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A proof of the Russo–Dye theorem for JB^*-algebras

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Abstract. We give a new and clever proof of the Russo–Dye theorem for JB^*-algebras, which depends on certain recent tools due to the present author. The proof given here is quite different from the known proof by J. D. M. Wright and M. A. Youngson. The approach adapted here is motivated by the corresponding C^*-algebra results due to L. T. Gardner, R. V. Kadison and G. K. Pedersen. Accordingly, it yields more precise information. Incidentally, we obtain an alternate proof of Russo–Dye Theorem for C^*-algebras. A couple of further results due to Kadison and Pedersen have been extended to JB^*-algebras as corollaries to the main results.

Contents

1. Introduction

In [9], R. V. Kadison obtained a characterization of the extreme points of the closed unit ball of a C^*-algebra A as the elements x such that

$$(e - xx^*)A(e - x^*x) = \{0\},$$

where e stands for the identity element of A. From this, it is seen that every unitary operator in a C^*-algebra is an extreme point of the unit ball. Subsequently, B. Russo and H. A. Dye proved in [19] that the closed unit ball of any C^*-algebra is the closed convex hull of its unitaries. This Russo–Dye theorem has been very useful in providing means of reducing the study of nonnormal operators to that of unitary (normal) operators and has been extensively used in unitary approximations (see [10, 15], for instance). Several

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simplifications and generalizations, e.g., [17, 12, 5], revealed that the underlying structure making the results hold was not the presence of an associative product but the presence of a Jordan product (or Jordan triple product). This provided one of the stimuli for the development of various Jordan algebra or Jordan triple product generalizations of C^*-algebras, these include JC-algebras [25], JB-algebras [1], JB^*-algebras [27] and JB^*-triples [5, 26] together with their subclasses which are Banach dual spaces. Similar results on linear isometries and extreme points of the unit ball have been proved, e.g., in [6, 7, 11, 28, 29]. In [28], J. D. M. Wright and M. A. Youngson presented a proof of Russo–Dye Theorem for Jordan C^*-algebras (original name of JB^*-algebras), which is a modification of the proof of Russo–Dye Theorem for J^*-algebras given by L. A. Harris (see [5]).

Russo and Dye [19] raised the point that little is known about the nonclosed convex hull of unitaries. Attention has been focused to this aspect with the appearance of L. T. Gardner's paper [4]; some such details are given in [10]. In [4], Gardner obtained an elementary proof of the Russo–Dye theorem by strengthening the fact that the open ball of radius one half in a C^*-algebra is contained in the nonclosed convex hull of unitaries. On this basis, R. V. Kadison and G. K. Pedersen [10, Theorem 2.1], proved that each element of the open unit ball in a C^*-algebra is a mean of unitary elements and as its immediate corollary they obtained Russo–Dye Theorem for C^*-algebras.

In this and four subsequent papers, we investigate the Russo–Dye Theorem and related geometric properties of general JB^*-algebras. In the sequel, our first main objective is to develop a new proof of the Russo–Dye Theorem for JB^*-algebras along the lines of Kadison and Pedersen [10]. Unfortunately, the proof of [10, Theorem 2.1] as given by the authors (see [10, page 251]) no longer works for general JB^*-algebras simply because the Jordan product generally is not associative and so the associative product of two unitary elements is not necessarily in a JB^*-algebra.

We present a new and clever proof of the Russo–Dye theorem for JB*-algebras. The proof depends on two very nice tools due to the present author from a recent publication [21]. The proof given here is quite different from the above mentioned known proof by Wright and Youngson. The approach adapted here is motivated essentially by the corresponding C^*-algebra results due to L. T. Gardner, R. V. Kadison and G. K. Pedersen (see [4, 10]). Accordingly, it yields more precise information. Incidentally, we obtain an alternate proof of Russo–Dye Theorem for C^*-algebras. A couple of further results due to Kadison and Pedersen have been extended to JB^*-algebras as corollaries to the main results.

1.1. Basics. We begin by recalling (from [8], for instance) the concept of homotopes of Jordan algebras. Let $\mathcal{J}$ be a Jordan algebra and $x \in \mathcal{J}$. The *x-homotope* of $\mathcal{J}$, denoted by $\mathcal{J}_{[x]}$, is the Jordan algebra consisting of the same elements and linear algebra structure as $\mathcal{J}$ but a different product, denoted by " $\cdot_x$", defined by $a \cdot_x b = \{axb\}$ for all a, b in $\mathcal{J}_{[x]}$. Here, $\{pqr\}$ denotes the *Jordan triple product* $\{pqr\} = (p \circ q) \circ r - (p \circ r) \circ q + (q \circ r) \circ p$ where "$\circ$" stands for the original Jordan product.

The homotopes of our interest are obtained if $\mathcal{J}$ has a unit e and x is *invertible*: this means that there exists $x^{-1} \in \mathcal{J}$, called the *inverse* of x, such that $x \circ x^{-1} = e$ and $x^2 \circ x^{-1} = x$. The set of all invertible elements of $\mathcal{J}$ will be denoted by $\mathcal{J}_{\text{inv}}$. Any invertible element x of (unital) Jordan algebra $\mathcal{J}$ acts as the unit for the homotope $\mathcal{J}_{[x^{-1}]}$ (see [14]).

If $\mathcal{J}$ is a unital Jordan algebra and $x \in \mathcal{J}_{\text{inv}}$ then *x-isotope* of $\mathcal{J}$, denoted by $\mathcal{J}^{[x]}$, is defined to be the x^{-1}-homotope $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$. Of course, x-isotope is defined only for invertible element x of the algebra $\mathcal{J}$. Our notation is motivated by the symmetry that x acts as the unit for the *x^{-1}-homotope* $\mathcal{J}_{[x^{-1}]}$ of $\mathcal{J}$, which is consistent with McCrimmon's concept of *isotopes* [14] and corresponds to the Jacobson's concept x^{-1}-isotope (see [8, p.57]).

Any two isotopes of an associative algebra are *isomorphic* to each other (see [8, p.56]. Thus in the associative case, *isotopy* basically just changes the unit element and does not produce new structures. However, it may cause convenience in doing calculations; such an example is given in [13, p. 617]. However, the x-isotope $\mathcal{J}^{[x]}$ of a Jordan algebra $\mathcal{J}$ need not be *isomorphic* to $\mathcal{J}$; for such details and examples see [13, 12]. Fortunately, some features of our interest in Jordan algebras are unaffected on passage to an isotope. Such a feature is stated in the following result:

Lemma 1.1 ([21, Lemma 4.2])**.** *For any invertible element a in a unital Jordan algebra $\mathcal{J}$,*

$$\mathcal{J}_{\text{inv}} = \mathcal{J}^{[a]}_{\text{inv}}.$$

A Jordan algebra $\mathcal{J}$ with product $\circ$ is called a *Banach Jordan algebra* if there is a norm $\|.\|$ on $\mathcal{J}$ such that $(\mathcal{J}, \|.\|)$ is a Banach space and $\|a \circ b\| \leq \|a\|\|b\|$. If, in addition, $\mathcal{J}$ has unit e with $\|e\| = 1$ then $\mathcal{J}$ is called a *unital* Banach Jordan algebra. For the basic theory of Banach Jordan algebras, we refer to the sources [1, 3, 20, 26, 27, 28, 30].

We are interested in a special class of unital Banach Jordan algebras, called *JB^*-algebras.* These include all C^*-algebras as a proper subclass: A complex Banach Jordan algebra $\mathcal{J}$ with *involution* $*$ is called a *JB^*-algebra* if $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$ (cf. [27]). It is easily seen that $\|x^*\| = \|x\|$ for all elements x of a JB^*-algebra [30].

Let $\mathcal{J}$ be a JB^*-algebra. An element u of a JB^*-algebra $\mathcal{J}$ is called *unitary* if $u^* = u^{-1}$, the inverse of u. The set of all unitary elements of $\mathcal{J}$ will be denoted by $\mathcal{U}(\mathcal{J})$.

Given any unitary element u of a JB^*-algebra $\mathcal{J}$, the isotope $\mathcal{J}^{[u]}$ is called a *unitary isotope* of $\mathcal{J}$. The following lemma is a well known result, originally due to Braun, Kaup and Upmeier [2, 12]:

Lemma 1.2. *Any unitary isotope $\mathcal{J}^{[u]}$ of a JB^*-algebra $\mathcal{J}$ is itself a JB^*-algebra having u as its unit with respect to the original norm and the involution "$*_u$" given as below:*

$$x^{*_u} = \{ux^*u\}.$$

Unfortunately, for nonunitary $x \in \mathcal{J}_{\text{inv}}$, the isotope $\mathcal{J}^{[x]}$ of the JB^*-algebra $\mathcal{J}$ may not be a JB^*-algebra with the "$*_u$" as involution.

In [21, 22, 23, 24], the author presented various results on unitary isotopes of JB^*-algebras. Some of these results are used in our subsequent work; in particular, the following result plays a key role in obtaining the main results of next section:

Lemma 1.3 ([21, Theorem 4.6])**.** *For any unitary u in JB^*-algebra $\mathcal{J}$,*

$$\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J}^{[u]}).$$

Recall that an element x of a JB^*-algebra $\mathcal{J}$ is called *positive* in $\mathcal{J}$ if $x^* = x$ (self-adjoint) and its spectrum $\sigma_{\mathcal{J}}(x)$ is contained in the set of nonnegative real numbers where $\sigma_{\mathcal{J}}(x) = \sigma_{\mathcal{J}}(x) = \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible in } \mathcal{J}\}$.

The following lemma is a main result of paper [21], which says that any invertible is positive in a certain unitary isotope, where the unitary comes from the polar decomposition of the invertible. This is a nice tricky result and its technical proof involves the well known Stone–Weierstrass Theorem and functional calculus.

Lemma 1.4 ([21, Theorem 4.12])**.** *Every invertible element x of the JB^*-algebra $\mathcal{J}$ is positive (in fact, positive invertible) in the isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$, where $u \in \mathcal{U}(\mathcal{J})$ and is given by the usual polar decomposition $x = u|x|$ of x considered as an operator in some $\mathcal{B}(\mathcal{H})$.*

2. Russo–Dye Theorem

We follow the lines of L. T. Gardner [4] and R. V. Kadison and G. K. Pedersen who proved similar result for C^*-algebras. In Theorem 2.1 of [10], Kadison and Pedersen by following Gardner [4] proved that each element of the open unit ball in a C^*-algebra is a mean of unitaries and then the Russo–Dye Theorem for C^*-algebras was immediately obtained . Their proof for Theorem 2.1 (as appeared in [10, page 251] no longer works for general JB^*-algebras simply because the Jordan product generally is not associative and so the associative product of two unitary elements is not necessarily in a JB^*-algebra.

Here, we adapt a new approach to resolve this difficulty, which is highly nontrivial. Given the nature one might expect that we are showing the result for special Jordan algebras given by C^*-algebras and for the exceptional case separately. Instead, we shall give a unified approach that is general enough to cover all exceptional as well as all special JB^*-algebras including all C^*-algebras. This also provides a different proof of [10, Theorem 2.1], hence an alternate proof of the Russo–Dye Theorem for C^*-algebras.

As our main tools, we shall use results that have been fixed in the previous section as lemmas to obtain the strict extensions of Russo–Dye Theorem [10, Theorem 2.1 and its Corollary on p. 251] to general JB^*-algebras.

We need another lemma which says symbolically that all invertible elements of norm at most one are the mean of two unitaries:

Lemma 2.1. *For any JB^*-algebra $\mathcal{J}$,*

$$\mathcal{J}_{\text{inv}} \cap (\mathcal{J})_1 \subseteq \frac{1}{2}(\mathcal{U}(\mathcal{J}) + \mathcal{U}(\mathcal{J})).$$

Proof. Let $x \in \mathcal{J}_{\text{inv}}$. Then, by Lemma 1.4, x is positive invertible in the isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$, for a certain $u \in \mathcal{U}(\mathcal{J})$. In particular, x is self-adjoint in $\mathcal{J}^{[u]}$. Hence, by [22, Theorem 2.11], $x \in \frac{1}{2}(\mathcal{U}(\mathcal{J}) + \mathcal{U}(\mathcal{J}))$. □

The next result claims the existence of certain unitaries:

Theorem 2.2. *Let $\mathcal{J}$ be a unital JB^*-algebra, $s \in (\mathcal{J})_1^\circ$ (the open unit ball) and $v \in \mathcal{U}(\mathcal{J})$. Then, for any positive integer n, $v + (n-1)s = \sum_{i=1}^n u_i$ where the u_i's are unitaries in $\mathcal{J}$.*

Proof. By Lemma 1.2, v is the identity of the v-isotope $\mathcal{J}^{[v]}$ of $\mathcal{J}$. But $\|s\| < 1$. Therefore, by [21, Lemma 2.1(iii)], $v+s$ and so $\frac{1}{2}(v+s)$ is invertible in $\mathcal{J}^{[v]}$. Hence, by Lemma 1.1, $\frac{1}{2}(v+s)$ is invertible in the original algebra $\mathcal{J}$. Also note that $\|\frac{1}{2}(v+s)\| \leq 1$. Thus, by the previous Lemma 2.1, there exist two unitaries y and z in $\mathcal{J}$ such that $v+s = y+z$. The assertion now follows by induction on n. □

The following result extends joint results of Kadison and Pedersen [10] to general JB^*-algebras. Its part (iii) is a JB^*-algebra strict analogue of the famous Russo–Dye Theorem [19].

Theorem 2.3 (Russo–Dye)**.** (i) *Let x be an element of a JB^*-algebra $\mathcal{J}$ with unit e such that $\|x\| < 1 - 2n^{-1}$ for some $n \geq 3$. Then there exist $u_i \in \mathcal{U}(\mathcal{J})$, $i = 1, 2, 3, \ldots, n$ such that $x = \frac{1}{n}\sum_{i=1}^n u_i$.*

(ii) *$(\mathcal{J})_1^\circ \subseteq co\mathcal{U}(\mathcal{J})$.*

(iii) *$\overline{co}\mathcal{U}(\mathcal{J}) = (\mathcal{J})_1$.*

Here, $co\mathcal{U}(\mathcal{J})$ and $\overline{co}\mathcal{U}(\mathcal{J})$ denote the convex hull of $\mathcal{U}(\mathcal{J})$ and its norm closure, respectively.

Proof. (i) Since $\|x\| < 1 - 2n^{-1}$, we have $\|(n-1)^{-1}(nx - e)\| < 1$. Hence, by taking $v = e$ and $s = (n-1)^{-1}(nx - e)$ in Theorem 2.2, we get $nx = \sum_{j=1}^n u_i$, for some unitaries u_i in $\mathcal{J}$. This proves part (i).

(ii) Suppose $x \in (\mathcal{J})_1^\circ$. Then there exists an integer $n \geq 3$ such that $\|x\| < 1 - 2n^{-1}$. Therefore, $x \in co\mathcal{U}(\mathcal{J})$ by part (i).

(iii) Clearly, $\overline{co}\mathcal{U}(\mathcal{J}) \subseteq (\mathcal{J})_1$. On the other hand, we have $(\mathcal{J})_1^\circ \subseteq co\mathcal{U}(\mathcal{J})$ by part (ii). Thus $(\mathcal{J})_1 \subseteq \overline{co}\mathcal{U}(\mathcal{J})$ because $(\bar{\mathcal{J}})_1^\circ = (\mathcal{J})_1$. □

Remark 2.4. There is no strict analogue of Russo–Dye Theorem for more general JB*-triples (cf. [26]) just because an arbitrary JB*-triple has no unit.

It is worth mentioning that the number of unitaries in part (i) of Theorem 2.3 is the least possible for general (unital) JB^*-algebras , for otherwise, [10, Proposition 3] provides a counterexample from C^*-algebras. As mentioned in the previous section, Wright and Youngson [28] also obtained the Russo–Dye Theorem for JB^*-algebras with an entirely different proof. Our approach to obtain the Russo–Dye Theorem for general JB^*-algebras gives more information about the number of unitaries required in the approximations.

Corollary 2.5. *Each element of a unital JB^*-algebra $\mathcal{J}$ is some positive multiple of a sum of three unitaries in $\mathcal{J}$.*

Proof. Let $x \in \mathcal{J}$ and $\epsilon > 0$. Let $y = (3\|x\| + \epsilon)^{-1}x$. Then $\|y\| < \frac{1}{3}$. Hence, by Theorem 2.3, there exist three unitaries u_1, u_2, u_3 in $\mathcal{J}$ such that $y = \frac{1}{3}(u_1 + u_2 + u_3)$. Thus, $x = (\|x\| + \frac{\epsilon}{3})(u_1 + u_2 + u_3)$. □

In the next result $u_m(x)$ denotes $\min\{n : x = \frac{1}{n}\sum_{j=1}^{n} u_j\}$, where the u_j are unitaries in the JB^*-algebra $\mathcal{J}$. We have the following relation between $u_m(x)$ and the distance from nx to the unitaries:

Corollary 2.6. *Let x be an element of a unital JB^*-algebra $\mathcal{J}$ and let d_n denote the distance from nx to $\mathcal{U}(\mathcal{J})$ with $n \geq 2$. If $d_n < n - 1$, then $u_m(x) \leq n$. On the other hand, if $u_m(x) \leq n$, then $d_n \leq n - 1$.*

Proof. Suppose $d_n < n - 1$. Then there exists $u \in \mathcal{U}(\mathcal{J})$ with

$$\|nx - u\| < n - 1$$

and so $\|(n-1)^{-1}(nx - u)\| < 1$. Hence, $(n-1)^{-1}(nx - u) \in (\mathcal{J})_1^\circ$. In Theorem 2.2, replacing s by $(n-1)^{-1}(nx - u)$ and v by u, we deduce that x is the mean of n elements of $\mathcal{U}(\mathcal{J})$ (as $x = \frac{1}{n}(v + (n-1)s)$). Thus $u_m(x) \leq n$.

For the other hand, suppose $u_m(x) \leq n$. Then $x = r^{-1}\sum_{i=1}^{r} u_i$ for some $1 \leq r \leq n$ with u_i's in $\mathcal{U}(\mathcal{J})$. Then $\|x\| \leq r^{-1}\sum_{i=1}^{r}\|u_i\| = 1$. Further, $\|rx - u_1\| = \|\sum_{i=2}^{r} u_i\| \leq r - 1$. Hence $\|nx - u_1\| = \|(n-r)x + rx - u_1\| \leq \|rx - u_1\| + \|(n-r)x\| \leq r - 1 + n - r = n - 1$ because $\|x\| \leq 1$. Thus $d_n = \inf_{u \in \mathcal{U}(\mathcal{J})}\|nx - u\| \leq \|nx - u_1\| \leq n - 1$. □

Remark 2.7. It should be emphasized that the strict inequality in first part of the above result is significant. For example, let Δ be the closed unit disk in the complex plane $\mathbb{C}$ and let n be any integer ≥ 2. Then for the function $f \in \mathcal{C}_{\mathbb{C}}(\Delta)$ defined by $f(z) = (1 - \frac{1}{n})z + \frac{1}{n}$ we have $\operatorname{dist}(nf, \mathcal{U}(\mathcal{C}_{\mathbb{C}}(\Delta)) = n - 1$ but f can not be the mean of n unitaries in $\mathcal{C}_{\mathbb{C}}(\Delta)$ so that $u_m(f) > n$; for more details see [16, pp. 374–375].

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