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Equivariant extensions of $*$-algebras

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Abstract. A bivariant functor is defined on a category of $*$-algebras and a category of operator ideals, both with actions of a second countable group G, into the category of abelian monoids. The elements of the bivariant functor will be G-equivariant extensions of a $*$-algebra by an operator ideal under a suitable equivalence relation. The functor is related with the ordinary Ext-functor for C^*-algebras defined by Brown–Douglas–Fillmore. Invertibility in this monoid is studied and characterized in terms of Toeplitz operators with abstract symbol.

Contents

Introduction

Extensions of C^*-algebras by stable C^*-algebras have been thoroughly studied (see [2], [3], [10], [14]) due to their close relation to Toeplitz operators and KK-theory (see [10], [14]). The starting point was the article [3] where an abelian monoid $\mathrm{Ext}(A)$ was associated to a C^*-algebra A. This monoid consists of extensions $0 \to \mathcal{K} \to E \to A \to 0$ under a certain equivalence relation, here $\mathcal{K}$ denotes the ideal of compact operators. The construction can be generalized to a bivariant theory by replacing $\mathcal{K}$ with an arbitrary stable C^*-algebra B and one obtains an abelian monoid $\mathrm{Ext}(A,B)$. In [14] this construction was put into the equivariant setting although only the invertible elements of $\mathrm{Ext}_G(A,B)$ were studied. We will study the full extension monoids.

As is shown in [10], and equivariantly in [14], an odd Kasparov $A-B$-module gives an extension of A by B which induces an additive mapping

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$KK^1_G(A,B) \to \mathrm{Ext}_G(A,B)$. It can be shown, as is done in [14] that this is a bijection to the group $\mathrm{Ext}^{-1}_G(A,B) \subseteq \mathrm{Ext}_G(A,B)$ of invertible elements. A more straightforward approach is the proof in [10] using the Stinespring representation theorem. As a corollary of this proof, if A is nuclear and separable the Choi–Effros lifting theorem implies that $\mathrm{Ext}_G(A,B)$ is a group if G is trivial. This is the main motivation of studying extension theory.

The reason for leaving the category of C^*-algebras is that most cohomology theories behave badly on C^*-algebras and one needs to look at dense subalgebras (see more in [11]). For example, if we use cohomology and the Atiyah–Singer index theorem to calculate the index of a Toeplitz operator this is easily done via an explicit integral in terms of the symbol and its derivatives if the symbol is smooth (see more in [7]).

With this as motivation we will extend the Ext_G-functor to $*$-algebras which embed into separable C^*-algebras and actions which extend to C^*-automorphisms. In the first part of this paper we define suitable categories for the first and the second variable of the functor. Then, similarly to the setting with C^*-algebras, we will construct a bivariant functor $\mathcal{E}xt_G$ to the category of abelian monoids. In particular there is a natural transformation

$$\Theta : \mathcal{E}xt_G \to \mathrm{Ext}_G$$

in the category of abelian monoids. An interesting question to study further is what types of elements are in the kernel of the Θ-mapping and if there is some way to make Θ surjective?

After that we will move on to study the invertible elements. A rather remarkable result is that the invertible elements are those extensions which arise from a G-equivariant algebraic $\mathcal{A} - \mathfrak{J}$-Kasparov modules. As an example, we will study the case of extensions of the smooth functions on a compact manifold by the Schatten class operators, in this case the Θ-mapping turns out to be a surjection. At the end of the paper we describe a certain type of elements in the kernel of the Θ-mapping which we will call linear deformations. The linear deformations are analytic in their nature. We end the paper by giving an explicit example of a linear deformation of the ordinary Toeplitz operators on the Hardy space that produces another $\mathcal{E}xt$-class but is homotopic to the $\mathcal{E}xt$-class defined by the ordinary Toeplitz operators.

1. Definitions and basic properties

To begin with we will define the suitable categories. From here on, let G be a second countable locally compact group. We will say that the group action $\alpha : G \to \mathrm{Aut}(A)$ acts continuously on the C^*-algebra A if $g \mapsto \alpha_g(a)$ is continuous for all $a \in A$.

Definition 1.1. *Let C^*A_G denote the category with objects consisting of pairs $(\mathcal{A}, A)$ where A is a separable C^*-algebra with a continuous G-action and $\mathcal{A}$ is a G-invariant dense $*$-subalgebra. A morphism in C^*A_G between*

$(\mathcal{A}, A)$ to $(\mathcal{A}', A')$ is a G-equivariant $*$-homomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ bounded in C^*-norm.

As an abuse of notation we will denote an object $(\mathcal{A}, A)$ in C^*A_G by $\mathcal{A}$ and its latin character A will denote the ambient C^*-algebra. Observe that a morphism in C^*A_G is the restriction of an equivariant $*$-homomorphism $\bar{\varphi} : A \to A'$ uniquely determined by φ. This follows from that if $\varphi : \mathcal{A} \to \mathcal{A}'$ is bounded in C^*-norm it extends to $\bar{\varphi} : A \to A'$ and since φ is equivariant $\bar{\varphi}$ will also be equivariant. Conversely, an equivariant $*$-homomorphism of C^*-algebras is always C^*-bounded. When a linear mapping $T : \mathcal{A} \to \mathcal{A}'$, not necessarily equivariant, between two objects is induced by a bounded mapping $\bar{T} : A \to A'$ we will say that T is C^*-bounded.

For a C^*-algebra B we will denote its multiplier C^*-algebra by $\mathcal{M}(B)$ and embed B as an ideal in $\mathcal{M}(B)$. If B has a G-action we will equip $\mathcal{M}(B)$ with the induced G-action.

Definition 1.2. *If $(\mathfrak{I}, I) \in C^*A_G$ satisfies that the C^*-algebra I is equivariantly stable, that is $I \otimes \mathcal{K} \cong I$ where $\mathcal{K}$ has trivial G-action, and $\mathfrak{I}$ is an ideal in $\mathcal{M}(I)$ the algebra $\mathfrak{I}$ is called a C^*-stable G-ideal. Let C^*SI_G denote the full subcategory of C^*A_G consisting of C^*-stable G-ideals.*

We will call a morphism $\psi : \mathfrak{I} \to \mathfrak{I}'$ of C^*-stable G-ideals an embedding of C^*-stable G-ideals if $\psi : I \to I'$ is an isomorphism.

Proposition 1.3. *For any C^*-stable G-ideal $\mathfrak{I}$ there is an equivariant isomorphism $M_2 \otimes I \cong I$ inducing an isomorphism $M_2 \otimes \mathfrak{I} \cong \mathfrak{I}$. The isomorphism is given by the adjoint action of a G-invariant unitary operator $V = V_1 \oplus V_2 : I \oplus I \to I$ between Hilbert modules.*

Notice that V being unitary is equivalent to $V_1, V_2 \in \mathcal{M}(I)$ being isometries satisfying

$$V_1V_1^* + V_2V_2^* = 1.$$

Proof. It is sufficient to construct two G-invariant isometries $V_1, V_2 \in \mathcal{M}(I)$ such that $V_1V_1^* + V_2V_2^* = 1$. Then $V := V_1 \oplus V_2$ is a G-invariant unitary. Thus V will be an isomorphism of Hilbert modules so $\mathrm{Ad}\, V : M_2 \otimes I \to I$ is an isomorphism and since $\mathfrak{I}$ is an ideal $\mathrm{Ad}\ V$ induces a isomorphism $M_2 \otimes \mathfrak{I} \cong \mathfrak{I}$.

Let K denote a separable Hilbert space with trivial G-action. Choose a unitary $V' : K \oplus K \to K$. Let $V_1', V_2' \in \mathcal{B}(K)$ be defined by $V'(x_1 \oplus x_2) := V_1'x_1 + V_2'x_2$. We may take the isometries V_1 and V_2 to be the image of V_1' and V_2' under the equivariant, unital embedding

$$\mathcal{B}(K) = \mathcal{M}(\mathcal{K}) \hookrightarrow \mathcal{M}(I \otimes \mathcal{K}) \cong \mathcal{M}(I). \qquad \square$$

One important class of C^*-stable G-ideals is the class of symmetrically normed operator ideals such as the Schatten class ideals and the Dixmier ideals (see more in [4]) over a separable Hilbert space H with a G-action. In order to get equivariant stability we need to stabilize the Hilbert space with another Hilbert space with trivial G-action. Let H' denote a separable Hilbert space and define

$$\mathcal{L}^p_H := (\mathcal{L}^p(H \otimes H'), \mathcal{K}(H \otimes H'))$$

and analogously for the Dixmier ideal $\mathcal{L}^{n+}_H$. The G-action on the algebras are the one induced from the G-action on H.

The main study of this paper are equivariant extensions

$$0 \to \mathfrak{I} \to \mathcal{E} \xrightarrow{\varphi} \mathcal{A} \to 0$$

where $\mathfrak{I}$ is a C^*-stable G-ideal and $\mathcal{A} \in C^*A_G$. In particular we are interested in when such extensions admit C^*-bounded splittings of Toeplitz type.

Consider for example the 0:th order pseudodifferential extension $\Psi^0(M)$ on a closed Riemannian manifold M. This extension is an extension of the smooth functions on the cotangent sphere S^*M by the classical pseudodifferential operators of order -1 given by the short exact sequence

$$0 \to \Psi^{-1}(M) \to \Psi^0(M) \to C^\infty(S^*M) \to 0.$$

The algebra $\Psi^{-1}(M)$ is not C^*-stable, but $\Psi^{-1}(M)$ is dense in $\mathcal{L}^p(L^2(M))$ for any $p > n$, so the pseudo-differential extension fits in our framework after some modifications. The pseudo-differential extension admits an explicit splitting $T : C^\infty(S^*M) \to \Psi^0(M)$ in terms of Fourier integral operators which is not C^*-bounded if $\dim M > 1$. Read more about this in Chapter 18.6 in [9]. In this setting however, the problem can be mended. In [8] a C^*-bounded splitting is constructed for real analytic manifolds M in terms of Grauert tubes and Toeplitz operators.

We will abuse the notation somewhat by referring both to the object $\mathcal{E}$ and the extension by $\mathcal{E}$. Observe that the definition implies that there exists a commutative diagram with equivariant, exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathcal{E} & \xrightarrow{\varphi} & \mathcal{A} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & E & \xrightarrow{\bar{\varphi}} & A & \longrightarrow & 0. \end{array}$$

The $*$-homomorphism $\bar{\varphi} : E \to A$ is the extension of φ to E.

Definition 1.4. *Two G-equivariant extensions $\mathcal{E}$ and $\mathcal{E}'$ of $\mathcal{A}$ by $\mathfrak{I}$ are said to be isomorphic if there exists a morphism $\psi : \mathcal{E} \to \mathcal{E}'$ in C^*A_G that fits into a commutative diagram*

$$(1) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathcal{E} & \xrightarrow{\varphi} & \mathcal{A} & \longrightarrow & 0 \\ & & \| & & \downarrow\psi & & \| & & \\ 0 & \longrightarrow & \mathfrak{I} & \longrightarrow & \mathcal{E}' & \xrightarrow{\varphi'} & \mathcal{A} & \longrightarrow & 0. \end{array}$$

Because of the five lemma, ψ is an isomorphism.

Choose a linear splitting $\tau : \mathcal{A} \to \mathcal{E}$ and identify $\mathfrak{I}$ with an ideal in $\mathcal{E}$. The mapping τ being a splitting of an equivariant mapping $\mathcal{E} \to \mathcal{A}$ implies that

$$\tau(ab) - \tau(a)\tau(b), \;\; \tau(a^*) - \tau(a)^* \in \mathfrak{I} \quad \text{and} \tag{2}$$

$$\tau(g.a) - g.\tau(a) \in \mathfrak{I} \; \forall g \in G. \tag{3}$$

Given a C^*-stable G-ideal $\mathfrak{I}$ we define the G-$*$-algebra $\mathcal{C}_{\mathfrak{I}} := \mathcal{M}(I)/\mathfrak{I}$ and denote by $q_{\mathfrak{I}} : \mathcal{M}(I) \to \mathcal{C}_{\mathfrak{I}}$ the canonical surjection. By the equations (2) and (3) the mapping $q_{\mathfrak{I}}\tau : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}$ is an equivariant $*$-homomorphism. We will call the mapping $\beta_{\mathcal{A}} := q_{\mathfrak{I}}\tau$ the Busby mapping for the extensions $\mathcal{E}$. A Busby mapping that is C^*-bounded after composing with $\mathcal{C}_{\mathfrak{I}} \to \mathcal{M}(I)/I$ is called bounded. A Busby mapping which can be lifted to a C^*-bounded G-equivariant $*$-homomorphism of $\mathcal{A}$ is called trivial.

For an equivariant $*$-homomorphism $\beta : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}$ we can define the $*$-algebra

$$\mathcal{E}_\beta := \{a \oplus x \in \mathcal{A} \oplus \mathcal{M}(I) : \beta(a) = q_{\mathfrak{I}}(x)\}.$$

The $*$-algebra $\mathcal{E}_\beta$ is closed under the G-action on $\mathcal{A} \oplus \mathcal{M}(I)$ so it is a G-$*$-algebra. Denote the norm closure of $\mathcal{E}_\beta$ in $A \oplus \mathcal{M}(I)$ by E_β. We have an injection $\mathfrak{I} \to \mathcal{E}_\beta$ and a surjection $\mathcal{E}_\beta \to \mathcal{A}$. The kernel of $\mathcal{E}_\beta \to \mathcal{A}$ is $\mathfrak{I}$, so the sequence $0 \to \mathfrak{I} \to \mathcal{E}_\beta \to \mathcal{A} \to 0$ is exact and the arrows are equivariant. The $*$-algebra $\mathcal{E}_\beta$ is a well defined object in C^*A_G, because Theorem 2.1 of [14] states that the induced G-action on E_β is continuous provided it is continuous on I and on A.

Proposition 1.5. *The equivariant $*$-homomorphism $\beta : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}$ determines the extension up to a isomorphism, i.e if $\mathcal{E}$ has Busby mapping β, $\mathcal{E}$ is isomorphic to $\mathcal{E}_\beta$.*

Proof. Suppose that β is Busby mapping for $\mathcal{E}$. Define $\psi : \mathcal{E} \to \mathcal{E}_\beta$ as

$$\psi(x) := \varphi(x) \oplus x.$$

Since φ is equivariant, so is ψ. This makes the diagram (1) commutative, thus ψ is an isomorphism of G-equivariant extensions. □

The most useful class of G-equivariant extensions are the ones arising from algebraic $\mathcal{A} - \mathfrak{I}$-Kasparov modules. This is defined as an algebraic generalization of Kasparov modules for C^*-algebras, see more in [10].

Definition 1.6. *A G-equivariant algebraic $\mathcal{A} - \mathfrak{I}$-Kasparov module is a C^*-bounded G-equivariant representation $\pi : \mathcal{A} \to \mathcal{M}(I)$ and an almost G-invariant symmetry $F \in \mathcal{M}(I)$ that is almost commuting with $\pi(\mathcal{A})$, that is:*

$$g.F - F \in \mathfrak{I} \;\; \forall\, g \in G \quad \textit{and} \quad [F, \pi(a)] \in \mathfrak{I} \;\; \forall\, a \in \mathcal{A}.$$

Since F is a grading we can define the projection $P := (F + 1)/2$. The pair (π, F) induces a $*$-homomorphism

$$\beta : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}, \; a \mapsto q_{\mathfrak{I}}(P\pi(a)P). \tag{4}$$

The requirement $[F, \pi(a)] \in \mathfrak{I}$ together with $g.F - F \in \mathfrak{I}$ implies that β is an equivariant $*$-homomorphism.

Let $B_G(\mathcal{A}, \mathfrak{I})$ denote the set of bounded G-equivariant Busby mappings on $\mathcal{A}$. This is the correct set to study extensions in. By Proposition 1.5 the set of G-equivariant Busby mappings is the same set as the set of isomorphism classes of G-equivariant extensions. But we need some useful notion of equivalence of extensions, or by the previous reasoning an equivalence relation on $B_G(\mathcal{A}, \mathfrak{I})$. For an object $\mathfrak{I} \in C^*SI_G$ we define the almost invariant weakly unitaries

$$U^{aw}(\mathfrak{I}) := q_{\mathfrak{I}}^{-1}(\{v \in \mathcal{C}_{\mathfrak{I}} : g.v = v,\ v^*v = vv^* = 1\}).$$

Let the almost invariant unitaries be defined as $U^a(\mathfrak{I}) := U^{aw}(\mathfrak{I}) \cap U(\mathcal{M}(\mathfrak{I}))$.

Definition 1.7. *Strong equivalence on $B_G(\mathcal{A}, \mathfrak{I})$ is the equivalence of Busby mappings by the adjoint $U^a(\mathfrak{I})$-action on $\mathcal{C}_{\mathfrak{I}}$. Weak equivalence on $B_G(\mathcal{A}, \mathfrak{I})$ is that of the adjoint $U^{aw}(\mathfrak{I})$-action on $\mathcal{C}_{\mathfrak{I}}$.*

Let $E_G(\mathcal{A}, \mathfrak{I})$ denote the set of strong equivalence classes of $B_G(\mathcal{A}, \mathfrak{I})$ and let $E_G^w(\mathcal{A}, \mathfrak{I})$ denote the set of weak equivalence classes. Similarly let $D_G(\mathcal{A}, \mathfrak{I})$ denote the set of strong equivalence classes of trivial Busby mappings and let $D_G^w(\mathcal{A}, \mathfrak{I})$ denote the set of weak equivalence classes of trivial Busby maps.

The isomorphism $\lambda : M_2 \otimes \mathcal{C}_{\mathfrak{I}} \to \mathcal{C}_{\mathfrak{I}}$ induced by Ad V from Proposition 1.3 can be used to define the sum of two G-equivariant Busby mappings $\beta_1, \beta_2 \in B_G(\mathcal{A}, \mathfrak{I})$ as

$$\beta_1 + \beta_2 := \lambda \circ (\beta_1 \oplus \beta_2) : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}.$$

Proposition 1.8. *The binary operation $+$ on $B_G(\mathcal{A}, \mathfrak{I})$ induces a well defined abelian semigroup structure on $E_G(\mathcal{A}, \mathfrak{I})$ independent of the choice of the unitary $V = V_1 \oplus V_2$. The set $D_G(\mathcal{A}, \mathfrak{I})$ is a subsemigroup.*

The proof of the above proposition is the same as the proof of Lemma 3.1 in [14] where the semigroup of equivariant extensions of a C^*-algebra is constructed. Two G-equivariant Busby mappings $\beta_1, \beta_2 \in B_G(\mathcal{A}, \mathfrak{I})$ are said to be stably equivalent if they differ by trivial Busby mappings. That is, if there exist C^*-bounded, G-equivariant $*$-homomorphisms $\pi_1, \pi_2 : \mathcal{A} \to \mathcal{M}(I)$ such that

$$\beta_1 \oplus q_{\mathfrak{I}}\pi_1 \equiv \beta_2 \oplus q_{\mathfrak{I}}\pi_2 : \mathcal{A} \to M_2 \otimes \mathcal{C}_{\mathfrak{I}}.$$

Stable equivalence induces a well defined equivalence relation on $E_G(\mathcal{A}, \mathfrak{I})$ and $E_G^w(\mathcal{A}, \mathfrak{I})$.

Definition 1.9. *We define $\mathcal{E}xt_G(\mathcal{A}, \mathfrak{I})$ as the monoid of stable equivalence classes of $E_G(\mathcal{A}, \mathfrak{I})$ and $\mathcal{E}xt_G^w(\mathcal{A}, \mathfrak{I})$ as the monoid of stable equivalence classes of $E_G^w(\mathcal{A}, \mathfrak{I})$. For $G = \{1\}$ we denote the $\mathcal{E}xt$-invariants by $\mathcal{E}xt(\mathcal{A}, \mathfrak{I})$ and $\mathcal{E}xt^w(\mathcal{A}, \mathfrak{I})$.*

The monoids $\mathcal{E}xt_G(\mathcal{A}, \mathfrak{I})$ and $\mathcal{E}xt_G^w(\mathcal{A}, \mathfrak{I})$ coincide with the semigroup quotients $E_G(\mathcal{A}, \mathfrak{I})/D_G(\mathcal{A}, \mathfrak{I})$, respectively $E_G^w(\mathcal{A}, \mathfrak{I})/D_G^w(\mathcal{A}, \mathfrak{I})$. It has a zero-element since the class of an element in $D_G(\mathcal{A}, \mathfrak{I})$ is zero.

If we are given a G-equivariant extension $\mathcal{E}$ of $\mathcal{A}$ we will denote the class in $\mathcal{E}xt_G(\mathcal{A}, \mathfrak{I})$ of its G-equivariant Busby mapping β by $[\mathcal{E}]$ or by $[\beta]$.

Proposition 1.10. *If $\mathfrak{I} = I$ there are isomorphisms*

$$\mathcal{E}xt_G^w(\mathcal{A}, I) \cong \mathcal{E}xt_G(\mathcal{A}, I) \cong \mathcal{E}xt_G(A, I) \equiv \mathrm{Ext}_G(A, I) \cong \mathrm{Ext}_G^w(A, I).$$

Proof. We will prove the existence of the first and the second isomorphism. The proof of the last isomorphism is a special case of the first isomorphism for $\mathcal{A} = A$.

To prove the existence of the first isomorphism it is sufficient to show that weakly equivalent G-equivariant Busby mappings are strongly equivalent up to stable equivalence. Assume that $\beta_1, \beta_2 \in B_G(\mathcal{A}, \mathfrak{I})$ are weakly equivalent via the almost invariant weakly unitary $U \in U^{aw}(\mathfrak{I})$. Then $\beta_1 \oplus 0$ and $\beta_2 \oplus 0$ are weakly equivalent via the almost invariant weakly unitary $U \oplus U^*$. But the operator $U \oplus U^*$ lifts to a unitary $\tilde{U} \in \mathcal{M}(M_2 \otimes I)$ since $\mathcal{C}_{\mathfrak{I}}$ is a C^*-algebra. In fact $\tilde{U} \in U^a(M_2 \otimes \mathfrak{I})$ since U is almost invariant. Thus $\beta_1 \oplus 0$ and $\beta_2 \oplus 0$ are strongly equivalent. For the proof that $U \oplus U^*$ lifts to a unitary, see Proposition 3.4.1 in [2].

The second isomorphism is given by the mapping

$$\mathcal{E}xt_G(\mathcal{A}, I) \to \mathcal{E}xt_G(A, I),$$
$$[\mathcal{E}] \mapsto [E].$$

In terms of the G-equivariant Busby mapping β the mapping is given by $[\beta] \mapsto [\bar{\beta}]$, since $\mathcal{A}$ is dense and β is bounded by assumption this is a surjection and $\bar{\beta}$ determines β uniquely. □

The constructions of Ext_G and Ext_G^w are the same as $\mathcal{E}xt_G$ and $\mathcal{E}xt_G^w$ but with C^*-algebras. These constructions can be found in [3], [10] and [14]. Proposition 1.10 is a mild generalization of Proposition 15.6.4 in [2]. The proof is the same although $\mathcal{A}$ does not need to be a C^*-algebra.

Since the two theories are very similar we will focus on $\mathcal{E}xt_G$. All results stated in this paper are easily verified to also hold for $\mathcal{E}xt_G^w$.

2. Functoriality of $\mathcal{E}xt_G$

In this section we will prove that $\mathcal{E}xt_G$ is a functor to the category Mo^{ab} of abelian monoids. We define this category to have objects of abelian monoids and a morphism is an additive mapping $k : M_1 \to M_2$ such that $k(0) = 0$. We know how $\mathcal{E}xt_G$ acts on the objects of C^*A_G and C^*SI_G. What needs to be defined is the action of $\mathcal{E}xt_G$ on the morphisms. We begin by showing that $\mathcal{E}xt_G$ depends covariantly on $\mathfrak{I}$.

Let $\psi : \mathfrak{I} \to \mathfrak{I}'$ be a morphism of C^*-stable G-ideals. By definition ψ can be extended to an equivariant mapping $\mathcal{M}(I) \to \mathcal{M}(I')$ which induces an equivariant mapping $q_\psi : \mathcal{C}_\mathfrak{I} \to \mathcal{C}_{\mathfrak{I}'}$. Define $\psi_* : E_G(\mathcal{A}, \mathfrak{I}) \to E_G(\mathcal{A}, \mathfrak{I}')$ by $\psi_*[\beta] := [q_\psi \circ \beta]$. Clearly, $\psi_*[\beta]$ is independent of the stable equivalence class of $[\beta]$. Hence ψ induces a well defined mapping

$$\psi_* : \mathcal{E}xt_G(\mathcal{A}, \mathfrak{I}) \to \mathcal{E}xt_G(\mathcal{A}, \mathfrak{I}').$$

Since ψ_* acting on a trivial extension gives a trivial extension we have a homomorphism of monoids.

Let us move on to proving that $\mathcal{E}xt_G$ depends contravariantly on $\mathcal{A}$. Let $\varphi : \mathcal{A} \to \mathcal{A}'$ be a morphism in C^*A_G. Take a G-equivariant Busby mapping β of $\mathcal{A}'$. Then we can define a G-equivariant Busby mapping $\varphi^*\beta := \beta \circ \varphi$ of $\mathcal{A}$. This clearly depends on neither strong equivalence class nor stable equivalence class of the G-equivariant Busby mapping. If β is trivial it follows that $\varphi^*\beta$ is trivial so we have a morphism of monoids

$$\varphi^* : \mathcal{E}xt_G(\mathcal{A}', \mathfrak{I}) \to \mathcal{E}xt_G(\mathcal{A}, \mathfrak{I}).$$

We have now proved the following proposition.

Proposition 2.1. *The functor $\mathcal{E}xt_G : C^*A_G \times C^*SI_G \to Mo^{\mathrm{ab}}$ is a well defined functor. It is covariant in $\mathfrak{I}$ and contravariant in $\mathcal{A}$.*

As noted above, an extension $\mathcal{E}$ of the algebra $\mathcal{A}$ by $\mathfrak{I}$ gives rise to an extension E of A by I. This procedure defines a mapping $E_G(\mathcal{A}, \mathfrak{I}) \to E_G(A, I)$ which respects stable equivalences.

Let C^*_G denote the category of separable C^*-algebras with a continuous G-action and SC^*_G the full subcategory of equivariantly stable objects in C^*_G. We can define an essentially surjective functor

$$\Gamma_1 : C^*A_G \times C^*SI_G \to C^*_G \times SC^*_G,$$
$$((\mathcal{A}, A), (\mathfrak{I}, I)) \mapsto (A, I).$$

Its right adjoint is the full and faithful functor

$$\Gamma_2 : C^*_G \times SC^*_G \to C^*A_G \times C^*SI_G$$
$$(A, I) \mapsto ((A, A), (I, I)).$$

Notice that $\Gamma_1\Gamma_2$ is the identity functor on $C^*_G \times SC^*_G$. Define the functor

$$\mathrm{Ext}_G : C^*_G \times SC^*_G \to Mo^{\mathrm{ab}} \quad \text{by} \quad \mathrm{Ext}_G := \mathcal{E}xt_G \circ \Gamma_2.$$

As noted above this definition coincides with the definition of the Ext_G-functor in [3] and [10].

Proposition 2.2. *The mapping Θ defines a natural transformation*

$$\Theta : \mathcal{E}xt_G \to \mathrm{Ext}_G \circ \Gamma_1.$$

Proof. The mapping $\Theta^{\mathcal{A}}_{\mathfrak{I}}$ merely extends Busby mappings to the object's C^*-closure, so $\Theta^{\mathcal{A}}_{\mathfrak{I}}$ commutes with composition of morphisms in $C^*A_G \times C^*SI_G$ since they are just equivariant C^*-bounded $*$-homomorphisms. Thus Θ is a natural transformation. □

3. Invertible extensions

Just as in the case of a C^*-algebra one can relate invertibility in the $\mathcal{E}xt_G$-monoid and properties of the splitting. In this section we will study invertibility in $\mathcal{E}xt_G$-monoid in terms of Toeplitz operators.

The main result to be obtained in this section tells us that there is a direct link between algebraic properties in the $\mathcal{E}xt_G$-monoid and analytical properties of the extension. But this tells us nothing about how to construct the inverse or give explicit expressions. We will study this in the case of G being the trivial group and for extensions admitting a C^*-bounded, completely positive splitting. Then these explicit constructions are possible in an ideal $\mathcal{J}_{\mathfrak{I}} \supseteq \mathfrak{I}$ such that $\mathfrak{I}$ is the linear span of $\{a^*a : a \in \mathcal{J}_{\mathfrak{I}}\}$. In this setting an explicit inverse can be given in $\mathcal{E}xt(\mathcal{A}, \mathcal{J}_{\mathfrak{I}})$.

Definition 3.1. *A G-equivariant extension which admits a splitting of the form $a \mapsto P\pi(a)P$, for a G-equivariant algebraic $\mathcal{A} - \mathfrak{I}$-Kasparov module (π, F) and $P = (F+1)/2$, is called a G-equivariant Toeplitz extension.*

We will sometimes identify the Toeplitz extension with the pair (P, π).

Theorem 3.2. *An extension $[\mathcal{E}] \in \mathcal{E}xt_G(\mathcal{A}, \mathfrak{I})$ is invertible if and only if $[\mathcal{E}]$ can be represented by a G-equivariant Toeplitz extension.*

For equivariant extensions of C^*-algebras this statement is proved in [14] (Lemma 3.2) and the case G trivial is well studied in [10] and [2]. Our proof of Theorem 3.2 is based upon the same ideas adjusted to our setting.

Lemma 3.3. *Every strong equivalence class of an invertible G-equivariant extension is stably equivalent to a G-equivariant Toeplitz extension.*

Proof. Assume that $\mathcal{E}$ is a G-equivariant extension of $\mathcal{A}$ by $\mathfrak{I}$ with equivariant Busby mapping $\beta_1 : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}$ which is invertible in $\mathcal{E}xt_G(\mathcal{A}, \mathfrak{I})$. By definition there is a mapping $\beta_2 : \mathcal{A} \to \mathcal{C}_{\mathfrak{I}}$ and a $U \in U^a(M_2 \otimes \mathfrak{I})$ such that

$$U^*(\beta_1 \oplus \beta_2)U : \mathcal{A} \to M_2 \otimes \mathcal{C}_{\mathfrak{I}}$$

can be lifted to an equivariant C^*-bounded representation

$$\pi : \mathcal{A} \to M_2 \otimes \mathcal{M}(I).$$

Let $P \in M_2 \otimes \mathcal{M}(I)$ denote the almost G-invariant projection

$$U^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U.$$

Define

$$\beta'(a) := q_{\mathfrak{I}}(P\pi(a)P), \quad \beta''(a) := q_{\mathfrak{I}}((1-P)\pi(a)(1-P)).$$

For $a \in \mathcal{A}$, we have

$$\begin{aligned}\beta_1(a) &= q_{\mathfrak{I}}(UPU^*)(\beta_1(a) \oplus \beta_2(a))q_{\mathfrak{I}}(UPU^*) \\ &= q_{\mathfrak{I}}(U)q(P\pi(a)P)q_{\mathfrak{I}}(U^*) = q_{\mathfrak{I}}(U)\beta'(a)q_{\mathfrak{I}}(U^*),\end{aligned}$$

which implies that up to strong equivalence β is the Busby mapping of the extension. By the same reasoning β'' is strongly equivalent β_2.

Define $\tau'(a) := P\pi(a)P$ and $\tau''(a) := (1-P)\pi(a)(1-P)$. We express the representation $\pi' := \mathrm{Ad}\, U^* \circ \pi$ as follows

$$\pi'(a) = \begin{pmatrix} U\tau'(a)U^* & \pi_{12}(a) \\ \pi_{21}(a) & U\tau''(a)U^* \end{pmatrix},$$

Since $q_{\mathfrak{I}}\pi' = \beta_1 \oplus \beta_2$, it follows that $\pi_{12}(a), \pi_{21}(a) \in \mathfrak{I}$. The calculation

$$[P, \pi(a)] = U^* \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \pi'(a) \right] U = U^* \begin{pmatrix} 0 & \pi_{12}(a) \\ -\pi_{21}(a) & 0 \end{pmatrix} U \in M_2 \otimes \mathfrak{I},$$

is a consequence of that $M_2 \otimes \mathfrak{I}$ is an ideal in $M_2 \otimes I$ and implies that τ defines a G-equivariant Toeplitz extension. □

Proof of Theorem 3.2. If $[\mathcal{E}]$ is invertible it is given by a Toeplitz extension by Lemma 3.3. Conversely assume that $\mathcal{E}$ is a G-equivariant Toeplitz extension (π, P) of $\mathcal{A}$. We define $P' := 1 - P$, $P_2 := P \oplus P'$, $\tau(a) := P\pi(a)P$ and $\tau'(a) := P'\pi(a)P'$. Then the claim from which the theorem will follow is that the Busby mapping $q_{\mathfrak{I}} \circ \tau'$ defines an inverse to $\mathcal{E}$. To prove this, we define the almost G-invariant symmetry

$$U := \begin{pmatrix} P & P' \\ P' & P \end{pmatrix}.$$

This symmetry satisfies $UP_2U = 1 \oplus 0$. We note that $(\pi \oplus \pi, P_2)$ and $(U\pi \oplus \pi U, P_2)$ define the same extension because of Proposition 1.5 and that the pair (π, P) are $\mathfrak{I}$-almost commuting. Since

$$\pi(a) \oplus 0 = UP_2U(\pi(a) \oplus \pi(a))UP_2U$$

it follows that

$$\begin{aligned} [q_{\mathfrak{I}} \circ \tau] + [q_{\mathfrak{I}} \circ \tau'] &= [q_{\mathfrak{I}} \circ (P_2(\pi \oplus \pi)P_2)] = [q_{\mathfrak{I}} \circ (UP_2U^2(\pi \oplus \pi)U^2P_2U)] \\ &= [q_{\mathfrak{I}} \circ (UP_2U(\pi \oplus \pi)UP_2U)] = [q_{\mathfrak{I}} \circ \pi \oplus 0] = 0. \end{aligned}$$ □

Suppose that we are in the situation $G = \{e\}$. In this case we are able to calculate an inverse to extensions admitting positive splitting if we enlarge the ideal somewhat. This should be thought of as passing from $\mathcal{L}^n(H)$ to $\mathcal{L}^{2n}(H)$. First we need an abstract notion of this procedure.

Proposition 3.4. *Suppose that $\mathfrak{I}$ is a C^*-stable G-ideal. The $*$-algebra*

$$\mathcal{J}_{\mathfrak{I}} := l.s.\{x \in I : x^*x \in \mathfrak{I} \quad \text{and} \quad xx^* \in \mathfrak{I}\}.$$

defines a C^*-stable G-ideal $(\mathcal{J}_{\mathfrak{I}}, I) \in C^*SI_G$. We will call $\mathcal{J}_{\mathfrak{I}}$ the square root of $\mathfrak{I}$.

Proof. Define the two $*$-invariant subsets $\mathcal{J}_{\mathfrak{I}}^+ := \{x \in I : x^*x \in \mathfrak{I}\}$ and $\mathcal{J}_{\mathfrak{I}}^- := \{x \in I : xx^* \in \mathfrak{I}\}$. For $x \in \mathcal{J}_{\mathfrak{I}}^+$ and $a \in \mathcal{M}(I)$, $(xa)^*xa \in \mathfrak{I}$ so $xa \in \mathcal{J}_{\mathfrak{I}}^+$. Since $\mathcal{J}_{\mathfrak{I}}^+$ is $*$-invariant, $ax \in \mathcal{J}_{\mathfrak{I}}^+$. Similarly, if $x \in \mathcal{J}_{\mathfrak{I}}^+$ and

$a \in \mathcal{M}(I)$ we have that $ax(ax)^* \in \mathfrak{I}$ so $ax \in \mathcal{J}_{\mathfrak{I}}^-$ and $xa \in \mathcal{J}_{\mathfrak{I}}^-$. The $*$-algebra $\mathcal{J}_{\mathfrak{I}} \equiv l.s.(\mathcal{J}_{\mathfrak{I}}^+ \cap \mathcal{J}_{\mathfrak{I}}^-)$ so $\mathcal{J}_{\mathfrak{I}}$ is an ideal in $\mathcal{M}(I)$. There is an embedding $\mathfrak{I} \subseteq \mathcal{J}_{\mathfrak{I}}$ because $\mathfrak{I}$ is a $*$-algebra, so $\mathcal{J}_{\mathfrak{I}}$ is dense in I. □

Theorem 3.5. *Let $\mathcal{E}$ be an extension of $\mathcal{A}$ by $\mathfrak{I}$ admitting a C^*-bounded splitting κ extending to a completely positive contraction $\kappa : A \to \mathcal{M}(I)$. If $i : \mathfrak{I} \to \mathcal{J}_{\mathfrak{I}}$ is the embedding of $\mathfrak{I}$ into its square root, $i_*[q_{\mathfrak{I}} \circ \kappa]$ is invertible in $\mathcal{E}xt(\mathcal{A}, \mathcal{J}_{\mathfrak{I}})$.*

Before proving this we need to review the useful construction of the Stinespring representation. This is a standard method for operator algebras and was first introduced by Stinespring in [13].

Theorem 3.6 (Stinespring Representation Theorem)**.** *Assume that A is a separable C^*-algebra, I is a stable C^*-algebra and that $\kappa : A \to \mathcal{M}(I)$ is a completely positive mapping such that $\|\kappa\| \leq 1$. Then there exists a $*$-homomorphism $\pi_\kappa : A \to M_2 \otimes \mathcal{M}(I)$ of A such that*

$$\begin{pmatrix} \kappa(a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pi_\kappa(a) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The $*$-homomorphism π_κ is called a Stinespring representation of κ. For proof see [10].

Lemma 3.7. *Assume that $\kappa : A \to \mathcal{M}(I)$ is a completely positive contraction. In the notation above*

$$\{a \in A : \kappa(a^2) - \kappa(a)^2 \in \mathfrak{I}\} = \{a \in A : [P, \pi_\kappa(a)] \in \mathcal{J}_{\mathfrak{I}}\},$$

where $P := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Proof. We express the representation as follows

$$\pi(a) = \begin{pmatrix} \kappa(a) & \pi_{12}(a) \\ \pi_{21}(a) & \pi_{22}(a) \end{pmatrix},$$

where $\pi_{12}(a) = P\pi(a)(1-P)$ and so on. This implies that $\pi_{12}(a)^* = \pi_{21}(a^*)$. Since π is a representation

$$\begin{pmatrix} \kappa(ab) & * \\ * & * \end{pmatrix} = \pi(ab) = \pi(a)\pi(b) = \begin{pmatrix} \kappa(a)\kappa(b) + \pi_{12}(a)\pi_{21}(b) & * \\ * & * \end{pmatrix}. \tag{5}$$

So

$$\kappa(ab) - \kappa(a)\kappa(b) = \pi_{12}(a)\pi_{21}(b).$$

Thus $\kappa(a^2) - \kappa(a)^2 \in \mathfrak{I}$ if and only if $\pi_{12}(a)\pi_{21}(a) \in \mathfrak{I}$. After polarization we only need to show that this is equivalent to the statement $[P, \pi_\kappa(a)] \in \mathcal{J}_{\mathfrak{I}}$ for self adjoint a. But

$$[P, \pi(a)] = \begin{pmatrix} 0 & \pi_{12}(a) \\ -\pi_{21}(a) & 0 \end{pmatrix}$$

implies

$$|[P,\pi(a)]|^2 = -[P,\pi(a)]^2 = \begin{pmatrix} \pi_{12}(a)\pi_{21}(a) & 0 \\ 0 & \pi_{21}(a)\pi_{12}(a) \end{pmatrix} \in M_2 \otimes \mathfrak{I} \tag{6}$$

It follows from (6) that $\pi_{12}(a)\pi_{21}(a) \in \mathfrak{I}$ if and only if $|[P,\pi_\kappa(a)]|^2 \in \mathfrak{I}$ if and only if $[P,\pi_\kappa(a)] \in \mathcal{J}_\mathfrak{I}$. □

This proves Theorem 3.5 since this implies that κ defines a Toeplitz extension of $\mathcal{A}$ by $\mathcal{J}_\mathfrak{I}$ and by Theorem 3.2 the element $i_*[q_\mathfrak{I} \circ \kappa]$ is invertible in $\mathcal{E}xt(\mathcal{A},\mathcal{J}_\mathfrak{I})$.

To see the square root of a C^*-stable ideal is needed sometimes, consider the Besov space $\mathcal{A} = \mathcal{B}_p^{1/p}$ on the circle S^1. This carries a representation

$$\pi : \mathcal{A} \to \mathcal{B}(L^2(S^1))$$

by multiplication as functions. Let P be the Hardy projection. By [12], if $a \in L^\infty(S^1)$ then $[P,\pi(a)] \in \mathcal{L}^p(L^2(S^1))$ if and only if $a \in \mathcal{A}$. Making a similar decomposition of π as in the proof of Lemma 3.7 one can show that the completely positive mapping $\tau(a) := P\pi(a)P$ is a splitting of an extension of $\mathcal{A}$ by $\mathcal{L}^{p/2}$. Since

$$\mathcal{A} \equiv \{a \in L^\infty(S^1) : [P,\pi(a)] \in \mathcal{L}^p(L^2(S^1))\}$$

it follows that $[q_{\mathcal{L}^{p/2}} \circ \tau] \in \mathcal{E}xt(\mathcal{A},\mathcal{L}^{p/2})$ is not invertible by Theorem 3.2. But if $i : \mathcal{L}^{p/2} \to \mathcal{L}^p$ denotes the inclusion mapping (which coincides with the mapping constructed in Proposition 3.4) the element $i_*[q_{\mathcal{L}^{p/2}} \circ \tau] \in \mathcal{E}xt(\mathcal{A},\mathcal{L}^p)$ is invertible by Theorem 3.2.

4. Example: Extensions of $C^\infty(M)$ by Schatten ideals

Commutative C^*-algebras have many good properties such as nuclearity and concrete realizations in geometry. The geometric interpretations of extensions of commutative C^*-algebras over a manifold, such as Toeplitz operators and pseudodifferential operators, are motivating for extension theory and allows for very concrete smooth $*$-subalgebras to do calculations in.

For example, the one-dimensional case $M = \mathbb{T}$ can be handled fairly straightforwardly by finding an invertible generator for $\mathcal{E}xt^{-1}(C^\infty(S^1),\mathcal{L}^p)$ for $p \geq 2$ precisely as is done for $C(S^1)$ in Chapter 7 in [6]. To find a set of generators in the general setting will be difficult. But a more abstract approach together with a topological description of K-homology of smooth manifolds shows that the Θ-mapping in fact is a surjection for $\mathcal{A} = C^\infty(M)$ and $\mathfrak{I}$ being a Schatten ideal or a Dixmier ideal.

Theorem 4.1. *Let $p > n$. Assume that M is a compact manifold of dimension n and $\mathcal{A} = C^\infty(M)$. Then the mappings*

$$\Theta^{\mathcal{A}}_{\mathcal{L}^{n+}} : \mathcal{E}xt(\mathcal{A}, \mathcal{L}^{n+}) \to \mathrm{Ext}(C(M), \mathcal{K}) = K_1(M) \quad \text{and}$$
$$\Theta^{\mathcal{A}}_{\mathcal{L}^p} : \mathcal{E}xt(\mathcal{A}, \mathcal{L}^p) \to \mathrm{Ext}(C(M), \mathcal{K})$$

are surjective.

Proof. Using the definition of topological K-homology, see [1], one sees that a class in $K_1^{\mathrm{top}}(M) \cong K^1(C(M)) \cong \mathrm{Ext}(C(M), \mathcal{K})$ can be represented as the Fredholm module associated to a 0:th order pseudodifferential operator F over M and the representation π being pointwise multiplication of functions on $L^2(M, E)$ for some vector bundle E. Since F is of order 0 the commutator $[F, \pi(a)]$ is of order -1 for $a \in \mathcal{A}$. Thus $[F, \pi(a)] \in \mathcal{L}^{n+}(L^2(M, E))$ so (F, π) is an $\mathcal{A}-\mathcal{L}^{n+}$-Kasparov module. Therefore $\mathcal{E}xt(\mathcal{A}, \mathcal{L}^{n+}) \to \mathrm{Ext}(C(M), \mathcal{K})$ is surjective. A similar argument to the above one implies that $\Theta^{\mathcal{A}}_{\mathcal{L}^p} : \mathcal{E}xt(\mathcal{A}, \mathcal{L}^p) \to \mathrm{Ext}(C(M), \mathcal{K})$ is surjective. □

5. Deformations of Toeplitz extensions

To end this paper we will look at a certain part of the set $\Theta^{-1}[(P, \pi)]$ for a Toeplitz extension (P, π). The part of $\Theta^{-1}[(P, \pi)]$ we will study are linear perturbations of the projection P. We will give an example of a smooth family of this type of linear deformations which gives a family of extensions $(x_\varepsilon)_{\varepsilon \in (1/2p, 2/p)} \subseteq \mathcal{E}xt(C^\infty(S^1), \mathcal{L}^p)$ such that the the endpoints are non-equivalent. This example shows that $\mathcal{E}xt$ is not a homotopy invariant but carries more analytic information than similar bivariant theories.

If (P, π) defines an $\mathfrak{I}$-summable Toeplitz extension we say $x \in \mathcal{E}xt(\mathcal{A}, \mathfrak{I})$ is a linear deformation of (P, π) by $T \in PIP$ if x can be represented by an extension with a splitting of the form $\tau_T : a \mapsto (P+T)\pi(a)(P+T)$. Observe that $T \in PIP \subseteq I$ implies that $\Theta(P, \pi) = \Theta(x)$. For $a, b \in \mathcal{A}$ we have that

$$\begin{aligned}
&\tau_T(ab) - \tau_T(a)\tau_T(b) \\
&\qquad = (P+T)\pi(ab)(P+T) - (P+T)\pi(a)(P+T)^2\pi(b)(P+T) \\
&\qquad = \pi(ab)(P+T)^2(P - (P+T)^2) + [P+T, \pi(ab)](P+T) \\
&\qquad\quad + (P+T)\pi(a)[\pi(b), (P+T)^2](P+T) \\
&\qquad\quad + [\pi(ab), (P+T)](P+T)^3,
\end{aligned}$$

so a sufficient condition for the operator T to define a linear deformation is that $T^* - T, T^2 + 2T \in \mathfrak{I}$ and $[T, \pi(a)] \in \mathfrak{I}$ for all $a \in \mathcal{A}$.

The main example of a linear deformation is when one considers different representatives of Toeplitz extensions via a pseudo-differential operator on a manifold. Assume that D is a self-adjoint, elliptic pseudo-differential operator on a smooth, compact manifold M without boundary and let us take P as the spectral projection onto the positive spectrum of D. The operator P is a pseudo-differential operator of order 0 so $[P, a] \in \mathcal{L}^p(L^2(M))$ for any

$a \in C^\infty(M)$ and any $p > n$. Therefore the linear mapping $\tau(a) := PaP$ defines an $\mathcal{L}^p$-summable Toeplitz extension of $C^\infty(M)$. Let us take one more self-adjoint, elliptic pseudo-differential operator K of order $\varepsilon > n/2p$ and consider the order $-\varepsilon$ operator

$$T = P(K(1+K^2)^{-1/2} - 1)P.$$

The operator T satisfies the identity

$$T^2 + 2T = (T+P)^2 - P = -P(1+K^2)^{-1}P.$$

So the operator T satisfies $T^2 + 2T \in \mathcal{L}^p$ since we choose K to have order bigger than $n/2p$. While T is of order $-\varepsilon$, $[T, \pi(a)] \in \mathcal{L}^p(L^2(M))$ and T is self-adjoint since K is self-adjoint. Therefore the linear mapping

$$\tau_T(a) := (P+T)a(P+T)$$

defines an extension which is a linear deformation of τ.

The model case of the above setting is $K = D$. In this case the operator $P + T$ is given by $PD(1+D^2)^{-1/2}P$. Up to a finite rank operator, we have that $P = \frac{1}{2}(D|D|^{-1} + 1)$ where the compact operator $|D|^{-1}$ can be defined as the inverse of $\sqrt{D^*D}$ on the range of D^*D and defined to be 0 on the finite-dimensional space $\ker(D^*D)$. Define the order 0 pseudo-differential operator

$$\tilde{P}_D := \frac{1}{2}(D(1+D^2)^{-1/2} + 1).$$

Since $t/|t| - t(1+t^2)^{-1/2} = \mathcal{O}(t^{-2})$ as $t \to \infty$ and the order of D is larger than $n/2p$ we have that

$$PD(1+D^2)^{-1/2}P - \tilde{P}_D \in \mathcal{L}^p(L^2(M)).$$

Therefore the linear deformation of τ by $P(D(1+D^2)^{-1/2} - 1)P$ coincides in $\mathcal{E}xt(C^\infty(M), \mathcal{L}^p)$ with the extension defined by the linear mapping $a \mapsto \tilde{P}_D a \tilde{P}_D$.

In general, we can not say more of T than $T \in \mathcal{L}^{n/\varepsilon}$ since the pseudo-differential operator $K(1+K^2)^{-1/2} - 1$ is of order $-\varepsilon$. As a consequence, if $\varepsilon < n/p$ one can not expect that the mappings $q_{\mathcal{L}^p} \circ \tau$ and $q_{\mathcal{L}^p} \circ \tau_T$ coincide. We will by an example show that the two mappings may even lie in different strong equivalence classes.

Lemma 5.1. *Let P be the Hardy projection on S^1 and assume that $T \in \mathcal{K}(H^2(S^1))$ is defined as $Tz^k := \lambda_k z^k$ for some positive sequence $(\lambda_k)_{k\in\mathbb{N}}$ converging to* 0. *If $a \in C^\infty(S^1)$ is given by $a(z) := z$ then for any $p \geq 1$ and any unitary $U \in \mathcal{B}(H^2(S^1))$ we have that*

$$\|U^*PaPU - (P+T)a(P+T)\|_{\mathcal{L}^p(H^2(S^1))} \geq \|T\|_{\mathcal{L}^p(H^2(S^1))}.$$

Proof. We will use the notation $e_k(z) := z^k$ for $k \geq 0$ and $f_k := Ue_k$. Our first observation is that

$$(P+T)a(P+T)e_k = (1 + \lambda_{k+1} + \lambda_k + \lambda_k\lambda_{k+1})e_{k+1}. \tag{7}$$

If we set $L = U^*PaPU - (P+T)a(P+T)$ we have that

$$L^*L = S_1 + S_2 - S_3 - S_4$$

where

$$\begin{aligned} S_1 &:= U^*Pa^*PaPU, \\ S_2 &:= (P+T)a^*(P+T)^2a(P+T), \\ S_3 &:= (P+T)a^*(P+T)U^*PaPU \quad \text{and} \\ S_4 &:= U^*Pa^*PU(P+T)a(P+T). \end{aligned}$$

Using (7) we obtain the following equalities:

$$\begin{aligned} \langle S_1e_k, e_k\rangle &= \|Paf_k\|^2 = 1, \\ \langle S_2e_k, e_k\rangle &= \|(P+T)a(P+T)e_k\|^2 = (1+\lambda_{k+1}+\lambda_k+\lambda_k\lambda_{k+1})^2, \\ \langle S_3e_k, e_k\rangle &= \overline{\langle S_3e_k, e_k\rangle} = (1+\lambda_{k+1}+\lambda_k+\lambda_k\lambda_{k+1})\langle af_k, f_{k+1}\rangle. \end{aligned}$$

Using these calculations the fact that $\lambda_k, \lambda_{k+1} \geq 0$ together with the elementary estimate $|\langle af_k, f_{k+1}\rangle| \leq 1$ implies that

$$\begin{aligned} \langle L^*Le_k, e_k\rangle &= 1 + (1+\lambda_{k+1}+\lambda_k+\lambda_k\lambda_{k+1})^2 \\ &\qquad - 2(1+\lambda_{k+1}+\lambda_k+\lambda_k\lambda_{k+1})\Re\langle af_k, f_{k+1}\rangle \\ &= 1 - |\langle af_k, f_{k+1}\rangle|^2 \\ &\qquad + |1 - \langle af_k, f_{k+1}\rangle + \lambda_{k+1} + \lambda_k + \lambda_k\lambda_{k+1}|^2 \\ &\geq (\lambda_{k+1}+\lambda_k+\lambda_k\lambda_{k+1})^2 \geq |\lambda_k|^2. \end{aligned}$$

After reordering the sequence λ_k into a decreasing sequence, we have that the singular values $(\mu_k(L))_{k\in\mathbb{N}}$ satisfies that $\mu_k(L) \geq \|Le_k\| \geq |\lambda_k|$, so by Lidskii's theorem

$$\|U^*PaPU - (P+T)a(P+T)\|^p_{\mathcal{L}^p(H^2(S^1))} = \sum_{k\in\mathbb{N}} \mu_k(L)^p \geq \sum_{k\in\mathbb{N}} |\lambda_k|^p. \quad \square$$

Proposition 5.2. *For any $p > 1$ there is a smooth family*

$$(T_\varepsilon)_{\varepsilon\in(1/2p,2/p)} \subseteq \mathcal{L}^{2p}(H^2(S^1))$$

such that the linear deformations of the Toeplitz extension on the Hardy space by T_ε defines a family $(x_\varepsilon)_{\varepsilon\in(1/2p,2/p)} \subseteq \mathcal{E}xt(C^\infty(S^1), \mathcal{L}^p)$ where $x_\varepsilon \neq x_{\varepsilon+1/p}$ for $\varepsilon \in (1/2p, 1/p)$.

If we would replace the $\mathcal{E}xt$-invariant by for instance kk-theory, see more in [5], one would not be able to separate the elements x_ε and $x_{\varepsilon+1/p}$ since the smooth family $(T_t)_{t\in[\varepsilon,\varepsilon+1/p]}$ can be used to construct a homotopy between the classification mappings of the extensions x_ε and $x_{\varepsilon+1/p}$.

Proof. Let us start by defining the smooth family $(T_\varepsilon)_{\varepsilon\in(1/2p,2/p)}$. We define T_ε for each $\varepsilon \in (1/2p, 2/p)$ in the same way as in Lemma 5.1 from the sequence

$$\lambda_{k,\varepsilon} := 1 - |k|^\varepsilon(1+|k|^{2\varepsilon})^{-1/2}.$$

This choice of $\lambda_{k,\varepsilon}$ coincides with that in the example above when $K = |\mathrm{d}/\mathrm{d}\theta|^{\varepsilon}$. Since $\varepsilon \mapsto \lambda_{k,\varepsilon}$ is smooth, so is $\varepsilon \mapsto T_\varepsilon$. The sequence $(\lambda_{k,\varepsilon})_{k\in\mathbb{Z}}$ behaves asymptotically as $|k|^{-\varepsilon}$ so $(\lambda_{k,\varepsilon})_{k\in\mathbb{Z}} \in \ell^{2p}(\mathbb{N})$ since $\varepsilon > 1/2p$.

When $\varepsilon \in (1/p, 2/p)$ the sequence $(\lambda_{k,\varepsilon})_{k\in\mathbb{Z}}$ is p-summable. Therefore $(T_\varepsilon)_{\varepsilon\in(1/p,2/p)} \subseteq \mathcal{L}^p(H^2(S^1))$ and τ_{T_ε} is isomorphic to the Toeplitz extension on the Hardy space for $\varepsilon \in (1/p, 2/p)$. However, when $\varepsilon < 1/p$ we have that $(\lambda_{k,\varepsilon})_{k\in\mathbb{Z}} \notin \ell^p(\mathbb{N})$. The norm estimate of the differences of the Toeplitz extension on the Hardy space and a deformation by T_ε in Lemma 5.1 implies that for any unitary $U \in \mathcal{B}(H^2(S^1))$

$$U^*PaPU - (P + T_\varepsilon)a(P + T_\varepsilon) \notin \mathcal{L}^p(H^2(S^1)).$$

Therefore τ is not strongly equivalent to τ_{T_ε} for $\varepsilon \in (1/2p, 1/p)$ and $x_\varepsilon \neq x_{\varepsilon+1/p}$ for $\varepsilon \in (1/2p, 1/p)$. □

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