leq r, d(y, t) \leq r$  imply  $d(z, t) < r$

An example of an M-convex metric space which is not strictly M-convex is the following

EXAMPLE 1.6. Consider the metric space  $(X, d)$  where  $d$  is defined as

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where  $x = (x_1, x_2), y = (y_1, y_2)$

$$\text{and } X = \{x, y \in R^2; x > 0, y > 0\} \cup (0, 0).$$

Then it is easy to see that  $(X, d)$  is M-convex. To see that it is not strictly M-convex, consider

$$\begin{array}{ll} x = (1/3, 1/3) & y = (2/3, 2/3) \\ t = (0, 0) & r = 2/3. \end{array}$$

and check that does not exist any  $z$  in  $X$  satisfying both the requirements of the definition.

DEFINITION 1.7. A metric space  $(X, d)$  is said to be uniformly M-convex if for every pair of positive numbers  $\varepsilon$  and  $r$ , there corresponds a positive number  $\delta$  such that for every triplet  $x, y, t$  in  $X$  all different and satisfying  $d(x, y) \geq \varepsilon, d(x, t) < r + \delta, d(y, t) < r + \delta$  there exists a  $z$  in  $X$  with the properties

- (1)  $d(x, y) = d(x, z) + d(z, y)$
- (2)  $d(z, t) < r.$

As an example of a uniformly M-convex metric space we site the following example.

EXAMPLE 1.8. Consider  $(M, d)$  with metric  $d$  defined as  $d(x, y) = |x - y|$ .

To see that it is uniformly M-convex, it is enough to check  $d(x, y) \geq \varepsilon$ ,  $d(x, 0) < r'$ ,  $d(y, 0) < r'$ , there exists a  $z$  in  $X$  defferent from  $x, y, t$  such that

- (1)  $d(x, y) = d(x, z) + d(z, y)$
- (2)  $d(z, 0) < r$ .

This can be verified easily.

One can readily see now.

PROPOSITION 1.9. Every uniformly M-convex metric space is strictly M-convex and not conversely.

DEFINITION 1.10. A metric space  $(X, d)$  is said to be *totally complete* if every bounded closed subset of  $X$  is compact.

We can now prove the following theorem.

THEOREM 1.11. *Every totally complete strictly M-convex metrix space is uniformly M-convex.*

PROOF. Let  $(X, d)$  be a totally complete M-convex metric space. Equip  $X \times X$  with a metric  $\rho$  defined as

$$\rho((x_1, y_1), (x_2, y_2)) = \{d_2(x_1, x_2) + d^2(y_1, y_2)\}^{\frac{1}{2}}$$

clearly,  $(X \times X, \rho)$  is totally complete.

Then  $S_t = \{(x, y) \in X \times X; d(x, t) \leq r\}$  is a closed and bounded subset of  $X \times X$  and hence compact for every  $t \in X$ .

Define  $\Phi_t: S_t \rightarrow R$  as

$$\Phi_t(x, y) = r - d(z, t) \text{ where } d(x, z) + d(z, y) = (x, y).$$

Then  $\Phi_t$  is continuous and positive on  $S_t$  and therefore there exists  $\delta > 0$  such that

$$\begin{aligned} r - d(z, t) &\geq \delta \text{ for all } t \text{ in } X \\ \text{i.e. } d(z, t) &\leq r - \delta < r. \end{aligned}$$

Hence the result.

2. We now define M-convex subsets of a metrix space.

DEFINITION 2.1. A subset  $G$  of a metric space  $(X, d)$  is said to be M-convex if for every  $x, y \in G$ ,  $x \neq y$ , there exists a  $z$  in  $G$  such that  $d(x, z) + d(z, y) = d(x, y)$ .

We remark here that there are metric spaces no subset of which is M-convex. As an example of such a metric space we refer to the following.

EXAMPLE 2.2. Consider  $X = R$  with the metric  $d$  defined as

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

It is worthy to note at this stage that there is no relation as such between convexity and M-convexity in metric linear spaces. The example 2.2 provides an illustration to the fact that in metric linear spaces there are convex sets which are not M-convex while example 2.3 below proves the other direction.

EXAMPLE 2.3. Consider the metric linear space  $(X, d)$  where  $X = R^2$  and  $d$  is defined as

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\} \text{ where } \begin{aligned} x &= (x_1, y_2) \\ y &= (y_1, y_2). \end{aligned}$$

Consider the set

$$G = \{(z_1, z_2); 0 \leq z_1 \leq 2, z_2 = 0\} \cup \{(z_1, z_2); z_1 = 2, 0 \leq z_2 \leq 1\}$$

Then it is easy to check that  $G$  is M-convex but not convex.

We however remark that in normed linear spaces this is not true.

DEFINITION 2.4. A subset  $G$  of a metric space  $(X, d)$  is said to be *promiminal* if for every  $x$  in  $X$  there exists at least one  $\xi$  in  $G$ , called the best approximating element of  $x$  in  $G$  such that

$$d(x, \xi) = d(x, G) \equiv \inf_{z \in G} d(x, z)$$

The set  $G$  is said to be *Chebyshev* if to every  $x \in X$  there exists exactly one  $\xi \in G$  such that

$$d(x, \xi) = d(x, G).$$

We shall denote by  $\Pi_{G(x)}$  the set of best approximating elements of  $x$  in  $G$ .

Thus we can define the setvalued map  $\Pi_G$  from  $X$  into subsets of  $G$  as

$$\Pi_G(x) = \{z \in G; d(x, z) = d(x, G)\}$$

We can define another real-valued function  $e_G$  on  $X$  as  $e_G(x) = d(x, G)$ .

Clearly  $e_G$  is uniformly continuous. Regarding continuity of  $\Pi_{G(x)}$ , it is known that  $\Pi_G$  is continuous at every point of  $G$  if  $G$  is Chebyshev.

We can prove the following theorems now.

**THEOREM 2.5.** *If  $(X, d)$  is an M-convex metric space and  $G$  is a Chebyshev subset of  $X$  then if  $z$  corresponds the M-convex element of  $x$  and  $\Pi_G(x)$*

$$\Pi_G(z) = \Pi_G(x).$$

**PROOF.** By the definition of  $z$ , we have

$$d(x, z) + d(z, \Pi_G(x)) = d(x, \Pi_G(x)).$$

Now, if  $\xi \in G$ , then

$$\begin{aligned} d(z, \xi) &\geq d(x, \xi) - d(x, z) \\ &= d(x, \Pi_G(x)) - d(x, z) \\ &= d(z, \Pi_G(x)) \end{aligned}$$

This implies that  $\Pi_G(x)$  is a best approximating element of  $z$ . Since  $G$  is Chebyshev,  $\Pi_G(x) = \Pi_G(z)$ .

**THEOREM 2.6.** *If  $(X, d)$  is a metric space,  $G$  is a subset of  $X$  and  $y_0 \in G$ , then  $\Pi_G^{-1}(y_0)$  is closed.*

*Further, if  $x_0 \in \Pi_G^1(y_0)$  and  $d(x, z) + d(z, y_0) = d(x_0, y_0)$  for some  $z \in X$ , then  $z \in \Pi_G^{-1}(y_0)$ .*

**PROOF.** By definition

$$\begin{aligned} \Pi_G^{-1}(y_0) &= \{x \in X; \quad d(x, y_0) = d(x, G)\} \\ &= \bigcap_{y \in G} \{x \in X; \quad d(x, y_0) \leq d(x, y)\} \end{aligned}$$

By the continuity of the metric  $d$ , the first part of the result is then obvious.

Now since  $x_0 \in \Pi_G^{-1}(y_0)$ , we have

$$d(x_0, y_0) \leq d(x_0, y) \quad \text{for all } y \in G.$$

Since  $z$  satisfies  $d(x_0, z) + d(z, y_0) = d(x_0, y_0)$ , we write

$$\begin{aligned} d(x, y_0) &= d(x_0, y_0) - d(x_0, z) \\ &\leq d(x_0, y_0) \leq d(x_0, y) \quad \text{for all } y \in G \end{aligned}$$

$\therefore z \in \Pi_G^{-1}(y_0)$ .

**COROLLARY 2.7.** *If  $G$  is Chebyshev in  $X$ , then  $\Pi_G^{-1}(\Pi_G(x))$  is closed for every  $x$  in  $X$ .*

In general, proximal sets or Chebyshev sets are neither convex nor M-convex. L.N.H. Bunt [4] and T.S. Motzkin have given conditions under which every Chebyshev set is convex. These conditions are however sufficient but not necessary. One such result is the following.

**THEOREM.** *In a finite dimensional smooth Banach Space, every Chebyshev set is convex and hence M-convex. But the problem whether the result is true for infinite dimensional Banach Spaces remains still open. Another interesting open problem is the following.*

Whether in a Hilbert Space, every Chebyshev set is convex? Under the present context, we can ask whether in a Hilbert space, every Chebyshev set is M-convex.

Before we prove our next theorem we need the following definition.

**DEFINITION 2.8.** In a metric space  $(X, d)$ , a Menger set denoted as  $M_{\langle x, y \rangle}$  for a pair of distinct points  $x, y$  is defined as the set of elements  $z$  in  $X$  such that

$$d(x, z) + d(z, y) = d(x, y)$$

i.e,  $M_{\langle x, y \rangle} = \{z \in X; \quad d(x, z) + d(z, y) = d(x, y)\}.$

One can immediately see that Menger sets can be emptysets, singleton sets or arbitrarily large sets. For example, we recall that  $(R, d)$  with  $d(x, y) = \frac{|x-y|}{1+|x-y|}$  has empty Menger sets for every pair of points of  $x, y$  while  $(R, d')$  with  $d'(x, y) = |x-y|$  is such that the Menger set for every pair of distinct points is uncountable.

One can immediattely note

**PROPOSITION 2.9.** Every Menger set is closed.

**DEFINITION 2.10.** If a metric space has only singleton Menger sets for every pair of distinct elements, then it will be called Mengerian. We can prove the following theorem now.

**THEOREM 2.11.** *Every M-convex proximal set in a strictly M-convex Mengerian metric space is Chebyshev.*

**PROOF.** Suppose  $G$  is an M-convex promiminal set in the strictly M-convex Mengerian metric space  $(X, d)$ .

If possible, for some  $x_0 \in X$ , let  $y_1, y_2 \in G$  be two best approximating elements i.e.  $\Pi_G(x_0) = \{y_1, y_2\}$ .

Then  $d(x_0, y_1) = d(x_0, y_2) = \inf_{\xi \in G} d(x_0, \xi) = r$  say.

Since  $X$  is strictly M-convex, there exists  $z \in X, x \neq z \neq y$  such that  $d(x, z) < r$

$$\text{and } d(y_1, z) + d(z, y_2) = d(y_1, y_2).$$

But since  $G$  is M-convex and  $X$  is Mengerian,  $z \in G$ .

This contradicts the definition of  $r$ . Hence the proof.

**DEFINITION 2.12.** A set  $G$  in a metric space  $(X, d)$  is said to be *approximately compact* if for every sequence  $y_n$  in  $G$  with  $\lim_n d(x, y_n) = d(x, G)$ , there exists a subsequence  $y_{n_k}$  converging to an element of  $G$ .

**THEOREM 2.13.** *In an uniformly M-convex Mengerian metric space every complete M-convex set is approximately compact.*

**PROOF.** Let  $G$  be an M-convex complete set in uniformly M-convex Mengerian metric space  $(X, d)$ .

Let  $y_n$  be a sequence in  $G$  satisfying

$$\lim d(x, y_n) = d(x, G) = r \quad (\text{say})$$

Let  $\varepsilon > 0$  be arbitrary.

Since  $X$  is uniformly M-convex, we can find a  $\delta > 0$  satisfying some inequality relations.

Since  $\lim_n d(x, y_n) = r$ , we can choose a positive integer  $N$  such that

$$d(x, y_n) > r + \delta \quad \text{whenever } n \geq N.$$

Let  $n, m \geq N$ . Then by the inequality relations, we get

$$\begin{aligned} d(x, y_n) &< r + \delta \\ d(x, y_m) &< r + \delta. \end{aligned}$$

If possible let  $d(y_n, y_m) \geq \varepsilon$ .

Then these imply that there exists a  $y \in X$  such that

$$d(y_n, y_0) + d(y_0, y_m) = d(y_n, y_m)$$

and

$$d(x, y_0) < r.$$

Since  $X$  is M-convex and Mengerian,  $y \in G$  and thus we arrive at a contradiction that  $r = d(x, G)$ .

Therefore  $d(y_n, y_m) < \varepsilon$  for  $m, n \geq N$ .

i.e.  $y_n$  is a Cauchy sequence.

By the completeness of  $G$ , the result follows then.

**DEFINITION 2.14.** A metric space  $(X, d)$  is said to have *P-property* if for a fixed  $p$  in  $X$ , every sequence  $y_n$  in a M-convex set  $G$  of  $X$  satisfying  $\lim_n d(p, y_n) = d(p, G)$  has a Cauchy subsequence.

The following theorem can now be proved.

**THEOREM 2.15.** *A complete  $M$ -convex subset  $G$  of a metric space  $(X, d)$  having  $P$ -property is Chebyshev.*

**PROOF.** Let  $p \in X$  and  $r = d(x, G)$ .

So there exists a sequence  $y_n$  in  $G$  such that

$$\lim_n d(p, y_n) = r$$

By  $P$ -property,  $y_n$  has a Cauchy subsequence  $y_{n_k}$  in  $G$ . Since  $G$  is complete,  $y_n$  converges to some  $y \in G$ .

By continuity of the metric, we then get

$$d(p, y) = r.$$

If possible now let  $y_1, y_2 \in G$  be such that

$$d(p, y_1) = d(p, y_2) = r.$$

Define a sequence  $z_n$  as follows

$$\begin{aligned} z_n &= y_1 \text{ if } n \text{ is even} \\ &= y_2 \text{ if } n \text{ is odd.} \end{aligned}$$

Then  $\lim_n d(p, z) = d(p, z_1) = r = d(p, z_2)$ .

By  $P$ -property  $z_n$  has a Cauchy subsequence  $z_{n_k}$  i.e., for given  $\xi > 0$ , there exists a positive integer  $N$  such that  $n_k, m_k \geq N$  implies  $d(z_{n_k}, z_{m_k}) < \xi$ .

Since  $\varepsilon$  is arbitrary,  $y_1 = y_2$ .

This proves that  $G$  is Chebyshev.

**THEOREM 2.16.** *Every uniformly  $M$ -convex Mengerian metric space has  $P$ -property.*

**PROOF.** Embodied in Theorem 2.14.

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