

# Heegaard splittings and virtually Haken Dehn filling. II

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ABSTRACT. We use Heegaard splittings to give a criterion for a tunnel number one knot manifold to be nonfibered and to have large cyclic covers. We also show that a knot manifold satisfying the criterion admits infinitely many virtually Haken Dehn fillings. Using a computer, we apply this criterion to the 2 generator, nonfibered knot manifolds in the cusped Snappea census. For each such manifold  $M$ , we compute a number  $c(M)$ , such that, for any  $n > c(M)$ , the  $n$ -fold cyclic cover of  $M$  is large.

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## 1. Introduction

This paper continues the project, begun in [18], of using Heegaard splittings to construct closed essential surfaces in finite covers of 3-manifolds. The idea, based on the work of Casson and Gordon [5], is to lift a Heegaard splitting of a 3-manifold to a finite cover in which there are disjoint compressing disks on each side. By compressing the lifted Heegaard surface along an appropriate choice of such disks, we hope to arrive at an essential surface.

By a *knot manifold* we mean a connected, compact, orientable 3-manifold whose boundary is a single torus. A *tunnel system* for a knot manifold  $M$  is a collection  $\{t_1, \dots, t_n\}$  where the  $t_i$ 's are disjoint, properly embedded arcs in  $M$ , such that  $\overline{M - N(\bigcup t_i)}$  is homeomorphic to a handlebody. The *tunnel number* of  $M$ , denoted  $t(M)$ , is the minimal cardinality of a tunnel system for  $M$ .

We focus attention on tunnel number one, nonfibered knot manifolds. These are obtained by attaching a single 2-handle to a genus two handlebody. We shall give a condition on the 2-handle which, if satisfied, ensures that in all large enough cyclic covers, the lifted Heegaard splitting can be compressed to obtain an essential surface. There is also a statement about incompressibility after Dehn surgery.

Freedman and Freedman ([9]) have already proved that for *any* nonfibered knot manifold, all but finitely many cyclic covers are large (i.e., contain closed essential surfaces). Cooper and Long [7] then proved a result about virtually Haken Dehn surgery for these manifolds, and also obtained a bound on the number of excluded covers in terms of the genus of the knot.

However, our results provide a computational benefit. We computed, for all of the 453 nonfibered, 2-generator knot manifolds in the SnapPea census, a covering degree past which all cyclic covers are large, and the bounds obtained are typically improvements over known bounds.

For other connections between Heegaard splittings and virtually Haken 3-manifolds, see [13], [14], [15], [17].

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## 2. Definitions, notation, and statement of results

Let  $F$  be a connected, closed, orientable surface of positive genus. Recall that a *compression body*  $W$  is a 3-manifold obtained from  $(F \times I)$  by first attaching a collection of 2-handles along disjoint curves in one component of  $\partial(F \times I)$ , and then capping off all resulting 2-sphere boundary components with 3-balls. One of the boundary components of  $W$  is homeomorphic to  $F$ , and is called the *outer boundary* of  $W$ , denoted  $\partial_+ W$ . The other components of  $\partial W$  form the *inner boundary*, denoted  $\partial_- W$ .

If  $X$  is a 3-manifold with boundary, and  $\mathcal{S} \subset X$  is a collection of disjoint compression disks for  $\partial X$ , we let  $X/\mathcal{S} = X - \mathring{N}(\mathcal{S})$ . If  $X$  is a 3-dimensional submanifold of a 3-manifold  $Y$ , and if  $\mathcal{S} \subset Y - \mathring{X}$  is a collection of disjoint compression disks for  $\partial X$ , then we let  $X[\mathcal{S}]$  denote  $X \cup N(\mathcal{S})$ , where  $N(\mathcal{S})$  is a regular neighborhood of  $\mathcal{S}$  in  $Y - \mathring{X}$ .

A *disk system*  $\mathcal{S}$  for a compression body  $W$  is a set of disjoint compressing disks for  $\partial W$  of minimal cardinality such that  $W/\mathcal{S}$  has incompressible boundary. We shall use some basic facts about compression bodies, which can be found in [2].

A *Heegaard splitting* of a compact 3-manifold is a decomposition

$$M = W_1 \cup_F W_2,$$

where the  $W_i$ 's are compression bodies with outer boundary homeomorphic to  $F$ . The *Heegaard genus* of  $M$ , denoted  $g(M)$ , is the minimal genus of  $F$  for all such decompositions. If  $M$  has boundary, then a *tunnel system* for  $M$  is a collection of properly embedded arcs in  $M$ , whose exterior is a handlebody. The *tunnel number* of  $M$ , denoted  $t(M)$ , is the minimal cardinality among all tunnel systems for  $M$ . It is an elementary fact that, if  $M$  is a knot manifold, then  $g(M) = t(M) + 1$ .

For the remainder of the paper,  $M$  will be a fixed knot manifold, with incompressible boundary, and with  $t(M) = 1$  and  $b_1(M) = 1$ . Thus there is a Heegaard splitting  $M = H \cup_F W$ , where  $H$  is a genus 2 handlebody, and  $W$  is a genus 2 compression body. Let  $\mathcal{D} = D_1 \cup D_2$  be a disk system for  $H$ , and let  $E$  be the unique (up to isotopy) nonseparating compression disk for  $W$ . We assume that  $E$  has been isotoped so that every component of  $\partial E - \mathring{N}(\mathcal{D})$  represents an essential arc in  $F - \mathring{N}(\mathcal{D})$ .

Since  $b_1(M) = 1$ , there is a unique surjective homomorphism  $\phi : \pi_1 M \rightarrow \mathbb{Z}$ , where  $\mathbb{Z}$  is the free factor of  $H_1(M)$ . Let  $M_n$  denote the corresponding  $n$ -fold cyclic cover, with  $M_\infty$  denoting the infinite cyclic cover. Let  $H_n, W_n$  and  $F_n$  be the preimages in  $M_n$  of  $H, W$  and  $F$ , respectively. Then

$$M_n = H_n \cup_{F_n} W_n$$

is a Heegaard splitting of  $M_n$  of genus  $n + 1$ .

Let  $\alpha_1, \alpha_2 \subset F$  be simple closed curves transverse to  $\partial \mathcal{D}$  such that

$$|\alpha_i \cap D_j| = \delta_{ij}$$

(the Kronecker delta function). We also assume that  $\alpha_1$  and  $\alpha_2$  intersect (nontransversely) in a single point  $p$ , which will be the base point for  $\pi_1 M$ , and we assign orientations to  $\alpha_i$  and  $\partial D_i$  so that the algebraic intersection numbers  $I(\alpha_i, \partial D_i)$  are both  $+1$ . We call such pair of curves  $\{\alpha_1, \alpha_2\}$  *dual curves* for the disk system  $\{D_1, D_2\}$  of  $H$ .

**Lemma 2.1.** *We may choose a disk system  $\mathcal{D}$  so that  $\phi(\alpha_1) = 0$  and  $\phi(\alpha_2) = 1$ .*

**Proof.** Suppose  $\phi(\alpha_i) = n_i$ , and that  $|n_1| \leq |n_2|$ . Let  $\delta$  be an oriented embedded arc in  $\alpha_1 \cup \alpha_2$  such that  $\delta \cap (D_1 \cup D_2) = \partial\delta$ , and that its orientation agrees with the orientation on  $\alpha_2$ , but disagrees with the orientation on  $\alpha_1$ . Let  $D'_1$  be a properly embedded disk in  $H$  obtained by band sum of  $D_1$  and  $D_2$  along the arc  $\delta$ , and let  $D'_2 = D_2$ . Then obviously we may assume that  $D'_1$  and  $D'_2$  are disjoint, and see that they form a disk system for  $H$ . Choose  $\alpha'_1 = \alpha_1$ ,  $\alpha'_2 = \alpha_2 \alpha_1^{-1}$ . Then up to an obvious homotopy of  $\alpha'_2$  in  $\partial H$ , we may consider  $\alpha'_2$  as a simple closed curve, and see that  $\{\alpha'_1, \alpha'_2\}$  form a dual curve pair for the disk system  $\mathcal{D}' = \{D'_1, D'_2\}$  (with a suitable choice of orientation for  $\partial D'_1$ ). Further we have that  $\phi(\alpha'_1) = n_1$ , and  $\phi(\alpha'_2) = n_2 - n_1$ .

Now we replace the disk system  $\mathcal{D}$  with the disk system  $\mathcal{D}'$ , and repeat the above procedure. Applying the Euclidean Algorithm, we may continue until we have a disk system  $\mathcal{D}$  for which  $\phi(\alpha_1) = 0$  (say) and  $\phi(\alpha_2) = \gcd(n_1, n_2)$ . Since  $\text{Image}(\phi) = \mathbb{Z}$ ,  $n_1$  and  $n_2$  are relatively prime, so the resulting disk system satisfies the requirements of Lemma 2.1.  $\square$

For the remainder of the paper, we shall assume that the disk system  $\mathcal{D}$  has been chosen as in Lemma 2.1.

Let  $p \in F - \partial\mathcal{D}$  be a base point for  $M$ , and let  $p^1, \dots, p^n$  be the lifts to  $M_n$  (where  $n \in \mathbb{Z}^+ \cup \{\infty\}$ ), with the natural indexing. Let  $\delta_1, \delta_2$  in  $F - \partial\mathcal{D}$  be arcs connecting  $p$  with  $\partial D_i$ . Let  $D_i^j \subset M_n$  denote the lift of  $D_i$  to  $M_n$  corresponding to the lift of  $\delta_i$  with base point  $p_j$ . (In our notation, we have suppressed the dependence of  $D_i^j$  on  $n$ , trusting the meaning to be clear from context.) Note that  $D_1^j$  is between  $D_2^{j-1}$  and  $D_2^j$ . See Figure 1.

Let  $I_{\text{geo}}(\cdot, \cdot)$  be the geometric intersection pairing, and for a loop  $\ell$  in  $F_\infty$ , define the *width* of  $\ell$  to be:

$$\text{width}(\ell) = \text{Max}(j \mid I_{\text{geo}}(\ell, D_2^j) \neq 0) - \text{Min}(j \mid I_{\text{geo}}(\ell, D_2^j) \neq 0) + 2;$$

in the special case where  $\ell$  is disjoint from all  $D_2^j$ 's, we define the width to be one. If  $\ell$  is a loop in  $F$  which lifts to  $F_\infty$ , and  $\tilde{\ell}$  and  $\tilde{\ell}'$  are any two lifts to  $F_\infty$ , then  $\text{width}(\tilde{\ell}) = \text{width}(\tilde{\ell}')$ . Thus for any such loop, we define  $\text{width}(\ell) = \text{width}(\tilde{\ell})$ , where  $\tilde{\ell}$  is any lift of  $\ell$  to  $F_\infty$ .

Since  $E$  bounds a disk in  $W$ , then  $\partial E$  lifts to  $F_\infty$ , and we set (for the remainder of the paper)

$$k = \text{width}(\partial E).$$

Since  $\partial M$  is incompressible, then  $E$  intersects  $D_1$  and  $D_2$  nontrivially. If  $n \geq k$ , let  $E^j \subset M_n$  denote the lift of  $E$  to  $M_n$  which intersects  $D_1^j$ , but is disjoint from  $D_2^j$ . Then  $D_2^1 \cup \bigcup_{j=1}^n D_1^j$  forms a disk system for  $H_n$  and  $\bigcup_{j=1}^n E^j$  forms a disk system for  $W_n$  (cf. Figure 1 for an example with  $k = 3$  and  $n = 4$ ).

Set  $H'_n = H_n/D_2^1$ , set  $\mathcal{E}_j = \{E^1, \dots, E^j\}$ , and  $\mathcal{E}_j^{(i)} = \mathcal{E}_j - E^i$ ,  $1 \leq i \leq j$ . Recall that a collection  $\mathcal{C}$  of disjoint simple closed curves in the boundary of a handlebody  $X$  is *disk busting* if  $\partial X - \mathcal{C}$  is incompressible in  $X$ .

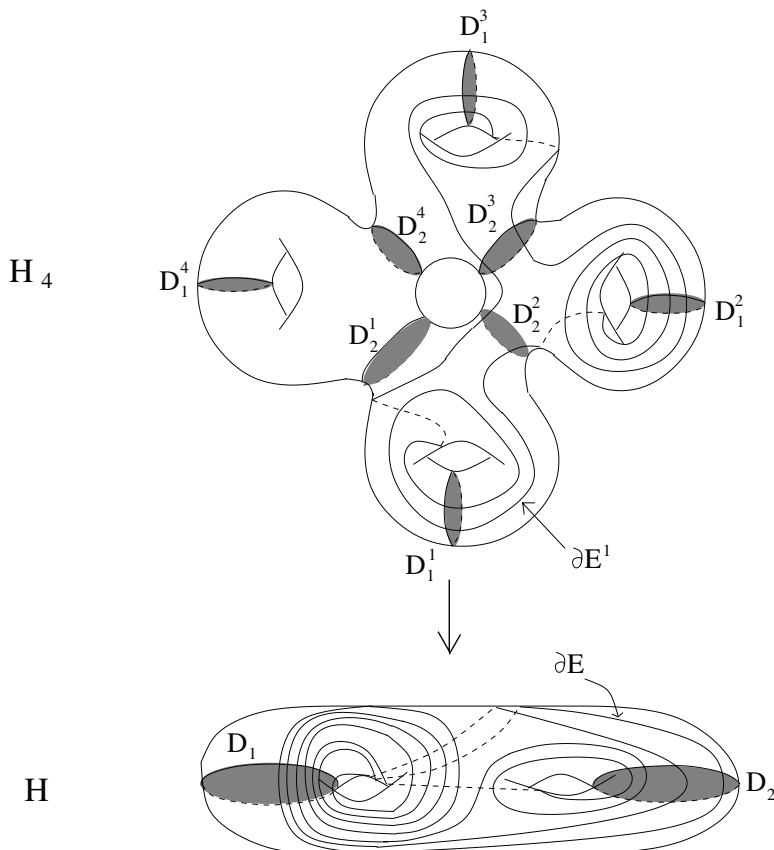


FIGURE 1. An example of a 4-fold cyclic cover

**Definition 2.2.** Let  $m \geq 1$  be an integer. We say that  $E$  satisfies the  $m$ -lift condition if:

- (1) The set  $\partial \mathcal{E}_m$  is disk-busting in  $H'_{m+k-1}$ .
- (2) The set  $\partial \mathcal{E}_m^{(i)}$  is not disk-busting in  $H'_{m+k-1}$ , for all  $1 \leq i \leq m$ .
- (3) For each  $1 \leq i \leq m$ , there is a compression disk

$$\Delta_i \subset H'_{m+k-1} - \partial \mathcal{E}_m^{(i)}$$

such that  $[\partial \Delta_i]$  is linearly independent from  $\{[\partial E^{i+1}], \dots, [\partial E^m]\}$  in  $H_1(\partial H'_{m+k-1}; \mathbb{Q})$ .

**Remark.** If  $M = H[E]$  is fibered, then it follows from Lemma 4.1 below that condition (1) fails, so  $E$  does not satisfy the  $m$ -lift condition for any  $m$ .

Recall that a 3-manifold  $M$  is *large* if it is irreducible, and contains a closed essential surface, i.e., an incompressible surface which is not parallel to a component of  $\partial M$ . We prove:

**Theorem 2.3.** *If  $E$  satisfies the  $m$ -lift condition, then  $M_n$  is large for any  $n \geq \text{Max}(m + k - 1, 2k - 2)$ .*

Let  $\lambda$  be a longitude for  $M$  (i.e., a simple, closed, essential curve in  $\partial M$  which lifts to a loop in  $M_\infty$ ). Fix a meridian  $\mu$  for  $M$  (i.e., a simple closed curve in  $\partial M$  which intersects  $\lambda$  exactly once). A slope  $p/q$  in  $\partial M$  means the pair of homology class  $\pm(p[\mu] + q[\lambda])$  and  $(p, q) = 1$ . We use  $M(p/q)$  to denote the closed manifold obtained by Dehn filling of  $M$  with slope  $p/q$ . Let  $b = |\phi(\mu)|$ . Then  $b > 0$  is a finite integer. Then we have:

**Theorem 2.4.** *If  $E$  satisfies the  $m$ -lift condition, then  $M(np/q)$  is virtually Haken for any  $p \geq 2$ ,  $n \geq \text{Max}\{m + k - 1, 2k - 2, \text{width}(\lambda) + b\}$ , and  $q$  with  $(pn, q) = 1$ .*

Given a 2-generator, 1-relator presentation of a 3-manifold group, there is an algorithm to decide if this presentation corresponds to a genus 2 Heegaard splitting (conjecturally it always does). From the data of such a geometric presentation, it is possible to check if the  $m$ -lift condition holds for a given  $m$ . Using the computer program GAP, we have shown:

**Theorem 2.5.** *Every 2-generator, 1-relator 3-manifold  $M$  in the SnapPea census of 1-cusped hyperbolic 3-manifolds has a genus 2 Heegaard splitting. Moreover, if  $b_1(M) = 1$  and  $M$  is nonfibered, then  $M$  has a genus 2 Heegaard splitting whose 2-handle satisfies the  $m$ -lift condition for some  $m$ .*

A complete table of the values of  $m$  is available at  
[www.math.buffalo.edu/~jdmaster](http://www.math.buffalo.edu/~jdmaster).

The first few values are given in Table 1.

To prove these theorems, consider the surface  $F_n$  (recall this is the preimage of the Heegaard surface  $F$  in  $M_n$ ). Then  $F_n$  is a Heegaard surface for  $M_n$  of genus  $n + 1$ , which we shall compress to both sides. On the handlebody side, we compress  $F_n$  along a single lift,  $D_2^1$ , of  $D_2$ ; on the compression body side we compress  $F_n$  along all the lifts of  $E$  which are disjoint from  $D_2^1$ . We shall show that if  $E$  satisfies the  $m$ -lift condition and  $n \geq m + k - 1$ , then the resulting surface is incompressible.

Let

$$\begin{aligned} X_n &= (W_n/\mathcal{E}_{n-k+1})[D_2^1], \\ Y_n &= (H_n/D_2^1)[\mathcal{E}_{n-k+1}], \text{ and} \\ S_n &= \partial Y_n. \end{aligned}$$

Note that  $M_n - \mathring{N}(S_n) \cong X_n \amalg Y_n$ .

**Lemma 2.6.** *The surface  $S_n$  is connected, has genus  $= k - 1$ , and is not parallel to  $\partial M_n$ .*

**Proof.** To prove that  $S_n$  is connected, it is enough to show that

$$[\partial D_2^1], [\partial E^1], \dots, [\partial E^n]$$

are linearly independent in  $H_1(F_n)$ .

Let  $I(\cdot, \cdot)$  denote the algebraic intersection pairing on the first homology group of a surface. Recall from Lemma 2.1 that  $\phi(\alpha_2) = 1$  is a generator of  $\phi(\pi_1(M)) = \mathbb{Z}$ . It follows that  $I([\partial E], [\partial D_2]) = 0$  and that  $\{[\partial E], [\partial D_2]\}$  are linearly independent. We may complete  $\{[\partial E], [\partial D_2]\}$  to a symplectic basis  $\{[\partial E], [\partial E]^*, [\partial D_2], [\partial D_2]^*\}$ , i.e., a basis satisfying

$$\begin{aligned} I([\partial E], [\partial E]^*) &= I([\partial D_2], [\partial D_2]^*) = 1, \\ I([\partial E]^*, [\partial D_2]) &= I([\partial E]^*, [\partial D_2]^*) = I([\partial E], [\partial D_2]) = I([\partial E], [\partial D_2]^*) = 0. \end{aligned}$$

Let  $\alpha \subset F$  be an embedded loop representing  $[\partial E]^*$ , and intersecting  $\partial E$  geometrically exactly once (such representative always exists). Then  $\alpha$  lifts homeomorphically to loops  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \subset F_n$  (since  $I([\alpha], [\partial D_2]) = 0$ ). As  $\partial E$  also lifts to  $F_n$  as  $\partial E^1, \dots, \partial E^n$ , it is easy to see that  $I(\tilde{\alpha}_i, \partial E^j) = \delta_{ij}$ . Also

$$\begin{aligned} &I([\tilde{\alpha}_i], [\partial D_2^1]) \\ &= I([\tilde{\alpha}_i], [\bigcup_j \partial D_2^j]/n) \text{ (since all lifts of } \partial D_2 \text{ are homologous in } F_n) \\ &= I([\alpha], [\partial D_2])/n \\ &= 0. \end{aligned}$$

Recall that we have a map  $\phi : \pi_1 M \rightarrow \mathbb{Z}$ ; let  $\beta \subset \partial M$  be a loop with  $\phi(\beta) \neq 0$ , and let  $\beta_n$  be the preimage in  $M_n$ . Then  $[\beta_n], [\tilde{\alpha}_1], \dots, [\tilde{\alpha}_n]$  are dual classes for  $[\partial D_2^1], [\partial E^1], \dots, [\partial E^n]$  in  $H_1(F_n, \mathbb{Q})$ , which proves the linear independence, and completes the proof that  $S_n$  is connected.

The linear independence of  $[\partial D_2^1], [\partial E^1], \dots, [\partial E^n]$  also allows us to compute:

$$\begin{aligned} \text{genus}(S_n) &= \text{genus}(H_n/D_2^1) - |\mathcal{E}_{n-k+1}| \\ &= n - (n - k + 1) = k - 1. \end{aligned}$$

Finally, note that  $S_n$  is not parallel into  $\partial M_n$ , since every loop in  $S_n$  projects to an element in  $\ker(\phi)$  (because  $S$  is disjoint from  $D_2^1$ ), but each component of  $\partial M_n$  contains a loop whose projection is not in  $\ker(\phi)$ .  $\square$

To prove Theorems 2.3 and Theorem 2.4, we shall show that when  $n \geq \text{Max}\{m + k - 1, 2k - 2\}$ ,  $S_n$  is incompressible in both  $X_n$  and  $Y_n$ , and that when  $n \geq \text{Max}\{m + k - 1, 2k - 2, \text{width}(\lambda)\}$ ,  $S_n$  remains incompressible in an equivariant Dehn filling of  $M_n$  along  $\partial M_n$  (which may have several components) which is a free cyclic cover of  $M(np/q)$ .

### 3. Background on 1-relator groups and 3-manifolds

We will require the following result of Magnus (see [16]). The statement given here is easily seen to be equivalent to the standard statement.

**Theorem 3.1** (Freiheitsatz for 1-relator groups). *Let*

$$G = \langle x_1, \dots, x_n \mid w(x_1, \dots, x_n) = 1 \rangle$$

*be a 1-relator group, where  $w$  is a freely reduced word. Let  $\mathcal{X} = \{x_1, \dots, x_n\}$ , let  $\mathcal{X}^* \subset \mathcal{X}$ , and suppose that some  $x_i \in \mathcal{X} - \mathcal{X}^*$  appears in  $w$ . Then  $\mathcal{X}^*$  freely generates a free subgroup of  $G$ .  $\square$*

**Corollary 3.2.** *Let  $w(x_1, \dots, x_k)$  be a word in which  $x_1$  and  $x_k$  appear nontrivially, and consider the group*

$$G = \langle \dots, x_{-1}, x_0, x_1, \dots \mid w(x_i, \dots, x_{i+k-1}) = 1, \forall i \in \mathbb{Z} \rangle.$$

*Then for any  $i$ , the set  $\{x_i, \dots, x_{i+k-2}\}$  freely generates a free subgroup of  $G$ .*

**Proof.** Let  $G_i = \langle x_i, \dots, x_{i+k-1} \mid w(x_i, \dots, x_{i+k-1}) \rangle$ , and let  $J_i = \langle x_{i+1} \rangle * \dots * \langle x_{i+k-1} \rangle$ . By repeated applications of Theorem 3.1, the group  $G$  has the structure of the following iterated amalgamated free product over free subgroups:  $G \cong \dots *_{J_{i-1}} G_i *_{J_i} G_{i+1} *_{J_{i+1}} \dots$ . By Theorem 3.1, each  $J_i$  injects into  $G_i$  and  $G_{i+1}$ . Each  $G_i$  thus injects into  $G$ , and so we obtain the corollary.  $\square$

Suppose  $G = \langle x, y \mid w(x, y) \rangle$  is a 1-relator group (where  $w$  is a cyclically reduced word) which admits a surjective homomorphism  $\psi : G \rightarrow \mathbb{Z}$ , such that  $\psi(y) = 0$ . Let  $x_i = x^{-i} y x^i$ . Then  $\ker(\psi)$  is generated by the  $x_i$ 's, and the relation  $w$  lifts to a relation  $\tilde{w}$  on the  $x_i$ 's, so that  $\ker(\psi)$  has a presentation as in the statement of Corollary 3.2. Write  $\tilde{w} = \prod_j x_{\mu_j}$ , and consider the finite integer sequence  $(\mu_j)$ . Then we have:

**Theorem 3.3** (Brown). *If  $(\mu_j)$  has a repeated minimum (or maximum) (i.e., it assumes its minimum (or maximum) value more than once), then  $\ker(\psi)$  is not finitely generated.*

The case of a repeated minimum is a special instance of Theorem 4.2 in [4], and the case of a repeated maximum follows from a trivial modification of the proof (which is an application of the Freiheitsatz).

We also need the following, a special case of Corollary 2.2 of [5], which is in turn a slight modification of a theorem proved by Jaco in [12].

**Theorem 3.4** (Handle Addition Lemma). *Let  $M$  be an irreducible 3-manifold with compressible boundary of genus at least 2, and suppose  $\alpha \subset \partial M$  is a simple closed curve, such that  $\partial M - \alpha$  is incompressible in  $M$ . Then the 3-manifold obtained by adding a 2-handle to  $M$  along  $\alpha$  is irreducible, and has incompressible boundary.*

## 4. 2-handles in nonfibered manifolds

The results presented in this section are essentially combinations of results due to Brown ([4]) and Bieri–Neumann–Strebel ([1]).



If  $D$  is a compressing disk for  $H_\infty$ , and  $\alpha$  is a simple closed curve in  $H_\infty$ , then let  $I_{\text{geo}}(\alpha, D)$  be the geometric intersection number of  $\alpha$  and  $D$ ; in other words,  $I_{\text{geo}}(\alpha, D)$  is the minimal cardinality of  $\alpha' \cap D$  over all curves  $\alpha'$  which is isotopic to  $\alpha$  in  $H_\infty$ .

If  $\mathcal{D}$  is a disk system for  $H$ , then there are dual simple loops which form a free basis for  $\pi_1 H$ . If  $D \in \mathcal{D}$  corresponds to the generator  $x_D$ , and if  $\alpha$  is a simple loop in  $H$ , then  $I_{\text{geo}}(\alpha, D)$  is the number of times the generator  $x_D$  appears in a cyclically reduced representative for the conjugacy class of  $[\alpha]$  in  $\pi_1 H$ .

**Lemma 4.1.** *Suppose  $M$  is a knot manifold with  $t(M) = 1$ , and suppose that  $E^i$  and  $D_i^j$  are as defined in Section 2.*

- (a) *If  $M$  is fibered, then  $I_{\text{geo}}(\partial E^1, D_1^1) = I_{\text{geo}}(\partial E^1, D_1^k) = 1$  in  $H_\infty$ .*
- (b) *If  $M$  is nonfibered, then  $I_{\text{geo}}(\partial E^1, D_1^1) \geq 2$  and  $I_{\text{geo}}(\partial E^1, D_1^k) \geq 2$  in  $H_\infty$ .*

**Proof.** (a) Suppose  $I_{\text{geo}}(\partial E^1, D_1^1) \geq 2$  or  $I_{\text{geo}}(\partial E^1, D_1^k) \geq 2$ . Then by Theorem 3.3 (together with the note in the preceding paragraph of the present lemma),  $\ker(\phi)$  is not finitely generated, and so by [21],  $M$  is not fibered.

(b) Suppose  $E^1$  intersects one of the disks, say  $D_1^k$ , exactly once. We shall show that in this case  $M$  is fibered.

Dual to each  $D_1^i$  is a generator  $x_i$  for the fundamental group of  $M_\infty$ . The boundary of the disk  $E^1$  gives a relation among these generators which involves  $x_k$  only once; therefore  $x_k \in \langle x_1, \dots, x_{k-1} \rangle \subset \pi_1 M_\infty$ . Similarly, using the relation corresponding to the disk  $E^2$ , we get that

$$x_{k+1} \in \langle x_1, \dots, x_k \rangle = \langle x_1, \dots, x_{k-1} \rangle \subset \pi_1 M_\infty.$$

Continuing in this way, we see that all of the generators  $x_i$  with  $i \geq k$  can be expressed in terms of  $x_1, \dots, x_{k-1}$ .

Let  $H^*$  be the component of  $H_\infty/D_2^1$  containing  $D_1^1, D_1^2, \dots$ , and let  $Q = H^*[E^1, E^2, \dots]$ , which is a submanifold of  $M_\infty$ . The argument we just gave shows that  $\pi_1(Q)$  is finitely generated.

Note that there is a nonseparating incompressible surface  $S$  in  $M$  with boundary slope  $\lambda$  such that  $M_\infty$  is the infinite cover dual to  $S$ . Let  $\tilde{S}$  be a lift of  $S$  to  $M_\infty$  which is disjoint from  $Q$ , and let  $Q^+$  be the component of  $M_\infty - \mathring{N}(\tilde{S})$  which contains  $Q$ . Then  $Q^+ - \mathring{N}(Q)$  is compact, and since  $\pi_1 Q$  is finitely generated,  $\pi_1 Q^+$  is finitely generated as well.

Let  $M_0^- = M - \mathring{N}(S)$ , let  $S_0$  and  $S_1$  be the two preimages of  $S$  in  $\partial M_0^-$ , let  $\tilde{S}_i$  be the preimages of  $S$  in  $M_\infty$ , and let  $M_i^-$  be the submanifold of  $Q$  bounded by  $\tilde{S}_0$  and  $\tilde{S}_i$ . Since  $\tilde{S}_i$  is incompressible,  $M_i^-$  is  $\pi_1$ -injective in  $Q$  for each  $i$ . If neither of the maps  $i_* \pi_1 S_j \rightarrow \pi_1 M_0^-$  is onto, then  $\{\pi_1 M_i^-\}$  forms an infinite sequence of subgroups of  $\pi_1 Q$ , with  $\pi_1 M_{i+1}^-$  properly containing  $\pi_1 M_i^-$  for each  $i$ , which is a contradiction, since  $\pi_1 Q$  is finitely

generated. Therefore, one of the induced maps  $\pi_1 S_j \rightarrow \pi_1 M_0$  is onto, and so, as in the proof of Theorem 2 in [21],  $M$  is fibered.  $\square$

**Corollary 4.2.** *If  $M$  is fibered, then  $E$  does not satisfy the  $m$ -lift condition for any  $m$ .*

**Proof.** Suppose  $M$  is fibered, let  $m \geq 1$  be an integer, and consider the cover  $F_{m+k-1}$  of  $F$ . In  $F_{m+k-1}$ , we have that  $E^j$  is disjoint from  $D_1^1$  for all  $2 \leq j \leq m-k+1$ , and by Lemma 4.1(a),  $|\partial E^1 \cap \partial D_1^1| = 1$ . Therefore there is a compressing disk  $\Delta$  in  $H_{m+k-1}$  (whose boundary is equal to  $\partial N(\partial E^1 \cup \partial D_1^1)$ ) with  $\partial \Delta \cap \partial \mathcal{E}_m = \emptyset$ , and so  $E$  fails condition (2).  $\square$

## 5. Proof of irreducibility of $M_n$

We shall now begin the proof of Theorem 2.3, which will occupy the next three sections.

**Lemma 5.1.**  *$M_n$  is irreducible for all  $n$ .*

**Proof.** By Theorem 3.4,  $M_1 = M$  is irreducible. By [19] (or [8]), the cover of an irreducible manifold is irreducible, so  $M_i$  is irreducible for all  $i \geq 2$ .  $\square$

## 6. Proof that $X_n$ has incompressible boundary

We remark that the  $m$ -lift condition is not needed in this case; we only use the assumption that  $\partial M$  is incompressible.

Let  $W'_n = W_n / \mathcal{E}_{n-k+1}$ . By Theorem 3.4 it is enough to prove:

**Lemma 6.1.** *If  $n \geq 2k - 2$ , then  $W'_n - \partial D_2^1$  has incompressible boundary.*

First we need:

**Lemma 6.2.** *For each  $n$  there is a loop  $\alpha_n \subset \partial M_n$  such that  $I(\alpha_n, D_2^1) \neq 0$ .*

**Proof.** By the exact sequence of the pair, there is a loop  $\alpha \in \partial M$  such that  $\phi[\alpha] = I(\alpha, D_2^1) \neq 0$ . Letting  $\alpha_n$  be the preimage of  $\alpha$  in  $\partial M_n$ , we have  $I(\alpha_n, \bigcup_{i=1}^n D_2^i) = nI(\alpha, D_2) \neq 0$ . Since  $[D_2^i] = [D_2^j] \in H_1(M_n)$  for all  $i, j$ , then we have  $I(\alpha_n, D_2^1) = I(\alpha, D_2) \neq 0$ .  $\square$

**Proof of Lemma 6.1.** Suppose otherwise that there is a compressing disk in  $W'_n - \partial D_2^1$ . First, if there is a compressing disk, we claim that there must be a nonseparating one. To see this, suppose that  $\Delta$  is a separating compressing disk. If there are no nonseparating compressing disks, then one of the components of  $W'_n / \Delta$  is homeomorphic to a surface cross an interval, and the other component is a handlebody containing  $\partial D_2^1$ . Every curve in  $\partial M_n$  lies on the surface cross interval side, but by Lemma 6.2, there is a curve in  $\partial M_n$  which has nontrivial intersection with  $D_2^1$ , yielding a contradiction. So we may assume that there is a nonseparating compressing disk  $\Delta$  in  $W'_n - \partial D_2^1$ .

Consider the Heegaard surface  $F_n$  and the curves  $\partial D_2^j$  and  $\partial E^j$ ,  $j = 1, \dots, n$ , in  $F_n$ . Note that since  $n \geq 2k - 2$ ,  $\{\partial E^{n-k+2}, \dots, \partial E^n\}$  are all disjoint from  $\partial D_2^{n-k+2}$  and  $\partial D_2^k$  (by considering the definition of  $k = \text{width}(E)$ ). The two simple closed curves  $\partial D_2^{n-k+2}$  and  $\partial D_2^k$  cut  $F_n$  into two components,  $F_n^1$  and  $F_n^2$ , one of which, say  $F_n^1$ , is disjoint from all  $\partial E^{n-k+2}, \dots, \partial E^n$ .

The curves  $\partial E^1, \dots, \partial E^{n-k+1}$  may intersect  $\partial D_2^{n-k+2}$  and  $\partial D_2^k$ . But by the property of the width  $k$  again, any arc component of  $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$  has either both endpoints in  $\partial D_2^{n-k+2}$  or both endpoints in  $\partial D_2^k$ . Further, every arc component of  $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$  is disjoint from  $\partial D_2^1$ . Let  $F_n^3$  be the subsurface of  $F_n$  which is the union of  $F_n^1$  and a small regular neighborhood of the arcs  $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$  in  $F_n^2$  (in other words,  $F_n^3$  is  $F_n^1$  with some bands attached, one for each arc component in  $F_n^2 \cap (\cup_{j=1}^{n-k+1} \partial E^j)$ ). Then  $F_n^3$  is a connected subsurface of  $F_n$  which contains all  $\partial E^1, \dots, \partial E^{n-k+1}$  but is disjoint from all  $\partial E^{n-k+2}, \dots, \partial E^n$  and  $\partial D_2^1$ .

Let  $F_n^4$  be the surface obtained from  $F_n^3$  by surgery along the curves  $\partial E^1, \dots, \partial E^{n-k+1}$  (i.e., cut  $F_n^3$  open along  $\{\partial E^1, \dots, \partial E^{n-k+1}\}$  and fill each of the new boundary circles with a disk), which may not be connected. Note that  $\{E^{n-k+2}, \dots, E^n\}$  is a disk system for the compression body  $W'_n$  and  $W'_n / \{E^{n-k+2}, \dots, E^n\}$  is an  $I$ -bundle over a surface. As  $F_n^4$  is disjoint from  $\{\partial E^{n-k+2}, \dots, \partial E^n\}$ ,  $F_n^4$  is contained in the horizontal boundary of the  $I$ -bundle. So  $\partial F_n^4 \times I$  are vertical annuli of this  $I$ -bundle. By standard cut-and-paste operations along arcs and circles of  $\Delta \cap (\partial F_n^4 \times I)$ , we get a nonseparating compressing disk, still denoted  $\Delta$ , which is disjoint from the annuli  $\partial F_n^4 \times I$ . It is easy to see that  $\Delta$  cannot be contained in  $F_n^4 \times I$ , so it follows that  $\partial \Delta$  is disjoint from all  $\partial D_2^1, \partial D_2^k, \dots, \partial D_2^{n-k+2}$ . Hence the width of  $\Delta$  is strictly less than  $k$ .

Let  $\delta = \partial \Delta$ , and let  $\tilde{\delta}$  be a lift of  $\delta$  in  $M_\infty$ . The group  $\pi_1 M_\infty$  has the following presentation:

$$\pi_1 M_\infty = \langle \dots, x_{-1}, x_0, x_1, \dots \mid w(x_i, \dots, x_{i+k-1}) = 1, \forall i \in \mathbb{Z} \rangle,$$

where the relations correspond to the lifts of  $E$ . Since  $\text{width}(\delta) < k$ , the loop  $\tilde{\delta}$  represents an element in the subgroup of  $\pi_1 M_\infty$  generated by the elements  $x_i, \dots, x_{i+k-2}$ , for some  $i$ . By Corollary 3.2, these elements are a basis for a free subgroup; since  $\tilde{\delta}$  represents a trivial element in  $\pi_1 M_\infty$ , we see that  $\tilde{\delta}$  represents the trivial word in  $x_i, \dots, x_{i+k-2}$ . Thus  $\tilde{\delta}$  bounds a disk in  $H_\infty$  by Dehn's lemma, and thus  $\delta$  bounds a disk in  $H_n$ . Thus there is a nonseparating sphere in  $M_n$ , contradicting Lemma 5.1.  $\square$

**Lemma 6.3.** *Suppose  $n \geq 2k - 2$ . Then  $X_n$  has incompressible boundary.*

**Proof.** This is a consequence of Lemma 6.1 and Theorem 3.4.  $\square$

## 7. Proof that $Y_n$ has incompressible boundary

In this section, we are under the assumption that  $\mathcal{E}$  satisfies the  $m$ -lift condition. Recall that  $H'_n = H_n/D_2^1$ .

**Lemma 7.1.** *The curve  $\partial E^{n-k+1}$  cannot be isotoped in  $H'_n[\mathcal{E}^{n-k}]$  to intersect  $D_1^n$  fewer than two times.*

**Proof.** We have

$$\begin{aligned} \pi_1 H'_n[\mathcal{E}^{n-k}] &= \langle x_1, \dots, x_n \mid w_1, \dots, w_{n-k} \rangle \\ &\cong \langle x_1, \dots, x_{n-1} \mid w_1, \dots, w_{n-k} \rangle * \langle x_n \rangle \end{aligned}$$

where  $x_j$  is dual to  $D_1^j$ , and the word  $w_j$  corresponds to  $\partial E^j$ .

The word  $w_{n-k+1}$  can be cyclically permuted to have the form

$$w_{n-k+1} = \mathcal{W}_1 x_n^{\ell_1} \mathcal{W}_2 x_n^{\ell_2} \dots \mathcal{W}_t x_n^{\ell_t},$$

where the  $\mathcal{W}$ 's are freely reduced words involving only  $x_j$ 's with  $n-k+1 \leq j \leq n-1$ , and each  $\mathcal{W}_j$  represents a nontrivial element in the group  $\langle x_1, \dots, x_n \mid w_1, \dots, w_{n-k} \rangle$ .

Suppose  $\partial E^{n-k+1}$  can be isotoped to be disjoint from  $D_1^n$  in  $H'_n[\mathcal{E}^{n-k}]$ . Then, using the relations  $w_j$ ,  $j \leq n-k$ , the word  $w_{n-k+1}$  can be rewritten entirely in terms of  $x_j$ 's,  $j \leq n-1$ , and this would imply that one of the  $\mathcal{W}_j$ 's must be trivial in  $\pi_1 H'_n[\mathcal{E}^{n-k}]$ . However,  $\mathcal{W}_j$  is a freely reduced word on  $x_{n-k+1}, \dots, x_{n-1}$ , which by Corollary 3.2 freely generate a free subgroup, for a contradiction.

Suppose  $\partial E^{n-k+1}$  can be isotoped to intersect  $D_1^n$  exactly once. Then, as in the proof of Lemma 4.1,  $M$  is fibered, and so by Corollary 4.2,  $\mathcal{E}$  does not satisfy the  $m$ -lift condition, for a contradiction.  $\square$

**Lemma 7.2.** *If  $n \geq m+k-1$ , then  $Y_n$  has incompressible boundary.*

**Proof.** Let  $n_0 = m+k-1$ . We first prove the result in the case where  $n = n_0$ .

**Claim.** *The manifold  $Y_{n_0}$  has incompressible boundary.*

**Proof.** Recall  $H'_{n_0} = H_{n_0}/D_2^1$ . By condition (2) of Definition 2.2, the manifold

$$H'_{n_0} - \partial \mathcal{E}_m^{(m)} = H'_{n_0} - \partial \mathcal{E}_{m-1}$$

has compressible boundary, and by condition (1),  $H'_{n_0} - \partial \mathcal{E}_m$  has incompressible boundary; therefore, by Theorem 3.4, the manifold  $(H'_{n_0} - \partial \mathcal{E}_{m-1})[E^m]$  has incompressible boundary.

Suppose for some  $i \in [1, m]$ , that  $H'_{n_0}[\overline{\mathcal{E}}_i] - \partial \mathcal{E}_i$  has incompressible boundary, where  $\overline{\mathcal{E}}_i = \mathcal{E}_m - \mathcal{E}_i$ . By condition (2) of Definition 2.2, there is a compression disk for  $H'_{n_0} - \partial \mathcal{E}_m^{(i)}$  which is also a compression disk for  $H'_{n_0}[\overline{\mathcal{E}}_i] - \partial \mathcal{E}_{i-1}$ . Therefore, by Theorem 3.4,  $H'_{n_0}[\overline{\mathcal{E}}_{i-1}] - \partial \mathcal{E}_{i-1}$  has incompressible boundary.

By induction on  $i$ , it follows that  $Y_{n_0} = H'_{n_0}[\mathcal{E}_m]$  has incompressible boundary.  $\square$

Now, suppose  $n > n_0$ . and proceed by induction on  $n$ . Suppose  $Y_n$  has incompressible boundary. The manifold  $Y_{n+1}$  is obtained from  $Y_n$  by adding a 1-handle  $Z$  to  $H'_n$ , and then attaching a 2-handle  $E^{n-k+2}$ . We claim that  $(Y_n \cup Z) - \partial E^{n-k+2}$  has incompressible boundary.

Suppose otherwise, and let  $\Delta$  be a compressing disk. Since  $Y_n$  has incompressible boundary, the maximal compression body for  $\partial(Y_n \cup Z)$  has a unique disk system consisting only of  $D_1^{n+1}$ . Therefore if  $\Delta$  is nonseparating, it is isotopic to  $D_1^{n+1}$ ; then  $\partial E^{n-k+2}$  can be isotoped off of  $\partial D_1^{n+1}$  in  $Y_n \cup Z$ , contradicting Lemma 7.1.

Suppose  $\Delta$  is separating, so it separates off a solid torus  $V \subset H'_{n+1}$ , with  $V \supset D_1^{n+1}$ . Since  $\partial E^{n-k+2} \cap D_1^{n+1} \neq \emptyset$ , we have  $\partial E^{n-k+2} \subset V$ .

So  $Y_{n+1}$  contains the punctured lens space  $V[E^{n-k+2}]$ . By Lemma 7.1, this lens space cannot be  $B^3$ . So  $Y_n$  is not irreducible, contradicting Lemma 5.1. This completes the proof that  $(Y_n \cup Z) - \partial E^{n-k+2}$  has incompressible boundary. Then by Theorem 3.4,  $Y_{n+1}$  has incompressible boundary.  $\square$

## 8. Proof of Theorem 2.3 and Theorem 2.4

**Proof of Theorem 2.3.** Assume  $M$  satisfies the hypotheses of Theorem 2.3. Then by Lemmas 6.3 and 7.2,  $M_n$  contains an incompressible closed surface  $S_n$ , which is not parallel into  $\partial M_n$  by Lemma 2.6.  $\square$

**Proof of Theorem 2.4.** Suppose the hypotheses of Theorem 2.4 are satisfied. Let  $b_n$  be the number of boundary components of  $M_n$ . Then it is easy to see that  $b_n$  is equal to the largest common divisor of  $n$  and  $b$  (recall from the proceeding paragraph of Theorem 2.4 that  $b = |\phi(\mu)|$  is a finite integer). Let each boundary component of  $\partial M_n$  have the coordinate basis induced from the basis  $\{\mu, \lambda\}$  of  $\partial M$ . Let  $M_n(b_n p/q)$  denote the closed manifold obtained by Dehn filling each component of  $\partial M_n$  with slope  $b_n p/q$ . Then it is easy to check that  $M(np/q)$  is cyclically covered by  $M_n(b_n p/q)$ . We have shown that  $M_n$  contains an incompressible surface  $S_n$ . When  $n \geq \text{width}(\lambda) + b$  as well, there are  $b$  successive lifts of  $\lambda$  contained in  $S_n = \partial Y_n$ , which implies that  $S_n$  has an annular compression to each component of  $\partial M_n$ , with slope 0. Since  $p > 1$ , then by repeatedly applying Theorem 2.4.3 of [6]  $b_n$  times, we see that  $S_n$  remains incompressible in  $M_n(b_n p/q)$ . Also, by [20],  $M_n(b_n p/q)$  is irreducible. Hence  $M(np/q)$  is virtually Haken.  $\square$

## 9. Brief description of algorithm

**A. Checking that SnapPea presentations are geometric.** Given an algebraic word on  $x$  and  $y$ , we need to know if it can be represented by a simple closed curve in the boundary of a genus 2 handlebody. To do this, we attempt to draw a Heegaard diagram. So we start with four disks in the

plane, corresponding to  $x, x^{-1}, y$  and  $y^{-1}$ . The word  $w$  is represented by a collection of edges connecting these disks, and we need to find a representation which embeds in the plane.

To program this on the computer, we start placing edges on the graph one by one, as indicated by the word  $w$ . When placing an edge, the initial point is determined from the previous step, but there may be a choice of terminal point. However, it is possible to keep track of the choices which are made, and if an impossible situation is arrived at, we retreat to the previous choice, and change it. In this way, the computer found for every given word, a geometric representation.

**B. Checking that the  $m$ -lift condition holds.** The only nonelementary step in checking the  $m$ -lift condition algorithmically is to find, for a given collection of loops in the fundamental group of a handlebody  $H$ , a specific compressing disk for  $H$  which is disjoint from the loops. An algorithm for this was given by Whitehead ([23], or see [22]), which we implemented on GAP.

Note that Whitehead's algorithm allows one to determine the existence of a compressing disk in polynomial time in the length of the word; however, to construct the disk explicitly requires exponential time. For our application, we are saved this difficulty, since for each compressing disk  $\Delta$  we only need to compute  $[\partial\Delta] \in H_1(\partial H)$ . This allows the algorithm to run in polynomial time.

TABLE 1. Data on SnapPea census manifolds

manifold name:	width of 2-handle:	satisfies $m$ -lift for:	$n$ -fold cyclic cover large for:
m006	$k = 3$	$m = 2$	$n \geq 4$
m007	$k = 3$	$m = 2$	$n \geq 4$
m015	$k = 4$	$m = 2$	$n \geq 6$
m017	$k = 3$	$m = 1$	$n \geq 4$
m029	$k = 3$	$m = 2$	$n \geq 4$
m030	$k = 3$	$m = 2$	$n \geq 4$
m032	$k = 3$	$m = 2$	$n \geq 4$
m033	$k = 3$	$m = 2$	$n \geq 4$
m035	$k = 3$	$m = 1$	$n \geq 4$
m037	$k = 3$	$m = 1$	$n \geq 4$
m045	$k = 3$	$m = 1$	$n \geq 4$
m046	$k = 3$	$m = 1$	$n \geq 4$
m053	$k = 3$	$m = 1$	$n \geq 4$
m054	$k = 3$	$m = 2$	$n \geq 4$
m058	$k = 3$	$m = 1$	$n \geq 4$

m059	$k = 3$	$m = 1$	$n \geq 4$
m061	$k = 3$	$m = 2$	$n \geq 4$
m062	$k = 3$	$m = 2$	$n \geq 4$
m066	$k = 3$	$m = 1$	$n \geq 4$
m067	$k = 3$	$m = 1$	$n \geq 4$
m073	$k = 3$	$m = 1$	$n \geq 4$
m074	$k = 3$	$m = 2$	$n \geq 4$
m076	$k = 3$	$m = 1$	$n \geq 4$
m077	$k = 3$	$m = 1$	$n \geq 4$
m079	$k = 3$	$m = 1$	$n \geq 4$
m080	$k = 3$	$m = 1$	$n \geq 4$
m084	$k = 3$	$m = 1$	$n \geq 4$
m085	$k = 3$	$m = 1$	$n \geq 4$
m089	$k = 3$	$m = 1$	$n \geq 4$
m090	$k = 3$	$m = 1$	$n \geq 4$
m093	$k = 3$	$m = 2$	$n \geq 4$
m094	$k = 3$	$m = 2$	$n \geq 4$
m104	$k = 3$	$m = 1$	$n \geq 4$
m105	$k = 3$	$m = 3$	$n \geq 5$
m110	$k = 3$	$m = 1$	$n \geq 4$
m111	$k = 3$	$m = 1$	$n \geq 4$
m137	$k = 3$	$m = 2$	$n \geq 4$
m139	$k = 4$	$m = 3$	$n \geq 6$
m148	$k = 3$	$m = 1$	$n \geq 4$
m149	$k = 3$	$m = 1$	$n \geq 4$
m202	$k = 4$	$m = 2$	$n \geq 6$
m203	$k = 4$	$m = 1$	$n \geq 6$
m208	$k = 4$	$m = 1$	$n \geq 6$
m249	$k = 5$	$m = 4$	$n \geq 8$
m259	$k = 5$	$m = 3$	$n \geq 8$
m260	$k = 5$	$m = 3$	$n \geq 8$
m261	$k = 3$	$m = 1$	$n \geq 4$
m262	$k = 3$	$m = 1$	$n \geq 4$
m285	$k = 3$	$m = 1$	$n \geq 4$
m286	$k = 3$	$m = 1$	$n \geq 4$
m287	$k = 5$	$m = 5$	$n \geq 9$
m288	$k = 5$	$m = 3$	$n \geq 8$
m292	$k = 5$	$m = 3$	$n \geq 8$
m319	$k = 3$	$m = 1$	$n \geq 4$

m320	$k = 3$	$m = 1$	$n \geq 4$
m328	$k = 4$	$m = 1$	$n \geq 6$
m329	$k = 4$	$m = 2$	$n \geq 6$
m340	$k = 5$	$m = 1$	$n \geq 8$
m357	$k = 4$	$m = 2$	$n \geq 6$
m366	$k = 4$	$m = 1$	$n \geq 6$
m388	$k = 4$	$m = 1$	$n \geq 6$

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