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The relative Riemann–Roch theorem from Hochschild homology

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Abstract. This paper attempts to clarify a preprint of Markarian (2001). Markarian's preprint proves the relative Riemann–Roch theorem using a result describing how the HKR map fails to respect comultiplication. This paper elaborates on the core computations in Markarian's preprint. These computations show that the HKR map twisted by the square root of the Todd genus "almost preserves" the Mukai pairing. This settles a part of a conjecture of Caldararu, 2005. The relative Riemann–Roch theorem follows from this and a result of Caldararu, preprint, 2003.

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Introduction

The purpose of this paper is to explain in detail an alternative approach to the relative Riemann–Roch theorem which first appeared in a very interesting but cryptic preprint [5] of Markarian. This approach leads to a proof of the relative Riemann–Roch theorem by a direct computation of the pairing on Hodge cohomology to which the Mukai pairing on Hochschild homology defined by Caldararu [1] descends via the Hochschild–Kostant–Rosenberg map multiplied by the square root of the Todd genus. In the framework of this approach, the fundamental reason for the appearance of the Todd genus in the Riemann–Roch theorem is the failure of the Hochschild–Kostant–Rosenberg map $\mathrm{I}_{\mathrm{HKR}}$ introduced in Section 2 to respect comultiplication. One of the main theorems in [5] (Theorem 1 of [5]) describes the Duflo like error term that measures by how much $\mathrm{I}_{\mathrm{HKR}}$ fails to respect comultiplication. The proof supplied in [5] however, has a nontrivial error. A "dual" version of this result equivalent to the original theorem has since been correctly proven by the author in [7]. A correct proof of another version of this result has been outlined by Markarian himself in a revised version [6] of [5]. Theorem 1 of [5] appears as Theorem $2'$ in this paper.

Let X be a smooth proper scheme over a field $\mathbb{K}$ of characteristic 0. We use Theorem $2'$ of this paper to prove the main result (Theorem 1) of the current paper. This explicitly interrelates the Hochschild–Kostant–Rosenberg map, the twisted HKR map introduced in Section 1 and a map which we call the duality map between $\mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$ and $\mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X)$. Here, S_X is the shifted line bundle on X tensoring with which is the Serre duality functor on $\mathrm{D}^b(X)$. Theorem 1 of this paper is equivalent to a corrected version of an erroneous result (Theorem 8 of [5]) that appears in [5].

Theorem 1 enables us to compute the pairing on Hodge cohomology to which the Mukai pairing on Hochschild homology defined by Caldararu [1] descends via the HKR-map twisted by the square root of the Todd genus. Given Theorem 1, doing this computation is fairly easy. This pairing on

Hodge cohomology (see Proposition 5 of this paper) is very similar to the generalized Mukai pairing Caldararu defined in [2]. In particular, it satisfies the adjointness property one expects from the Mukai pairing. Moreover, it coincides with the Mukai pairing defined by Caldararu [2] on Mukai vectors (see [2]) of elements of $\mathrm{D}^b(X)$. However, the pairing obtained in Proposition 5 is not exactly the same as Caldararu's Mukai pairing on Hodge cohomology. This settles a part of Caldararu's conjecture in [2] regarding the equivalence between the Hochschild and Hodge structures of a smooth proper complex variety — to be able to say that the HKR map twisted by the square root of the Todd genus preserves the Mukai pairing, one has to replace the Mukai pairing defined by Caldararu [2] with the pairing that shows up in Proposition 5. Since pairing in Proposition 5 does not in general coincide with the Mukai pairing on Hodge cohomology for K3-surfaces, we stay contented by saying that the HKR map twisted by the square root of the Todd genus "almost preserves" the Mukai pairing.

The relative Riemann–Roch theorem follows from Proposition 5 and the adjointness property of the Mukai pairing on Hochschild homology. The adjointness property of the Mukai pairing on Hochschild homology was proven in a paper [1] of Caldararu.

In order to prove Theorem 1, we elaborate upon the core computations in [5]. These computations appeared in [5] in a very cryptic way. Some of these computations do not appear in [6], whose approach differs in some details from [5]. In particular, unlike [5], [6] does not contain a result equivalent to Theorem 1 and does not compute the Mukai pairing on Hochschild homology at the level of Hodge cohomology.

The key steps in this computational approach are covered by Theorem $2'$ and Lemmas 2, 3 and 4 of this paper. The most crucial computations, Theorem $2'$ and Lemma 4 are related to very familiar computations in elementary Lie theory. Theorem $2'$ is related to computing the pullback of a left invariant 1-form on a Lie group G via the exponential map. Similarly, Lemma 4 is related computing the pullback of a left invariant volume form on G via the the map $\overline{\exp}$ where $\overline{\exp}(Z) = \exp(-Z)$ for any element Z of the Lie algebra $\mathfrak{g}$ of G. We aim to make these relations transparent by developing a "dictionary" in this paper in three separate subsections containing remarks meant for this purpose only.

The layout of this paper. Section 1 begins by introducing the notations and conventions that shall be used for the rest of this paper. It then goes on to state Theorem 1 after defining the various maps involved in Theorem 1. The pairing on Hodge cohomology to which the Mukai pairing on Hochschild homology descends (via the HKR map twisted by the square root of the Todd genus) is then computed. Finally, Section 1 uses this computation to prove the relative Riemann–Roch theorem. The remaining sections of this paper are devoted to proving Theorem 1.

Section 2 introduces two “connections” on the complex of completed Hochschild chains of a smooth scheme X. Their properties are proven in various propositions in this section. This section also proves Theorem $2'$. Sections 2.3 and 2.5 develop the “dictionary” making the analogy between Theorem $2'$ and its counterpart in elementary Lie theory more transparent.

Section 3 consists of a number of definitions, technical propositions and two lemmas (Lemma 1 and Lemma 2) pertaining mainly to linear algebra. These are used in later sections at various points. The definitions of this section are important to understand later sections. Proofs of propositions in later sections time and again refer to propositions in this section.

The key result of Section 4 is Lemma 3. This in turn follows from Lemma 4 and Lemma 2. Besides proving Lemma 3 and Lemma 4, Section 4 has a subsection (Section 4.3) which explains the analogy between Lemma 4 and its counterpart in basic Lie theory. Section 4.3 is the last of the three sections (2.3, 2.5 and 4.3) developing the “dictionary” in this paper.

Section 5 undertakes the final computations leading to the proof of Theorem 1.

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1. The main theorem, the Mukai pairing and the relative Riemann–Roch theorem

We begin this section by clarifying some notation and conventions that shall be followed throughout this paper. Immediately after that, in Section 1.1, we state the main theorem (Theorem 1) of this paper after describing the maps involved. This section then goes on to explain in detail why Theorem 1 implies the relative Riemann–Roch theorem. This is done in Sections 1.2 and 1.3.

Notation and conventions. Let X be a smooth proper scheme over a field $\mathbb{K}$ of characteristic 0. All schemes and complex varieties that we encounter in this paper are assumed to be proper. Let $\Delta : X \to X \times X$ denote the diagonal embedding. Let p_1 and p_2 denote the projections from $X \times X$ onto the first and second factors respectively. As usual, $\mathcal{O}_X$ denotes the structure sheaf of X.

$\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$ (resp. $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$ and $\mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod})$) denotes the category of bounded (resp. bounded above and bounded below) chain complexes of $\mathcal{O}_X$-modules with coherent cohomology. $\mathrm{D}^b(X)$ denotes the bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology. Similarly, $\mathrm{D}^b(X \times X)$ denotes the bounded derived category of complexes of $\mathcal{O}_{X\times X}$-modules with coherent cohomology.

Whenever $f : X \to Y$ is a morphism of schemes, $f^* : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ denotes the left derived functor of the pullback via f. Similarly, $f_* : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ denotes the right derived functor of the push-forward via f. $\mathcal{O}_\Delta$ will denote $\Delta_*\mathcal{O}_X$. Similarly, when we refer to a tensor product, we will mean the corresponding left-derived functor unless stated otherwise explicitly. Also, if $\mathcal{E}$ is an object of $\mathrm{D}^b(X)$ and φ is a morphism in $\mathrm{D}^b(X)$, $\mathcal{E} \otimes \varphi$ shall denote the morphism $\mathbf{1}_{\mathcal{E}} \otimes \varphi$. At times, $\mathcal{E}$ shall be used to denote the morphism $\mathbf{1}_{\mathcal{E}}$.

If $\mathcal{E}$ and $\mathcal{F}$ are objects of $\mathrm{D}^b(X)$, we shall denote $\mathrm{RHom}_{\mathrm{D}^b(X)}(\mathcal{E}, \mathcal{F})$ by $\mathrm{RHom}_X(\mathcal{E}, \mathcal{F})$. Also, we denote $\mathrm{RHom}_{\mathrm{D}^b(X\times X)}(-,-)$ by $\mathrm{RHom}_{X\times X}(-,-)$.

Ω_X denotes the cotangent bundle of X. $\mathbf{S}^\bullet(\Omega_X[1])$ will denote the symmetric algebra generated over $\mathcal{O}_X$ by Ω_X concentrated in degree -1. Note that $\mathbf{S}^\bullet(\Omega_X[1]) = \oplus_i \wedge^i \Omega_X[i]$. We shall often denote $\wedge^i\Omega_X$ by Ω^i_X. From Section 2 onwards, Ω_X and $\mathbf{S}^\bullet(\Omega_X[1])$ shall be denoted by Ω and $\mathbf{S}^\bullet(\Omega[1])$ respectively. The tangent bundle of X shall be denoted by T_X in this section and T from Section 2 onwards.

Where convenient,we shall denote the Hodge cohomology $\oplus_{p,q}\mathrm{H}^q(X, \Omega^p_X)$ by $\mathrm{H}^*(X)$ and $\mathrm{H}^q(X, \Omega^p_X)$ by $\mathrm{H}^{p,q}(X)$. Note that

$$\mathrm{H}^*(X) = \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^\bullet(\Omega_X[1])).$$

The product on $\mathrm{H}^*(X)$ induced by the wedge product on $\mathbf{S}^\bullet(\Omega_X[1])$ will be denoted by $\wedge$. However, we shall often suppress the $\wedge$: If $a, b \in \mathrm{H}^*(X)$, ab should be understood to mean $a \wedge b \in \mathrm{H}^*(X)$.

Despite our attempts to minimize abuse of notation, it does happen at times. There are many situations in this paper where we encounter maps between tensor products of various objects in $\mathrm{D}^b(X)$ that rearrange factors. Very often, such maps are denoted by the symbol τ. In each such situation, we specify what τ means unless we feel it is obvious to the reader.

1.1. The crux of this paper. Recall from Yekutieli [9] that the completed complex of Hochschild chains $\widehat{C}^\bullet(X) \in \mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$ is a complex of flat $\mathcal{O}_X$-modules that represents $\Delta^*\mathcal{O}_\Delta$ in $\mathrm{D}^b(X)$. Upper indexing is used here to convert what was originally a chain complex into a cochain complex.

The Hochschild–Kostant–Rosenberg (HKR) map $\mathrm{I}_{\mathrm{HKR}}$ from $\widehat{C}^\bullet(X)$ to $\mathbf{S}^\bullet(\Omega_X[1])$ is a map of complexes of $\mathcal{O}_X$ modules. We describe this in greater detail in Section 2. We identify $\Delta^*\mathcal{O}_\Delta$ with $\widehat{C}^\bullet(X)$. Thus, the HKR map $\mathrm{I}_{\mathrm{HKR}}$ can be thought of as a map in $\mathrm{D}^b(X)$ from $\Delta^*\mathcal{O}_\Delta$ to $\mathbf{S}^\bullet(\Omega_X[1])$.

Let S_X denote the object $\Omega^n[n]$ in $\mathrm{D}^b(X)$. Let π_n denote the projection from $\mathbf{S}^\bullet(\Omega_X[1])$ to the direct summmand $\Omega^n[n]$. Consider the pairing

$$\langle -, - \rangle : \mathbf{S}^\bullet(\Omega_X[1]) \otimes \mathbf{S}^\bullet(\Omega_X[1]) \to S_X$$

given by the composite

$$\mathbf{S}^\bullet(\Omega_X[1]) \otimes \mathbf{S}^\bullet(\Omega_X[1]) \xrightarrow{(-\wedge-)} \mathbf{S}^\bullet(\Omega_X[1]) \xrightarrow{\pi_n} S_X$$

of morphisms in $\mathrm{D}^b(X)$. One also has a *twisted HKR map* from $\mathbf{S}^\bullet(\Omega_X[1])$ to $\mathrm{R}\mathcal{H}om_X(\Delta^*\mathcal{O}_\Delta, S_X)$. This arises out of the composite

$$\Delta^*\mathcal{O}_\Delta \otimes \mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\mathrm{I}_{\mathrm{HKR}} \otimes \mathbf{1}_{\mathbf{S}^\bullet(\Omega_X[1])}} \mathbf{S}^\bullet(\Omega_X[1]) \otimes \mathbf{S}^\bullet(\Omega_X[1]) \xrightarrow{\langle,\rangle} S_X. \tag{1}$$

of morphisms in $\mathrm{D}^b(X)$. We denote the twisted HKR map by $\widehat{\mathrm{I}_{\mathrm{HKR}}}$.

The duality map. The material in this paragraph is recalled from Caldararu [1]. Recall that $\Delta^* : \mathrm{D}^b(X \times X) \to \mathrm{D}^b(X)$ is the left adjoint of $\Delta_* : \mathrm{D}^b(X) \to \mathrm{D}^b(X \times X)$. Also recall that the functor of tensoring by the shifted line bundle S_X is the Serre duality functor on $\mathrm{D}^b(X)$. Similarly, tensoring by the shifted line bundle $S_{X\times X}$ is the Serre duality functor on $\mathrm{D}^b(X \times X)$. We denote the functor given by tensoring by a shifted line bundle $\mathcal{L}$ by $\mathcal{L}$ itself. The left adjoint $\Delta_! : \mathrm{D}^b(X) \to \mathrm{D}^b(X \times X)$ of Δ^* is given by ${S_{X\times X}}^{-1}\Delta_* S_X$.

Since $\Delta_!$ is the left adjoint of Δ^* we have an isomorphism

$$\mathcal{I} : \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) \simeq \mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_X, \mathcal{O}_\Delta). \tag{2}$$

Now, $\Delta_!\mathcal{O}_X = {S_{X\times X}}^{-1}\Delta_* S_X \simeq \Delta_* S_X^{-1}$. We also have an isomorphism

$$\mathcal{T} : \mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_X, \mathcal{O}_\Delta) \simeq \mathrm{RHom}_{X\times X}(\mathcal{O}_\Delta, \Delta_* S_X) \tag{3}$$

given by tensoring an element of $\mathrm{RHom}_{X\times X}(\Delta_!\mathcal{O}_X, \mathcal{O}_\Delta)$ on the right by the shifted line bundle $p_2^* S_X$ and making the obvious identifications. Now, since Δ^* is the left adjoint of Δ_* we have an isomorphism

$$\mathcal{J} : \mathrm{RHom}_{X\times X}(\mathcal{O}_\Delta, \Delta_* S_X) \simeq \mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X). \tag{4}$$

Let $D_\Delta : \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) \to \mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X)$ denote the composite $\mathcal{J} \circ \mathcal{T} \circ \mathcal{I}$. We refer to D_Δ as the *duality map*.

The main theorem. The main theorem of this paper relates the HKR, twisted HKR and duality maps. This is a corrected version of Theorem 8 of Markarian's preprint [5].

Note that $\mathrm{I}_{\mathrm{HKR}}$ induces a map

$$\mathrm{I}_{\mathrm{HKR}} : \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) \to \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^\bullet(\Omega_X[1])) = \mathrm{H}^*(X).$$

Similarly, $\widehat{\mathrm{I}_{\mathrm{HKR}}}$ induces a map

$$\widehat{\mathrm{I}_{\mathrm{HKR}}} : \mathrm{H}^*(X) = \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^\bullet(\Omega_X[1])) \to \mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X).$$

Let J denote the endomorphism on $\mathbf{S}^\bullet(\Omega_X[1])$ that multiplies $\wedge^i\Omega_X[i]$ by $(-1)^i$. J induces an endomorphism on $\mathrm{H}^q(X, \Omega_X^p)$. Let

$$\mathrm{td}(T_X) \in \oplus_i \mathrm{H}^i(X, \Omega_X^i)$$

denote the Todd genus of the tangent bundle of X. Recall that the wedge product $(\text{ - } \wedge \text{ - }) : \mathbf{S}^\bullet(\Omega_X[1])^{\otimes 2} \to \mathbf{S}^\bullet(\Omega_X[1])$ induces a product on $\mathrm{H}^*(X)$. We are now in a position to state the main theorem.

Theorem 1. *The following diagram commutes.*

$$\begin{array}{ccc} \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta) & \xrightarrow{D_\Delta} & \mathrm{RHom}_X(\Delta^*\mathcal{O}_\Delta, S_X) \\ \Big\downarrow \mathrm{I_{HKR}} & & \widehat{\mathrm{I_{HKR}}} \Big\uparrow \\ \mathrm{H}^*(X) & \xrightarrow{(-\wedge \mathrm{td}(T_X))\circ J} & \mathrm{H}^*(X). \end{array}$$

The map in the bottom row of the above diagram takes an element $\alpha \in \mathrm{H}^*(X)$ to

$$J(\alpha) \wedge \mathrm{td}(T_X).$$

Theorem 1 can be thought of as an explicit computation of the duality map.

1.2. The Mukai pairing. We now try to understand how Theorem 1 leads to the relative Riemann–Roch theorem. It is in this attempt that we see how Theorem 1 helps us calculate what the Mukai pairing on Hochschild homology [1] descends to in Hodge cohomology. This settles a part of a conjecture by Caldararu in [2].

Let $\mathrm{HH}_i(X)$ denote $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta[i])$. $\mathrm{HH}_i(X)$ is called the i-th *Hochschild homology* of X. Let $\mathcal{I}$, $\mathcal{T}$ and $\mathcal{J}$ be as in (2), (3), and (4) respectively. Let tr_X and $\mathrm{tr}_{X\times X}$ denote the canonical identifications of $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, S_X)$ and $\mathrm{Hom}_{\mathrm{D}^b(X\times X)}(\Delta_* S_X^{-1}, \Delta_* S_X)$ with $\mathbb{K}$ respectively. We recall that Caldararu [1] defined a Mukai pairing on Hochschild homology. This was a pairing

$$\begin{aligned} &\mathrm{HH}_i(X) \times \mathrm{HH}_{-i}(X) \to \mathbb{K} \\ &(v, w) \mapsto \mathrm{tr}_{X\times X}(\mathcal{T}(\mathcal{I}(v)) \circ \mathcal{I}(w)). \end{aligned} \tag{5}$$

On the other hand we can consider the pairing

$$\begin{aligned} &\mathrm{HH}_i(X) \times \mathrm{HH}_{-i}(X) \to \mathbb{K} \\ &(v, w) \mapsto \mathrm{tr}_X(D_\Delta(v) \circ w). \end{aligned} \tag{6}$$

Proposition 1. *The pairings on Hochschild homology defined in* (5) *and* (6) *are identical.*

Proof. By definition, $D_\Delta(v) = \mathcal{J}(\mathcal{T}(\mathcal{I}(v)))$. The proposition would follow if we can check that

$$\mathrm{tr}_X(\mathcal{J}(\alpha) \circ \beta) = \mathrm{tr}_{X\times X}(\alpha \circ \mathcal{I}(\beta)) \tag{7}$$

for any $\alpha \in \mathrm{Hom}_{\mathrm{D}^b(X\times X)}(\mathcal{O}_\Delta, \Delta_* S_X[i])$ and $\beta \in \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X[i], \Delta^*\mathcal{O}_\Delta)$. This is just saying that $\mathcal{I}$ is the map "dual" to the map $\mathcal{J}$ in (4).

We remark here that the assertion (7) is similar to the last part of Proposition 3.1 of Caldararu's paper [1]. Proposition 3.1 of [1] describes the construction of a right adjoint to a functor from $\mathrm{D}^b(X)$ to $\mathrm{D}^b(Y)$ given a left adjoint (via Serre duality). In our situation, $\Delta_!$ is a left adjoint to $\Delta^* : \mathrm{D}^b(X \times X) \to \mathrm{D}^b(X)$. $\Delta_!$ was constructed in [1] using the right adjoint Δ_* of Δ^* and Serre duality. $\square$

Moreover, let $\int_X : \mathrm{H}^*(X) \to \mathbb{K}$ denote the linear functional that is 0 on $\mathrm{H}^{p,q}(X)$ if $(p,q) \neq (n,n)$ and coincides with the identification of $\mathrm{H}^n(X, \Omega_X^n) = \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, S_X)$ with $\mathbb{K}$ on $\mathrm{H}^{n,n}(X)$. Recall the definition of the twisted HKR map $\widehat{\mathrm{I}_{\mathrm{HKR}}}$. The following proposition is immediate from the definition of $\widehat{\mathrm{I}_{\mathrm{HKR}}}$.

Proposition 2. *If $a \in \mathrm{H}^*(X)$ and $b \in \mathrm{RHom}_X(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta)$, then*

$$\mathrm{tr}_X(\widehat{\mathrm{I}_{\mathrm{HKR}}}(a) \circ b) = \int_X \mathrm{I}_{\mathrm{HKR}}(b) \wedge a.$$

Let J be as in Theorem 1. Let $\langle , \rangle$ denote the Mukai pairing on Hochschild homology *in this subsection only*. The following proposition is immediate from Proposition 1, Proposition 2 and Theorem 1.

Proposition 3. *If $a \in \mathrm{HH}_i(X)$ and $b \in \mathrm{HH}_{-i}(X)$ then*

$$\langle a, b \rangle = \int_X \mathrm{I}_{\mathrm{HKR}}(b) \wedge J(\mathrm{I}_{\mathrm{HKR}}(a)) \wedge \mathrm{td}(T_X).$$

Note that the product on $\mathrm{H}^*(X)$ is graded commutative. Also note that $\int_X$ is nonvanishing only on $\mathrm{H}^{2n}(X)$. Therefore,

$$\int_X v \wedge w = \int_X \overline{w} \wedge v$$

where $\overline{w}$ is obtained from w by multiplying its component in $\mathrm{H}^k(X)$ by $(-1)^k$. Note that if $w \in \mathrm{H}^*(X)$, $\overline{J(w)} = K(w)$ where K is the endomorphism on $\mathrm{H}^*(X)$ multiplying $\mathrm{H}^q(X, \Omega_X^p)$ by $(-1)^q$. Since $\mathrm{td}(T_X) \in \oplus_i \mathrm{H}^{2i}(X)$, $\mathrm{td}(T_X)$ commutes with every element of $\mathrm{H}^*(X)$. Proposition 3 may therefore, be rewritten as

$$\langle a, b \rangle = \int_X K(\mathrm{I}_{\mathrm{HKR}}(a)) \wedge \mathrm{I}_{\mathrm{HKR}}(b) \wedge \mathrm{td}(T_X). \tag{8}$$

A Mukai like pairing on Hodge cohomology. Now suppose that X is a *smooth complex variety*. Recall that a generalized Mukai pairing $\langle , \rangle_C$ has been defined by Caldararu [2] on the Hodge cohomology $\mathrm{H}^*(X)$. Let $\omega_X = \Omega_X^n$ and let τ denote the endomorphism on $\mathrm{H}^*(X)$ that is multiplication by $\sqrt{(-1)}^{p+q}$ on $\mathrm{H}^q(X, \Omega_X^p)$. Let $\mathrm{ch} : \mathrm{D}^b(X) \to \mathrm{H}^*(X)$ denote the Chern character. Recall from [2] that $\sqrt{\mathrm{ch}(\omega_X)}$ is a well-defined element of $\mathrm{H}^*(X)$. Then, if $v, w \in \mathrm{H}^*(X)$,

$$\langle v, w \rangle_C = \int_X \frac{\tau(v)}{\sqrt{\mathrm{ch}(\omega_X)}} \wedge w. \tag{9}$$

Let $\overline{\tau}$ denote the endomorphism on $\mathrm{H}^*(X)$ given by multiplication by $\sqrt{(-1)}^{q-p}$ on $\mathrm{H}^q(X, \Omega_X^p)$. Then, $K = \tau \circ \overline{\tau} = \overline{\tau} \circ \tau$. Define a pairing $\langle , \rangle_M$ on $\mathrm{H}^*(X)$ by setting

$$\langle v, w \rangle_M = \langle \overline{\tau}(v), w \rangle_C = \int_X \frac{K(v)}{\sqrt{\mathrm{ch}(\omega_X)}} \wedge w. \tag{10}$$

Proposition 4. *If $f : X \to Y$ is a proper morphism of smooth complex varieties, then*

$$\langle f^*(v), w \rangle_M = \langle v, f_*(w) \rangle_M$$

for all $v \in \mathrm{H}^*(Y)$ and $w \in \mathrm{H}^*(X)$.

Proof. We recall from Caldararu [2] that $\langle f^*(v), w \rangle_C = \langle v, f_*(w) \rangle_C$ for all $v \in \mathrm{H}^*(Y)$ and $w \in \mathrm{H}^*(X)$. Now, $f^*(\overline{\tau} v) = \overline{\tau}(f^*(v))$ for any $v \in \mathrm{H}^*(Y)$. Thus,

$$\begin{aligned} \langle f^*(v), w \rangle_M &= \langle \overline{\tau}(f^*(v)), w \rangle_C = \langle f^*(\overline{\tau}(v)), w \rangle_C \\ &= \langle \overline{\tau}(v), f_*(w) \rangle_C = \langle v, f_*(w) \rangle_M \end{aligned}$$

for all $v \in \mathrm{H}^*(Y)$ and $w \in \mathrm{H}^*(X)$. □

Proposition 5. *If $a \in \mathrm{HH}_i(X)$ and $b \in \mathrm{HH}_{-i}(X)$ then*

$$\langle a, b \rangle = \left\langle \mathrm{I}_{\mathrm{HKR}}(a) \wedge \sqrt{\mathrm{td}(T_X)}, \mathrm{I}_{\mathrm{HKR}}(b) \wedge \sqrt{\mathrm{td}(T_X)} \right\rangle_M .$$

Proof. Since $\sqrt{\mathrm{td}(T_X)}$ is a linear combination of elements in $\mathrm{H}^i(X, \Omega_X^i)$, it commutes with other elements in $\mathrm{H}^*(X)$. The RHS of the equation in this proposition is therefore,

$$\int_X \frac{K(\mathrm{I}_{\mathrm{HKR}}(a) \wedge \sqrt{\mathrm{td}(T_X)})}{\sqrt{\mathrm{ch}(\omega_X)}} \wedge \sqrt{\mathrm{td}(T_X)} \wedge \mathrm{I}_{\mathrm{HKR}}(b).$$

But K is a ring endomorphism of $\mathrm{H}^*(X)$. Thus

$$K(\mathrm{I}_{\mathrm{HKR}}(a) \wedge \sqrt{\mathrm{td}(T_X)}) = K(\mathrm{I}_{\mathrm{HKR}}(a)) \wedge K(\sqrt{\mathrm{td}(T_X)}).$$

But $K(\sqrt{\mathrm{td}(T_X)}) = \tau(\sqrt{\mathrm{td}(T_X)})$ since both K and τ are multiplication by $(-1)^i$ on $\mathrm{H}^i(X, \Omega_X^i)$. It has also been shown in Caldararu [2] that

$$\frac{\tau(\sqrt{\mathrm{td}(T_X)})}{\sqrt{\mathrm{ch}(\omega_X)}} = \sqrt{\mathrm{td}(T_X)}.$$

It follows that

$$\begin{aligned} &\int_X \frac{K(\mathrm{I}_{\mathrm{HKR}}(a) \wedge \sqrt{\mathrm{td}(T_X)})}{\sqrt{\mathrm{ch}(\omega_X)}} \wedge \sqrt{\mathrm{td}(T_X)} \wedge \mathrm{I}_{\mathrm{HKR}}(b) \\ &= \int_X K(\mathrm{I}_{\mathrm{HKR}}(a)) \wedge \mathrm{td}(T_X) \wedge \mathrm{I}_{\mathrm{HKR}}(b) \\ &= \int_X K(\mathrm{I}_{\mathrm{HKR}}(a)) \wedge \mathrm{I}_{\mathrm{HKR}}(b) \wedge \mathrm{td}(T_X). \end{aligned}$$

The desired proposition now follows from (8). $\square$

Remark 1. Recall that if $\mathcal{E} \in \mathrm{D}^b(X)$, then $\mathrm{ch}(\mathcal{E}).\sqrt{\mathrm{td}(T_X)}$ is called the *Mukai vector* of $\mathcal{E}$ (Caldararu [2]). The pairing $\langle , \rangle_M$ is slightly different from the generalized Mukai pairing $\langle , \rangle_C$ defined by Caldararu [2]. However, if v and w are Mukai vectors of elements of $\mathrm{D}^b(X)$, then $\langle v, w\rangle_M = \langle v, w\rangle_C$.

Remark 2. Let X and Y be smooth complex varieties. Recall the definition of an integral transform $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ from Caldararu [2], [1]. An integral transform Φ induces a map $\Phi_* : \mathrm{H}^*(X) \to \mathrm{H}^*(Y)$. We remark that the pairing $\langle , \rangle_M$ satisfies the *adjointness* one expects from a Mukai pairing. More precisely, if $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ and $\Psi : \mathrm{D}^b(Y) \to \mathrm{D}^b(X)$ are integral transforms such that Ψ is a left adjoint of Φ, then

$$\langle \Psi_* v, w\rangle_M = \langle v, \Phi_* w\rangle_M$$

for all $v \in \mathrm{D}^b(Y)$ and $w \in \mathrm{D}^b(X)$. This follows from the analogous property for the pairing $\langle , \rangle_C$ and the fact (see Caldararu [2]) that integral transforms preserve the columns of the Hodge diamond. We are thus justified when we refer to the pairing $\langle , \rangle_M$ as a *Mukai like pairing*.

Remark 3. A part of the main conjecture of Caldararu [2] was that the HKR map twisted by the square root of the Todd genus of X preserves the Mukai pairing. However, instead of taking the Mukai pairing on Hochschild homology to $\langle , \rangle_C$, it takes it to $\langle , \rangle_M$ by Proposition 5. The latter pairing is itself a Mukai like pairing and is very similar to the former pairing. However, $\langle , \rangle_M$ does not coincide with the Mukai pairing on the Hodge cohomology of a K3-surface in general. In particular, if $v \in \mathrm{H}^{2,0}(X)$ and $w \in \mathrm{H}^{0,2}(X)$ then $\langle v, w\rangle_M \neq \langle v, w\rangle_C$. This is why we do not go so far as to call $\langle , \rangle_M$ a generalized Mukai pairing. We can however, justifiably say that the HKR map twisted by the square root of the Todd genus of X "almost preserves" the Mukai pairing.

1.3. The relative Riemann–Roch theorem. Recall that Caldararu [1] defined a Chern character

$$\mathrm{Ch} : \mathrm{D}^b(X) \to \mathrm{HH}_0(X).$$

He also showed in [1] that if $f : X \to Y$ is a proper morphism of smooth schemes, then

$$\mathrm{Ch}(f_*\mathcal{E}) = f_*\mathrm{Ch}(\mathcal{E}) \tag{11}$$

for any $\mathcal{E} \in \mathrm{D}^b(X)$. Also, in [2], it was shown that

$$\mathrm{I}_{\mathrm{HKR}} \circ \mathrm{Ch}(\mathcal{E}) = \mathrm{ch}(\mathcal{E}).$$

The relative Riemann–Roch theorem follows from Proposition 5.

Relative Riemann–Roch Theorem. *Let $f : X \to Y$ be a proper morphism of smooth proper complex varieties. Then, if $\mathcal{E}$ is a vector bundle on X,*

$$\int_X f^*(l)\mathrm{ch}(\mathcal{E})\mathrm{td}(T_X) = \int_Y l\mathrm{ch}(f_*\mathcal{E})\mathrm{td}(T_Y)$$

for any $l \in \mathrm{H}^*(Y)$.

Proof. Note that $\mathrm{I}_{\mathrm{HKR}} : \mathrm{HH}_*(X) \to \mathrm{H}^*(X)$ is an isomorphism of complex vector spaces. Let

$$a = \mathrm{I}_{\mathrm{HKR}}^{-1}(K(l)) \in \mathrm{HH}_*(Y).$$

Then,

$$\langle f^*a, \mathrm{Ch}(\mathcal{E})\rangle = \langle a, f_*\mathrm{Ch}(\mathcal{E})\rangle = \langle a, \mathrm{Ch}(f_*\mathcal{E})\rangle. \tag{12}$$

The first equality in (12) is due to the adjointness property of the Mukai pairing (see Caldararu [1]). The second equality in (12) is due to (11). By Proposition 5 and the fact that $\mathrm{td}(T_X)$ commutes with other elements of $\mathrm{H}^*(X)$,

$$\begin{aligned} \langle f^*a, \mathrm{Ch}(\mathcal{E})\rangle &= \int_X K(\mathrm{I}_{\mathrm{HKR}}(f^*a))\mathrm{I}_{\mathrm{HKR}}(\mathrm{Ch}(\mathcal{E}))\mathrm{td}(T_X) \\ \langle a, \mathrm{Ch}(f_*\mathcal{E})\rangle &= \int_Y K(\mathrm{I}_{\mathrm{HKR}}(a))\mathrm{I}_{\mathrm{HKR}}(\mathrm{Ch}(f_*\mathcal{E}))\mathrm{td}(T_Y). \end{aligned} \tag{13}$$

Now recall that $\mathrm{I}_{\mathrm{HKR}} \circ f^* = f^* \circ \mathrm{I}_{\mathrm{HKR}}$ (Theorem 7 of [5]) and note that $K \circ f^* = f^* \circ K$. Also $\mathrm{I}_{\mathrm{HKR}} \circ \mathrm{Ch} = \mathrm{ch}$. Now, applying these facts to (13) and using (12), the desired theorem follows. □

Remark 4. Note that the Chern character to Hochschild homology actually commutes with push-forwards as shown in Caldararu [1]. The Todd genus in the relative Riemann–Roch theorem thus occurs as a consequence of the fact that the Mukai pairing on Hochschild homology does not correspond to a Mukai like pairing on Hodge cohomology under $\mathrm{I}_{\mathrm{HKR}}$. For the Mukai pairing on Hochschild homology to be "preserved" in any sense, one has to twist $\mathrm{I}_{\mathrm{HKR}}$ by $\sqrt{\mathrm{td}(T_X)}$.

2. Two "connections" on the Hochschild chain complex $\widehat{C}^\bullet(X)$

2.1. The completed bar and Hochschild chain complexes. Let $U = \operatorname{Spec} R$ be an open affine subscheme of X. The restriction of $\mathcal{O}_\Delta$ to $U \times U$ has a free $R \otimes R$-module resolution given by the *bar resolution:*

$$B^{-n}(R) = R^{\otimes(n+2)}$$

$$\begin{aligned} d(r_0 \otimes \cdots \otimes r_{n+1}) = {} & r_0r_1 \otimes \cdots \otimes r_{n+1} - r_0 \otimes r_1r_2 \otimes \cdots \otimes r_{n+1} \\ & + \cdots + (-1)^n r_0 \otimes \cdots \otimes r_nr_{n+1} \end{aligned}$$

$r_i \in R$.

The $R \otimes R$-module structure is given by multiplication with the extreme factors. Let I_n denote the kernel of the $(n+2)$-fold multiplication $R^{\otimes(n+2)} \to R$. Let

$$\widehat{B}^{-n}(R) = \lim_k \frac{B^{-n}(R)}{I_n^k}.$$

Note that each summand of the differential d takes I_n to I_{n-1}. It follows that the differential on $B^\bullet(R)$ extends to yield a differential on $\widehat{B}^\bullet(R)$. Yekutieli [9] shows that completing the bar resolution in this manner yields a complex $\widehat{B}^\bullet(X)$ of coherent sheaves on $X \times X$. He also shows that $\widehat{B}^\bullet(X)$ is a resolution of $\mathcal{O}_\Delta$ by flat $\mathcal{O}_{X\times X}$-modules.

It follows that $\Delta^*\mathcal{O}_\Delta$ is represented by the complex

$$\widehat{C}^\bullet(X) := \mathcal{O}_X \otimes_{\Delta^{-1}\mathcal{O}_{X\times X}} \Delta^{-1}\widehat{B}^\bullet(X).$$

$\widehat{C}^\bullet(X)$ is called the complex of completed *Hochschild chains* on X.

On an open subscheme $U = \operatorname{Spec} R$ of X before completion,

$$C^{-n}(R) = R^{\otimes(n+1)}$$

$$\begin{aligned} d(r_0 \otimes \cdots \otimes r_n) = {} & r_0r_1 \otimes \cdots \otimes r_n - r_0 \otimes r_1r_2 \otimes \cdots \otimes r_n \\ & + \cdots + (-1)^{n-1} r_0 \otimes \cdots \otimes r_{n-1}r_n \\ & + (-1)^n r_nr_0 \otimes \cdots \otimes r_{n-1} \end{aligned}$$

$$\widehat{C}^{-n}(R) = \lim_k \frac{B^{-n}(R)}{I_n^k} \otimes_{R^{\otimes(n+2)}} C^{-n}(R).$$

Yekutieli [9] also showed that $\mathrm{RD}(\widehat{C}^\bullet(X))$ is represented in $\mathrm{D}^b(X)$ by the complex $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ of poly-differential operators on X equipped with Hochschild coboundary.

Let us describe some operations on $\widehat{C}^\bullet(X)$ that endow it with the structure of a Hopf algebra in $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$ (and therefore in $\mathrm{D}^b(X)$).

Product on $\widehat{C}^\bullet(X)$: The product $m : \widehat{C}^\bullet(X) \otimes_{\mathcal{O}_X} \widehat{C}^\bullet(X) \to \widehat{C}^\bullet(X)$ is given by the *signed shuffle product*. Recall that a (p,q)-shuffle σ is a permutation of $\{1,\dots,p+q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. Denote the set of (p,q)-shuffles by $\mathrm{Sh}_{p,q}$. On an open subscheme $U = \operatorname{Spec} R$ of X before completion, this product is given by

$$\begin{aligned} &(r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (r_0' \otimes r_{p+1} \otimes \cdots \otimes r_{p+q}) \\ &\qquad \mapsto \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma) r_0r_0' \otimes r_{\sigma^{-1}(1)} \otimes \cdots \otimes r_{\sigma^{-1}(p+q)}. \end{aligned}$$

This is easily seen to be a (graded) commutative product.

Coproduct on $\widehat{C}^\bullet(X)$. The coproduct $\widehat{C}^\bullet(X) \to \widehat{C}^\bullet(X) \otimes_{\mathcal{O}_X} \widehat{C}^\bullet(X)$ is given by the *cut coproduct* . Contrary to the usual practise, we denote the coproduct by $\mathbf{C}$ to avoid confusion with Δ which denotes the diagonal map $X \to X \times X$ in this paper. On an open subscheme $U = \operatorname{Spec} R$ of X before completion,

$$\mathbf{C}(r_0 \otimes \cdots \otimes r_n) = \sum_{p+q=n} r_0 \otimes r_1 \otimes \cdots \otimes r_p \otimes_R 1 \otimes r_{p+1} \otimes \cdots \otimes r_n.$$

Unit for $\widehat{C}^\bullet(X)$. There is a unit map $\epsilon : \mathcal{O}_X \to \widehat{C}^\bullet(X)$. On an open subscheme $U = \operatorname{Spec} R$ of X before completion, ϵ is given by the composite

$$R \simeq C^0(R) \hookrightarrow C^\bullet(R).$$

Counit for $\widehat{C}^\bullet(X)$. There is a counit $\eta : \widehat{C}^\bullet(X) \to \mathcal{O}_X$. On an open subscheme $U = \operatorname{Spec} R$ of X before completion, this is given by the projection from $C^\bullet(R)$ to $C^0(R)$.

Proposition 6. *$\widehat{C}^\bullet(X)$ is a Hopf algebra in $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$ via M, $\mathbf{C}$, ϵ and η.*

Proof. Yekutieli [9] showed that applying the functor $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ ($\mathcal{O}_X$ is given the discrete topology) to $\widehat{C}^\bullet(X) \in \mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$ yields

$$\mathrm{D}^\bullet_{\mathrm{poly}}(X) \in \mathrm{Ch}^+(\mathcal{O}_X - \mathrm{mod}).$$

It is easy to verify that this functor takes the product, coproduct, unit and counit of $\widehat{C}^\bullet(X)$ to the coproduct, product, counit and unit of $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ respectively. The desired proposition then follows from Proposition 2 of [7]. $\square$

Antipode on $\widehat{C}^\bullet(X)$. $\widehat{C}^\bullet(X)$ also comes equipped with an *antipode* map. We will denote this map by S. On $U = \operatorname{Spec} R$ before completion,

$$S(r_0 \otimes \cdots \otimes r_n) = (-1)^{\frac{n(n+1)}{2}} r_0 \otimes r_n \otimes r_{n-1} \otimes \cdots \otimes r_1.$$

The Hochschild–Kostant–Rosenberg (HKR) map. There is a quasi-isomorphism (see Yekutieli [9])

$$\mathrm{I}_{\mathrm{HKR}} : \widehat{C}^\bullet(X) \to \mathbf{S}^\bullet(\Omega[1])$$

of complexes of $\mathcal{O}_X$-modules. On an open subscheme $U = \operatorname{Spec} R$ of X before completion,

$$\mathrm{I}_{\mathrm{HKR}}(r_0 \otimes \cdots \otimes r_n) = \frac{1}{n!} r_0 dr_1 \wedge \cdots \wedge dr_n.$$

2.2. Two connections on $\widehat{C}^\bullet(X)$. Let $\pi_k : \mathbf{S}^\bullet(\Omega[1]) \to \Omega^k[k]$ denote the natural projection. Denote by α_R the composite

$$\widehat{C}^\bullet(X) \xrightarrow{\mathbf{C}} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \xrightarrow{(\mathbf{1}_{\widehat{C}^\bullet(X)} \otimes \pi_1 \circ \mathrm{I}_{\mathrm{HKR}})} \widehat{C}^\bullet(X) \otimes \Omega[1].$$

More concretely, on an open subscheme $U = \operatorname{Spec} R$ of X before completion,

$$\alpha_R(r_0 \otimes \cdots \otimes r_n) = r_0 \otimes \cdots \otimes r_{n-1} \otimes dr_n.$$

Let $\alpha_L : \widehat{C}^\bullet(X) \to \widehat{C}^\bullet(X) \otimes \Omega[1]$ be the map such that

$$\alpha_L(r_0 \otimes \cdots \otimes r_n) = (-1)^{n-1} r_0 \otimes r_2 \otimes \cdots \otimes r_n \otimes dr_1$$

on any open subscheme $U = \operatorname{Spec} R$ of X before completion.

Then, $\alpha_L = -(S \otimes \Omega[1]) \circ \alpha_R \circ S$. Let $\alpha_R \otimes \widehat{C}^\bullet(X)$ denote the composite

$$\begin{array}{l} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \\ \big\downarrow {\scriptstyle \alpha_R \otimes \widehat{C}^\bullet(X)} \\ \widehat{C}^\bullet(X) \otimes \Omega[1] \otimes \widehat{C}^\bullet(X) \xrightarrow{\widehat{C}^\bullet(X) \otimes \tau} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \otimes \Omega[1] \end{array}$$

where $\tau : \Omega[1] \otimes \widehat{C}^\bullet(X) \to \widehat{C}^\bullet(X) \otimes \Omega[1]$ is the map that swaps factors. Similarly, let $\alpha_L \otimes \widehat{C}^\bullet(X)$ denote the composite

$$\begin{array}{l} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \\ \big\downarrow {\scriptstyle \alpha_L \otimes \widehat{C}^\bullet(X)} \\ \widehat{C}^\bullet(X) \otimes \Omega[1] \otimes \widehat{C}^\bullet(X) \xrightarrow{\widehat{C}^\bullet(X) \otimes \tau} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \otimes \Omega[1]. \end{array}$$

We now have the following proposition.

Proposition 7. *The following diagrams commute in* $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{tikzcd} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \arrow[r, "\alpha_R \otimes \widehat{C}^\bullet(X) + \widehat{C}^\bullet(X) \otimes \alpha_R"] \arrow[d, "m"] & \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \otimes \Omega[1] \arrow[d, "m \otimes \Omega[1]"'] \\ \widehat{C}^\bullet(X) \arrow[r, "\alpha_R"] & \widehat{C}^\bullet(X) \otimes \Omega[1] \end{tikzcd} \tag{14}$$

$$\begin{tikzcd} \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \arrow[r, "\alpha_L \otimes \widehat{C}^\bullet(X) + \widehat{C}^\bullet(X) \otimes \alpha_L"] \arrow[d, "m"] & \widehat{C}^\bullet(X) \otimes \widehat{C}^\bullet(X) \otimes \Omega[1] \arrow[d, "m \otimes \Omega[1]"'] \\ \widehat{C}^\bullet(X) \arrow[r, "\alpha_L"] & \widehat{C}^\bullet(X) \otimes \Omega[1]. \end{tikzcd} \tag{15}$$

Proof. This proposition is proven by a combinatorial argument. On an open subscheme $U = \operatorname{Spec} R$ of X before completion,

$$\begin{aligned}(16)\quad &(\alpha_R \otimes \widehat{C}^\bullet(X) + \widehat{C}^\bullet(X) \otimes \alpha_R)\\ &\quad ((r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q}))\\ &\quad = (-1)^q (r_0 \otimes r_1 \otimes \cdots \otimes r_{p-1}) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q}) \otimes dr_p\\ &\qquad + (r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q-1}) \otimes dr_{p+q}.\end{aligned}$$

Note that if σ is a (p,q)-shuffle, then $\sigma^{-1}(p+q) = p$ or $\sigma^{-1}(p+q) = p+q$. Let $\mathrm{Sh}^1_{p,q}$ denote the set of all (p,q)-shuffles σ such that $\sigma^{-1}(p+q) = p$. Let $\mathrm{Sh}^2_{p,q}$ denote $\mathrm{Sh}_{p,q} \setminus \mathrm{Sh}^1_{p,q}$.

Also note that there is a sign preserving bijection from $\mathrm{Sh}_{p,q-1}$ to $\mathrm{Sh}^2_{p,q}$ the inverse of which takes an element σ of $\mathrm{Sh}^2_{p,q}$ to its restriction to the set $\{1, \dots, p+q-1\}$. Denote this bijection by $I : \mathrm{Sh}_{p,q-1} \to \mathrm{Sh}^2_{p,q}$.

We also have a bijection from $\mathrm{Sh}^1_{p,q}$ to $\mathrm{Sh}_{p-1,q}$. Let

$$\psi : \{1, \dots, p+q-1\} \to \{1, \dots, p-1, p+1, \dots, p+q-1\}$$

be the unique order-preserving map.The permutation in $\mathrm{Sh}_{p-1,q}$ corresponding to a permutation σ in $\mathrm{Sh}^1_{p,q}$ is given by the composite

$$\{1, \dots, p+q-1\} \xrightarrow{\psi} \{1, \dots, p-1, p+1, \dots, p+q\} \xrightarrow{\sigma} \{1, \dots, p+q-1\}.$$

This bijection from $\mathrm{Sh}^1_{p,q}$ to $\mathrm{Sh}_{p-1,q}$ however, changes the sign by $(-1)^q$. Denote the inverse of this bijection by $J : \mathrm{Sh}_{p-1,q} \to \mathrm{Sh}^1_{p,q}$.

Then

$$\begin{aligned}&m((r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q}))\\ &= \sum_{\sigma \in \mathrm{Sh}_{p,q}} \mathrm{sgn}(\sigma) r_0 \otimes r_{\sigma^{-1}(1)} \otimes \dots r_{\sigma^{-1}(p+q)}\\ &= \sum_{\sigma \in \mathrm{Sh}^1_{p,q}} \mathrm{sgn}(\sigma) r_0 \otimes r_{\sigma^{-1}(1)} \otimes \dots r_{\sigma^{-1}(p+q)}\\ &\quad + \sum_{\sigma \in \mathrm{Sh}^2_{p,q}} \mathrm{sgn}(\sigma) r_0 \otimes r_{\sigma^{-1}(1)} \otimes \dots r_{\sigma^{-1}(p+q)}\\ &= \sum_{\sigma \in \mathrm{Sh}_{p-1,q}} (-1)^q \mathrm{sgn}(\sigma) r_0 \otimes r_{J(\sigma)^{-1}(1)} \otimes \cdots \otimes r_{J(\sigma)^{-1}(p+q-1)} \otimes r_p\\ &\quad + \sum_{\sigma \in \mathrm{Sh}_{p,q-1}} \mathrm{sgn}(\sigma) r_0 \otimes r_{I(\sigma)^{-1}(1)} \otimes \cdots \otimes r_{I(\sigma)^{-1}(p+q-1)} \otimes r_{p+q}\\ &= (-1)^q m((r_0 \otimes r_1 \otimes \cdots \otimes r_{p-1}) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q})) \otimes r_p\\ &\quad + m((r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q-1})) \otimes r_{p+q}.\end{aligned}$$

It follows that

$$\begin{aligned}&\alpha_R(m((r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q})))\\ &= (-1)^q m((r_0 \otimes r_1 \otimes \cdots \otimes r_{p-1}) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q})) \otimes dr_p\\ &\quad + m((r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q-1})) \otimes dr_{p+q}.\end{aligned}$$

It follows from (16) that this is precisely

$$(m \otimes \Omega[1])((\alpha_R \otimes \widehat{C}^\bullet(X) + \widehat{C}^\bullet(X) \otimes \alpha_R)) \\ ((r_0 \otimes r_1 \otimes \cdots \otimes r_p) \otimes_R (1 \otimes r_{p+1} \otimes \cdots \otimes r_{p+q})).$$

This proves that the diagram (14) commutes. Proving that the diagram (15) commutes is very similar and left to the reader. □

Let $\alpha_R^{\circ i}$ denote the composite

$$\begin{tikzcd}
\widehat{C}^\bullet(X) \arrow[d, "\alpha_R"'] \\
\widehat{C}^\bullet(X) \otimes \Omega[1] \arrow[d, "\alpha_R \otimes \Omega[1]"'] \\
\widehat{C}^\bullet(X) \otimes \Omega[1]^{\otimes 2} \arrow[r] & \cdots \arrow[r, "\alpha_R \otimes \Omega[1]^{\otimes i-1}"] & \widehat{C}^\bullet(X) \otimes \Omega[1]^{\otimes i}.
\end{tikzcd}$$

Let $p : \Omega[1]^{\otimes i} \to \Omega^i[i]$ be the standard projection. On an open subscheme $U = \operatorname{Spec} R$, of X,

$$p(r_0 dr_1 \otimes \cdots \otimes dr_i) = r_0 dr_1 \wedge \cdots \wedge dr_i.$$

Let α_R^i denote the composite

$$\widehat{C}^\bullet(X) \xrightarrow{\alpha_R^{\circ i}} \widehat{C}^\bullet(X) \otimes \Omega[1]^{\otimes i} \xrightarrow{\widehat{C}^\bullet(X) \otimes p} \widehat{C}^\bullet(X) \otimes \Omega^i[i].$$

Let $\exp(\alpha_R)$ denote the sum

$$\sum_i \frac{1}{i!}\alpha_R^i : \widehat{C}^\bullet(X) \to \widehat{C}^\bullet(X) \otimes \mathbf{S}^\bullet(\Omega[1]).$$

We then have the following proposition.

Proposition 8.

$$(\widehat{C}^\bullet(X) \otimes \mathrm{I}_{\mathrm{HKR}}) \circ \mathbf{C} = \exp(\alpha_R).$$

Proof. On an open subscheme $U = \operatorname{Spec} R$ before completion,

$$\begin{aligned}
&\exp(\alpha_R)(r_0 \otimes \cdots \otimes r_n)\\
&= \frac{1}{i!}\sum_i r_0 \otimes \cdots \otimes r_{n-i} \otimes_R dr_{n-i+1} \wedge \cdots \wedge dr_n\\
&= \sum_i r_0 \otimes \cdots \otimes r_{n-i} \otimes_R \mathrm{I}_{\mathrm{HKR}}(1 \otimes r_{n-i+1} \otimes \cdots \otimes r_n)\\
&= (\widehat{C}^\bullet(X) \otimes \mathrm{I}_{\mathrm{HKR}})\Big(\sum_i r_0 \otimes \cdots \otimes r_{n-i} \otimes_R 1 \otimes r_{n-i+1} \otimes \cdots \otimes r_n\Big)\\
&= (\widehat{C}^\bullet \otimes \mathrm{I}_{\mathrm{HKR}}) \circ \mathbf{C}(r_0 \otimes \cdots \otimes r_n).
\end{aligned}$$

This verifies the desired proposition. □

Let $\tau : \Omega[1] \otimes \Omega[1] \to \Omega[1] \otimes \Omega[1]$ denote the swap map. The following proposition tells us that α_L and α_R "commute" with each other.

Proposition 9.

$$(\alpha_R \otimes \Omega[1]) \circ \alpha_L - (\widehat{C}^\bullet(X) \otimes \tau) \circ (\alpha_L \otimes \Omega[1]) \circ \alpha_R = 0.$$

Proof. On an open subscheme $U = \operatorname{Spec} R$ before completion,

$$(\alpha_R \otimes \Omega[1]) \circ \alpha_L(r_0 \otimes \cdots \otimes r_n) = (-1)^{n-1}(r_0 \otimes r_2 \otimes \cdots \otimes r_{n-1}) \otimes_R dr_n \otimes_R dr_1$$

$$(\alpha_L \otimes \Omega[1]) \circ \alpha_R(r_0 \otimes \cdots \otimes r_n) = (-1)^{n-2}(r_0 \otimes r_2 \otimes \cdots \otimes r_{n-1}) \otimes_R dr_1 \otimes_R dr_n$$

$$\begin{aligned}
&(\widehat{C}^\bullet(X) \otimes \tau)((r_0 \otimes r_2 \otimes \cdots \otimes r_{n-1}) \otimes_R dr_1 \otimes_R dr_n)\\
&\qquad\qquad = -(r_0 \otimes r_2 \otimes \cdots \otimes r_{n-1}) \otimes_R dr_n \otimes_R dr_1.
\end{aligned}$$

The desired proposition is now immediate. □

2.3. Remark — the beginning of a dictionary. For reasons that will become clear later in this section, the reader should think of $\widehat{C}^\bullet(X)$ as analogous to the ring of functions on an open "symmetric" neighborhood U_G of the identity of a Lie group G. By "symmetric" we mean that if $g \in U_G$ then $g^{-1} \in U_G$. $T[-1]$ is the analog of the Lie algebra $\mathfrak{g}$ of the Lie group G. Thus, $\Omega[1]$ is the analog of $\mathfrak{g}^*$. Proposition 7 says that in this picture, both α_L and α_R are analogs of "connections" on the ring of functions of G.

In the same picture, $\mathbf{S}^\bullet(\Omega[1])$ is to be thought of as analogous to the ring of functions on a neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$ that is diffeomorphic to U_G via the exponential map. The Hochschild–Kostant–Rosenberg map is then the analog of the pullback by the exponential map $\exp^*$.

The antipode map S is the analog of the pullback by the map which takes an element of G to its inverse.

2.4. More on the maps α_R and α_L. The question that arises at this stage is, "Can α_L and α_R be described by explicit formulae as maps in $\mathrm{D}^b(X)$ from $\mathbf{S}^\bullet(\Omega[1])$ to $\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]$?". Markarian sought to answer this in Theorem 1 of [5]. The proof there was however erroneous. A result dual to what we want is available in [7]. A later version [6] of [5] also contains a result equivalent to Theorem 1 of [5].

Recall from Kapranov [3] that $T[-1]$ is a Lie algebra object in $\mathrm{D}^b(X)$. The Lie bracket of $T[-1]$ is given by the Atiyah class

$$\mathrm{At}_T : T[-1] \otimes T[-1] \to T[-1]$$

of the tangent bundle of X. It is also known that the universal enveloping algebra of $T[-1]$ in $\mathrm{D}^b(X)$ is represented by the complex $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$. This was proven in [7]. Equivalent results have been proven using methods different from that in [7] by Markarian [6] and Roberts and Willerton [8].

Let μ denote the wedge product on $\mathbf{S}^\bullet(T[-1])$. Let

$$\delta : \mathbf{S}^\bullet(T[-1]) \otimes T[-1] \to \mathbf{S}^\bullet(T[-1]) \otimes T[-1] \otimes T[-1]$$

be the map

$$\delta(v_1 \wedge \cdots \wedge v_k \otimes y) = \sum_{i=1}^{i=k} (-1)^{k-i} v_1 \wedge \widehat{\cdots i \cdots} \wedge v_k \otimes v_i \otimes y$$

for sections $v_1, \dots, v_k, y$ of T over an open subscheme $U = \operatorname{Spec} R$ of X.

We have a map

$$\overline{\omega} : \mathbf{S}^\bullet(T[-1]) \otimes T[-1] \to \mathbf{S}^\bullet(T[-1]) \otimes T[-1]$$
$$\overline{\omega} = \big(\mathbf{1}_{\mathbf{S}^\bullet(T[-1])} \otimes \mathrm{At}_T\big) \circ \delta.$$

Note that $\mu \circ \overline{\omega}$ yields the right adjoint action of the Lie algebra object $T[-1]$ on $\mathbf{S}^\bullet(T[-1])$.

The Hochschild–Kostant–Rosenberg map $\mathrm{I}_{\mathrm{HKR}} : \mathbf{S}^\bullet(T[-1]) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$ is the "dual" of the HKR map $\mathrm{I}_{\mathrm{HKR}} : \widehat{C}^\bullet(X) \to \mathbf{S}^\bullet(\Omega[1])$. The following theorem, which figures as Corollary 1 in [7] and Theorem 2 in [6], describes the Duflo-like error term that measures how the map

$$\mathrm{I}_{\mathrm{HKR}} : \mathbf{S}^\bullet(T[-1]) \to \mathrm{D}^\bullet_{\mathrm{poly}}(X)$$

fails to commute with multiplication.

Theorem 2 (Recalled from [7]; [6] has a similar result)**.** *The following diagram commutes in* $\mathrm{D}^b(X)$.

$$\begin{tikzcd}
\mathrm{D}^\bullet_{\mathrm{poly}}(X) \otimes \mathrm{D}^\bullet_{\mathrm{poly}}(X) \arrow[r, "m"] & \mathrm{D}^\bullet_{\mathrm{poly}}(X) \\
\mathbf{S}^\bullet(T[-1]) \otimes T[-1] \arrow[u, "\mathrm{I}_{\mathrm{HKR}} \otimes \mathrm{I}_{\mathrm{HKR}}"] \arrow[r, "\mu \circ \frac{\overline{\omega}}{1 - e^{-\overline{\omega}}}"] & \mathbf{S}^\bullet(T[-1]). \arrow[u, "\mathrm{I}_{\mathrm{HKR}}"']
\end{tikzcd}$$

Note that applying the functor RD to $\overline{\omega}$ gives us a morphism in $\mathrm{D}^b(X)$

$$\overline{\omega} : \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1].$$

Further, denote the comultiplication on $\mathbf{S}^\bullet(\Omega[1])$ by $\mathbf{C}_\Omega$. Denote the map

$$(\mathbf{S}^\bullet(\Omega[1]) \otimes \pi_1) \circ \mathbf{C}_\Omega : \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]$$

by $\overline{\mathbf{C}}$.

We denote the map

$$\frac{\overline{\omega}}{e^{\overline{\omega}} - 1} \circ \overline{\mathbf{C}} : \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]$$

by Φ_L. The map

$$\frac{\overline{\omega}}{1 - e^{-\overline{\omega}}} \circ \overline{\mathbf{C}} : \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]$$

will be denoted by Φ_R.

We can now state the following theorem. This can be thought of as the starting point for the computations leading to Theorem 1.

Theorem $2'$**.** *The following diagrams commute in* $\mathrm{D}^b(X)$.

$$\begin{array}{ccc} \widehat{C}^\bullet(X) & \xrightarrow{\alpha_R} & \widehat{C}^\bullet(X) \otimes \Omega[1] \\ \Big\downarrow \mathrm{I_{HKR}} & & \mathrm{I_{HKR}}\otimes\Omega[1] \Big\downarrow \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\Phi_R} & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \end{array} \tag{17}$$

$$\begin{array}{ccc} \widehat{C}^\bullet(X) & \xrightarrow{\alpha_L} & \widehat{C}^\bullet(X) \otimes \Omega[1] \\ \Big\downarrow \mathrm{I_{HKR}} & & \mathrm{I_{HKR}}\otimes\Omega[1] \Big\downarrow \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\Phi_L} & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]. \end{array} \tag{18}$$

Proof. The fact that the diagram (17) commutes in $\mathrm{D}^b(X)$ is obtained by applying the functor RD to the diagram in Theorem 2.

Note that since $\alpha_L = -(S \otimes \Omega[1]) \circ \alpha_R \circ S$ the following diagram commutes in $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$ (and therefore in $\mathrm{D}^b(X)$).

$$\begin{array}{ccc} \widehat{C}^\bullet(X) & \xrightarrow{\alpha_R} & \widehat{C}^\bullet(X) \otimes \Omega[1] \\ \Big\downarrow S & & S\otimes\Omega[1] \Big\downarrow \\ \widehat{C}^\bullet(X) & \xrightarrow{-\alpha_L} & \widehat{C}^\bullet(X) \otimes \Omega[1]. \end{array} \tag{19}$$

Recall that J is the endomorphism of $\mathbf{S}^\bullet(\Omega[1])$ multiplying $\wedge^i\Omega[i]$ by $(-1)^i$. Then,

$$\mathrm{I_{HKR}} \circ S = J \circ \mathrm{I_{HKR}}. \tag{20}$$

To see (20), note that on an open subscheme $U = \operatorname{Spec} R$ of X before completion,

$$\begin{aligned}\mathrm{I}_{\mathrm{HKR}} \circ S(r_0 \otimes \cdots \otimes r_n) &= (-1)^{\frac{n(n+1)}{2}} \frac{1}{n!} r_0 dr_n \wedge \cdots \wedge dr_1 \\ &= (-1)^{\frac{n(n+1)}{2}} (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!} r_0 dr_1 \wedge \cdots \wedge dr_n \\ &= (-1)^n \mathrm{I}_{\mathrm{HKR}}(r_0 \otimes \cdots \otimes r_n).\end{aligned}$$

Further, note that applying the functor RD to J yields the endomorphism I of $\mathbf{S}^\bullet(T[-1])$ that multiplies $\wedge^i T_X[-i]$ by $(-1)^i$. For sections $v_1, \ldots, v_k, y$ of T over an open subscheme $U = \operatorname{Spec} R$ of X,

$$\begin{aligned}&\delta(I(v_1 \wedge \cdots \wedge v_k) \otimes y) \\ &= (-1)^k \sum_{i=1}^{i=k} (-1)^{k-i} v_1 \wedge \widehat{\cdots i \cdots} \wedge v_k \otimes v_i \otimes y \\ &= (-1)^k (-1)^{k-1} \sum_{i=1}^{i=k} (-1)^{k-i} I(v_1 \wedge \widehat{\cdots i \cdots} \wedge v_k) \otimes v_i \otimes y.\end{aligned}$$

It follows from the above computation and the fact that

$$\overline{\omega} = (\mathbf{S}^\bullet(T[-1]) \otimes \mathrm{At}_T) \circ \delta$$

that the following diagram commutes in $\mathrm{D}^b(X)$.

$$\begin{array}{ccc} \mathbf{S}^\bullet(T[-1]) \otimes T[-1] & \xrightarrow{\overline{\omega}} & \mathbf{S}^\bullet(T[-1]) \otimes T[-1] \\ \big\downarrow{\scriptstyle I \otimes T[-1]} & & \big\downarrow{\scriptstyle I \otimes T[-1]} \\ \mathbf{S}^\bullet(T[-1]) \otimes T[-1] & \xrightarrow{-\overline{\omega}} & \mathbf{S}^\bullet(T[-1]) \otimes T[-1]. \end{array} \tag{21}$$

Applying the functor RD to the diagram (21) we obtain the following diagram.

$$\begin{array}{ccc} \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] & \xrightarrow{\overline{\omega}} & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \\ \big\downarrow{\scriptstyle J \otimes \Omega[1]} & & {\scriptstyle J \otimes \Omega[1]}\big\downarrow \\ \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] & \xrightarrow{-\overline{\omega}} & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]. \end{array} \tag{22}$$

A calculation similar to the one made to verify (21) also shows that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\overline{\mathbf{C}}} & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \\ \big\downarrow{\scriptstyle J} & & {\scriptstyle J \otimes \Omega[1]}\big\downarrow \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{-\overline{\mathbf{C}}} & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]. \end{array} \tag{23}$$

Combining (23) and (22) we obtain the following commutative diagram.

$$\begin{array}{ccc} \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\frac{\overline{\omega}}{1-e^{-\overline{\omega}}}\circ\overline{\mathbf{C}}} & \mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1] \\ \Big\downarrow{\scriptstyle J} & & \Big\downarrow{\scriptstyle J\otimes\Omega[1]} \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{-\frac{\overline{\omega}}{e^{\overline{\omega}}-1}\circ\overline{\mathbf{C}}} & \mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1]. \end{array} \tag{24}$$

It follows from (17) and (24) that all squares in the diagram below commute in $\mathrm{D}^b(X)$.

$$\begin{array}{ccc} \widehat{C}^\bullet(X) & \xrightarrow{\alpha_R} & \widehat{C}^\bullet(X)\otimes\Omega[1] \\ \Big\downarrow{\scriptstyle \mathrm{I_{HKR}}} & & \Big\downarrow{\scriptstyle \mathrm{I_{HKR}}\otimes\Omega[1]} \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\frac{\overline{\omega}}{1-e^{-\overline{\omega}}}\circ\overline{\mathbf{C}}} & \mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1] \\ \Big\downarrow{\scriptstyle J} & & \Big\downarrow{\scriptstyle J\otimes\Omega[1]} \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{-\frac{\overline{\omega}}{e^{\overline{\omega}}-1}\circ\overline{\mathbf{C}}} & \mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1]. \end{array} \tag{25}$$

The diagram (25) says that

$$-\Phi_L\circ J\circ\mathrm{I_{HKR}} = [(J\circ\mathrm{I_{HKR}})\otimes\Omega[1]]\circ\alpha_R.$$

But by (20)

$$J\circ\mathrm{I_{HKR}} = \mathrm{I_{HKR}}\circ S \implies -\Phi_L\circ\mathrm{I_{HKR}}\circ S = (\mathrm{I_{HKR}}\otimes\Omega[1])\circ(S\otimes\Omega[1])\circ\alpha_R.$$

Since $\alpha_L = -(S\otimes\Omega[1])\circ\alpha_R\circ S$ and $S\circ S = \mathbf{1}_{\widehat{C}^\bullet}(X)$,

$$\Phi_L\circ\mathrm{I_{HKR}} = (\mathrm{I_{HKR}}\otimes\Omega[1])\circ\alpha_L.$$

This proves that the diagram (18) commutes. $\square$

2.5. A long remark — enlarging the dictionary. This subsection is a continuation of Section 2.3. Recall that $\widehat{C}^\bullet(X)$ should be thought of as analogous to the ring of functions on a "symmetric" neighborhood U_G of the identity in a Lie group G. $T[-1]$ is analogous to the Lie algebra $\mathfrak{g}$ of G. $\mathbf{S}^\bullet(\Omega[1])$ is analogous to the ring of functions on a neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$ that is diffeomorphic to U_G via the exponential map.

$$\mathrm{I_{HKR}}:\widehat{C}^\bullet(X)\to\mathbf{S}^\bullet(\Omega[1])$$

is analogous to the pullback of functions by the map $\exp:\mathfrak{g}\to G$.

2.5.1. The classical picture. I. Let us look at the classical picture for now. Choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ and a basis $\{Y_1, \dots, Y_n\}$ of $\mathfrak{g}^*$ dual to $\{X_1, \dots, X_n\}$. Then, $\sum_{i=1}^{i=n} X_i \otimes Y_i$ yields an element $\mathfrak{g} \otimes \mathfrak{g}^*$. Denote the ring of functions on U_G by $C(G)$. We identify elements of $\mathfrak{g}$ (resp. elements of $\mathfrak{g}^*$) with left-invariant vector fields (resp. left-invariant 1-forms) on U_G whenever necessary. Given any element of $\mathfrak{g} \otimes \mathfrak{g}^*$, letting $\mathfrak{g}$ act on $C(G)$ as a differential operator yields us a map from $C(G)$ to $C(G) \otimes \mathfrak{g}^*$ satisfying the Leibniz rule. Therefore, any element of $\mathfrak{g} \otimes \mathfrak{g}^*$ yields a global connection on $C(G)$. It is easy to verify that the connection on $C(G)$ yielded by $\sum_{i=1}^{n} X_i \otimes Y_i$ is precisely d_G, the de Rham differential from $C(G)$ to the sections of the cotangent bundle of U_G.

Denote the ring of functions on $\mathcal{V}$ by $C(\mathfrak{g})$. Note that replacing $C(G)$ by $C(\mathfrak{g})$ in the previous paragraph enables us to conclude $\sum_{i=1}^{n} X_i \otimes Y_i$ yields the connection $d_{\mathfrak{g}}$ on $C(\mathfrak{g})$ where $d_{\mathfrak{g}}$ is precisely the de Rham differential from $C(\mathfrak{g})$ to the sections of the cotangent bundle of $\mathcal{V}$.

Note that elements of $C(G) \otimes \mathfrak{g}^*$ are smooth functions from U_G to $\mathfrak{g}^*$, i.e, sections of the cotangent bundle of U_G. Similarly, elements of $C(\mathfrak{g}) \otimes \mathfrak{g}^*$ are smooth functions from $\mathcal{V}$ to $\mathfrak{g}^*$, i.e, sections of the cotangent bundle of $\mathcal{V}$.

Given a smooth function A from $\mathcal{V}$ to $\mathrm{End}(\mathfrak{g})$ and a smooth function h from $\mathcal{V}$ to $\mathfrak{g}$ (resp. $\mathfrak{g}^*$), one can obtain a smooth function $A(h)$ from $\mathcal{V}$ to $\mathfrak{g}$ (resp. $\mathfrak{g}^*$) by setting

$$A(h)(Z) = A(Z)h(Z)$$

for every $Z \in \mathcal{V}$. Denote the smooth function $Z \mapsto \mathrm{ad}(Z)$ from $\mathcal{V}$ to $\mathrm{End}(\mathfrak{g})$ by $\overline{\mathrm{ad}}$.

Consider the connection Φ on $C(\mathfrak{g})$ such that the following diagram commutes.

$$\begin{array}{ccc} C(G) & \xrightarrow{d_G} & C(G) \otimes \mathfrak{g}^* \\ \Big\downarrow{\scriptstyle \exp^*} & & \Big\downarrow{\scriptstyle \exp^* \otimes \mathfrak{g}^*} \\ C(\mathfrak{g}) & \xrightarrow{\Phi} & C(\mathfrak{g}) \otimes \mathfrak{g}^*. \end{array}$$

We are interested in comparing Φ with $d_{\mathfrak{g}}$. This is done as follows:

Let f be any function on U_G. Then, $d_G(f)$ is a 1-form on U_G. The pullback of the 1-form $d_G(f)$ via the exponential map is precisely the 1-form $d_{\mathfrak{g}}(\exp^*(f))$. Recall the formula

$$d(\exp)_Z = \frac{1 - \mathrm{e}^{-\mathrm{ad}(Z)}}{\mathrm{ad}(Z)} \implies (\exp^* \otimes d(\exp)^*) = \frac{1 - \mathrm{e}^{-\overline{\mathrm{ad}}}}{\overline{\mathrm{ad}}} \circ (\exp^* \otimes \mathfrak{g}^*).$$

By definition,

$$\Phi(\exp^*(f)) = (\exp^* \otimes \mathfrak{g}^*) d_G(f).$$

The fact that $d_{\mathfrak{g}}(\exp^*(f))$ is the pullback of $d_G(f)$ via the exponential map implies that

$$d_{\mathfrak{g}}(\exp^*(f)) = (\exp^* \otimes d(\exp)^*)(d_G(f)) = \frac{1-\mathrm{e}^{-\overline{\mathrm{ad}}}}{\overline{\mathrm{ad}}} \circ (\exp^* \otimes \mathfrak{g}^*)(d_G(f))$$
$$= \frac{1-\mathrm{e}^{-\overline{\mathrm{ad}}}}{\overline{\mathrm{ad}}} \circ \Phi(\exp^*(f)).$$

Since any smooth function on $\mathcal{V}$ is of the form $\exp^*(f)$,

$$d_{\mathfrak{g}} = \frac{1-\mathrm{e}^{-\overline{\mathrm{ad}}}}{\overline{\mathrm{ad}}} \circ \Phi \implies \Phi = \frac{\overline{\mathrm{ad}}}{1-\mathrm{e}^{-\overline{\mathrm{ad}}}} \circ d_{\mathfrak{g}}. \tag{26}$$

Now assume that U_G and $\mathcal{V}$ are chosen so that $\mathcal{V}$ is a sufficiently small open disc in $\mathfrak{g}$ containing 0. Also assume that G is not 1-dimensional. Then, any closed 1-form on U_G is also exact. Let $Y \in \mathfrak{g}^*$. Since the left-invariant 1-form $1 \otimes Y \in C(G) \otimes \mathfrak{g}^*$ is closed, it is exact as well. Thus, there is a function f_Y on U_G such that $d_G(f_Y) = 1 \otimes Y$. Then, $\Phi(\exp^*(f_Y)) = 1 \otimes Y$. On the other hand, $d_{\mathfrak{g}}(\exp^*(f_Y))$ is the pullback of the 1-form $1 \otimes Y$ via the exponential map. The formula (26) thus amounts to the formula for the pullback of a left-invariant 1-form on U_G via the exponential map.

Note that the bracket $\mathrm{ad} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ induces a map $\mathrm{ad} : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$. Let $\mu : C(\mathfrak{g}) \otimes \mathfrak{g}^* \to C(\mathfrak{g})$ denote the product taken after treating an element of $\mathfrak{g}^*$ as a function on $\mathcal{V}$. We now claim that as an endomorphism in the space of smooth sections of $\mathcal{V} \times \mathfrak{g}^*$, $\overline{\mathrm{ad}}$ is given by the following composite map

$$C(\mathfrak{g}) \otimes \mathfrak{g}^* \xrightarrow{C(\mathfrak{g})\otimes \mathrm{ad}} C(\mathfrak{g}) \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \xrightarrow{\mu \otimes \mathfrak{g}^*} C(\mathfrak{g}) \otimes \mathfrak{g}^*.$$

Denote the above composite map by $\omega_{\mathfrak{g}}$.

To verify this, choose a basis $\{X_i\}$ of $\mathfrak{g}$ and a basis $\{Y_i\}$ of $\mathfrak{g}^*$ dual to $\{X_i\}$. Suppose that $[X_i, X_j] = \sum_k C_{ij}^k X_k$. Then, $\mathrm{ad}(Y_k) = \sum_{i,j} C_{ij}^k Y_i \otimes Y_j$. Therefore, if $f \in C(\mathfrak{g})$, then

$$(\mu \otimes \mathfrak{g}^*) \circ (C(\mathfrak{g}) \otimes \mathrm{ad})(f \otimes Y_k)\left(\sum_i a_i X_i\right) = f\left(\sum_i a_i X_i\right) \sum_{i,j} a_i C_{ij}^k Y_j.$$

On the other hand,

$$\overline{\mathrm{ad}}(f \otimes Y_k)\left(\sum_i a_i X_i\right) = f\left(\sum_i a_i X_i\right) \mathrm{ad}\left(\sum_i a_i X_i\right)(Y_k).$$

But $\mathrm{ad}(X_i)(Y_k) = \sum_j C_{ij}^k Y_j$. Therefore,

$$\overline{\mathrm{ad}}(f \otimes Y_k)\left(\sum_i a_i X_i\right) = f\left(\sum_i a_i X_i\right) \sum_{i,j} a_i C_{ij}^k Y_j.$$

It follows that

$$\frac{\overline{\mathrm{ad}}}{1-\mathrm{e}^{-\overline{\mathrm{ad}}}} = \frac{\omega_{\mathfrak{g}}}{1-\mathrm{e}^{-\omega_{\mathfrak{g}}}}$$

and that

$$\Phi = \frac{\omega_{\mathfrak{g}}}{1 - \mathrm{e}^{-\omega_{\mathfrak{g}}}} \circ d_{\mathfrak{g}}.$$

2.5.2. The classical picture. II. Let $\overline{\exp}$ denote the map from $\mathcal{V}$ to U_G such that $Z \mapsto \exp(Z)^{-1} = \exp(-Z)$. The discussion in Section 2.5.1 with exp replaced by $\overline{\exp}$ together with the fact that

$$d(\overline{\exp})_Z = -\frac{1 - \mathrm{e}^{-\mathrm{ad}(-Z)}}{\mathrm{ad}(-Z)} = -\frac{\mathrm{e}^{\mathrm{ad}(Z)} - 1}{\mathrm{ad}(Z)}$$

tells us that if

$$\Psi = -\frac{\omega_{\mathfrak{g}}}{\mathrm{e}^{\omega_{\mathfrak{g}}} - 1} \circ d_{\mathfrak{g}}$$

then the following diagram commutes.

$$\begin{array}{ccc} C(G) \otimes \mathfrak{g}^* & \xrightarrow{d_G} & C(G) \otimes \mathfrak{g}^* \\ \Big\downarrow \overline{\exp}^* & & \overline{\exp}^* \otimes \mathfrak{g}^* \Big\downarrow \\ C(\mathfrak{g}) & \xrightarrow{\Psi} & C(\mathfrak{g}) \otimes \mathfrak{g}^*. \end{array}$$

This is equivalent to the formula,

$$d_{\mathfrak{g}} = -\frac{\mathrm{e}^{\overline{\mathrm{ad}}} - 1}{\overline{\mathrm{ad}}} \circ \Psi.$$

The above formula is equivalent to giving a formula for the pullback by $\overline{\exp}$ of a left-invariant 1-form on U_G.

2.5.3. Enlarging our dictionary. The discussion in the previous two subsections helps us understand Theorem $2'$ better. The discussion in Section 2.5.1 says that if

$$\Phi = \frac{\omega_{\mathfrak{g}}}{1 - \mathrm{e}^{-\omega_{\mathfrak{g}}}} \circ d_{\mathfrak{g}} : C(\mathfrak{g}) \otimes \mathfrak{g}^* \to C(\mathfrak{g}) \otimes \mathfrak{g}^*$$

then the diagram

$$\begin{array}{ccc} C(G) & \xrightarrow{d_G} & C(G) \otimes \mathfrak{g}^* \\ \Big\downarrow \exp^* & & \exp^* \otimes \mathfrak{g}^* \Big\downarrow \\ C(\mathfrak{g}) & \xrightarrow{\Phi} & C(\mathfrak{g}) \otimes \mathfrak{g}^* \end{array}$$

commutes. The analogy between this and the diagram (17) of Theorem $2'$ is now fairly explicit. $\widehat{C}^{\bullet}(X)$ is analogous to $C(G)$. The map α_R is analogous to the connection d_G. $\mathrm{I}_{\mathrm{HKR}}$ is analogous to $\exp^*$. $T[-1]$ is analogous to $\mathfrak{g}$. $\Omega[1]$ is analogous to $\mathfrak{g}^*$. $\mathbf{S}^{\bullet}(\Omega[1])$ is analogous to $C(\mathfrak{g})$. The map $\overline{\mathbf{C}}$ is analogous to $d_{\mathfrak{g}}$. The map $\overline{\omega} : \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \to \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1]$ is analogous to $\omega_{\mathfrak{g}}$, and the map Φ_R is analogous to Φ. *In short, the diagram in* (17) *is analogous to the computation of the Duflo-like term describing the correction that needs to be applied to* $d_{\mathfrak{g}}$ *to yield the map* Φ. *This in turn is* *equivalent to the formula for the pullback of a left-invariant* 1*-form on* U_G *via the exponential map.*

$S : \widehat{C}^\bullet(X) \to \widehat{C}^\bullet(X)$ is analogous to the map $S_G : C(G) \to C(G)$ given by pulling back an element of $C(G)$ by the map $g \mapsto g^{-1}$. The map

$$-\alpha_L = (S \otimes \Omega[1]) \circ \alpha_R \circ S$$

is therefore analogous to $(S_G \otimes \mathfrak{g}^*) \circ d_G \circ S_G$. Moreover, $\exp^* \circ S_G = \overline{\exp}^*$. Therefore $\mathrm{I}_{\mathrm{HKR}} \circ S$ is analogous to $\overline{\exp}^*$. It follows that Φ_L is analogous to the map $-\Psi$ of the previous subsection. *In short, the diagram* (18) *is analogous to the computation of the term describing the correction that has to be applied to* $d_{\mathfrak{g}}$ *to describe the map* $-\Psi$*. This in turn is equivalent to the formula for the pullback of a left-invariant* 1*-form on* U_G *via the map* $\overline{\exp}$.

Finally we recall from [7] that the universal enveloping algebra in $\mathrm{D}^b(X)$ of the Lie algebra object $T[-1]$ was shown to be represented by the complex $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ of polydifferential operators with Hochschild coboundary. Results equivalent to this statement were obtained using different methods by Roberts and Willerton [8] and Markarian [6] as well. Yekutieli [9] showed that the functor RD applied to $\mathrm{D}^\bullet_{\mathrm{poly}}(X)$ yields $\widehat{C}^\bullet(X)$. It follows that the Hopf algebra object $\widehat{C}^\bullet(X)$ of $\mathrm{D}^b(X)$ is the "dual" in $\mathrm{D}^b(X)$ of the universal enveloping algebra in $\mathrm{D}^b(X)$ of the Lie algebra object $T[-1]$ of $\mathrm{D}^b(X)$.

We warn the reader that the material in the remaining part of this subsection is rather hazy. It can be verified that $\widehat{C}^\bullet(X)$ satisfies all the formal properties that a ring of functions on a Lie group is required to satisfy. It might therefore be possible to say that $\widehat{C}^\bullet(X)$ corresponds to a "Lie group object" in $\mathrm{D}^b(X)$.

One has to be very careful here. Since the concept of a geometric object like a manifold in $\mathrm{D}^b(X)$ does not make sense by itself, the best we can do is to try to define such a notion by attempting to define a *ring of functions on a manifold* in $\mathrm{D}^b(X)$. This has to be a commutative algebra object in $\mathrm{D}^b(X)$. If such a definition is possible, $T[-1]$ thought of as a manifold in $\mathrm{D}^b(X)$ should correspond to the algebra object $\mathbf{S}^\bullet(\Omega[1])$ of $\mathrm{D}^b(X)$.

The Lie algebra of the Lie group object $\widehat{C}^\bullet(X)$ should be $T[-1]$. The diagrams (17) and (18) in Theorem 2′ could then be thought of as being equivalent to "computing the pullback of a 1-form on on the Lie group object $\widehat{C}^\bullet(X)$ via the maps exp and $\overline{\exp}$ respectively". Of course, exp and $\overline{\exp}$ are defined solely by what they are as maps from $\widehat{C}^\bullet(X)$ to $\mathbf{S}^\bullet(\Omega[1])$. However, for this to make any sense, one should be able to define the notion of a differential form on a manifold in $\mathrm{D}^b(X)$.

3. Some essential linear algebra

The first subsection of this section proves some propositions and a lemma in linear algebra. The second subsection describes their extensions pertaining to the object $\mathbf{S}^\bullet(\Omega[1])$ of $\mathrm{D}^b(X)$. We remind the reader that maps between tensor products of graded $\mathbb{K}$-vector spaces (or graded $\mathcal{O}_X$-modules) that rearrange factors are assumed to take the appropriate signs into account. If x is a homogenous element of a graded $\mathbb{K}$-vector space W, $|x|$ will denote the degree of x.

3.1. Some propositions and a lemma of linear algebra. In this subsection, we will work with differential graded vector spaces over a field $\mathbb{K}$ of characteristic 0. Almost every dg-vector space in this section has 0 differential. We shall assume that the differential on a graded vector space is 0 unless we explicitly say otherwise. Let V be a finite-dimensional vector space over $\mathbb{K}$. Denote the dual of V by V^*. *Denote the dimension of V in this subsection by m.*

As usual $\mathbf{S}^\bullet(V[1])$ denotes the symmetric algebra $\oplus_i \wedge^i V[i]$ generated over $\mathbb{K}$ by V concentrated in degree -1. Similarly, $\mathbf{S}^\bullet(V^*[-1])$ denotes the symmetric algebra $\oplus_i \wedge^i V^*[-i]$ generated over $\mathbb{K}$ by V^* concentrated in degree 1. Note that the products on $\mathbf{S}^\bullet(V[1])$ and $\mathbf{S}^\bullet(V^*[-1])$ are wedge products.

There is a map

$$\mathbf{i}_V : \mathbf{S}^\bullet(V[1]) \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$$
$$Z \mapsto (W \mapsto W \wedge Z).$$

$\mathbf{i}_V$ takes an element Z of $\mathbf{S}^\bullet(V[1])$ to the endomorphism of $\mathbf{S}^\bullet(V[1])$ given by multiplication by Z on the right.

We will often denote the related map

$$\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V[1])$$
$$W \otimes Z \mapsto W \wedge Z$$

by $(\text{ - } \wedge \text{ - })_V$. The subscript V may be dropped at times when it is obvious.

Choose a basis $\{x_1, \dots, x_m\}$ of V. Let $\{y_1, \dots, y_m\}$ be a basis of V^* dual to $\{x_1, \dots, x_m\}$. Let $\mathbf{j}_{V^*}(y_i)$ be the endomorphism of $\mathbf{S}^\bullet(V[1])$ given by

$$\mathbf{j}_{V^*}(y_i)(x_j) = \delta_{ij}$$
$$\mathbf{j}_{V^*}(y_i)(1) = 0$$

$$\begin{aligned}(27)\quad & \mathbf{j}_{V^*}(y_i)(x_{i_1} \wedge \dots \wedge x_{i_k}) \\ & \qquad = \delta_{ii_k} x_{i_1} \wedge \dots \wedge x_{i_{k-1}} - \mathbf{j}_{V^*}(y_i)(x_{i_1} \wedge \dots \wedge x_{i_{k-1}}) \wedge x_{i_k}.\end{aligned}$$

Note that $\mathbf{j}_{V^*}$ extends by linearity to a map from $V^*[-1]$ to $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$. Let

$$\circ : \mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\otimes 2} \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$$

denote the composition map. Extend $\mathbf{j}_{V^*}$ to a map

$$\mathbf{j}_{V^*} : \mathbf{S}^\bullet(V^*[-1]) \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$$

by setting

$$(28)\qquad \mathbf{j}_{V^*}(Y_1 \wedge Y_2) = \mathbf{j}_{V^*}(Y_2) \circ \mathbf{j}_{V^*}(Y_1)\ \forall\ Y_1, Y_2 \in \mathbf{S}^\bullet(V^*[-1]).$$

We will often denote the related map

$$\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V^*[-1]) \to \mathbf{S}^\bullet(V[1])$$

$$W \otimes Y \mapsto \mathbf{j}_{V^*}(Y)(W)$$

by $(\text{ - } \bullet \text{ - })_V$. The subscript V may be dropped at times when it is obvious.

Remark 5 (A geometric analogy)**.** One can think of $\mathbf{S}^\bullet(V[1])$ as the ring of functions on an odd supermanifold M_V. For an element Z of $\mathbf{S}^\bullet(V[1])$, $\mathbf{i}_V(Z)$ is just the operator given by "multiplication on the right by Z". All operators on $\mathbf{S}^\bullet(V[1])$ act on the right in our viewpoint. From this viewpoint, elements of $V^*[-1]$ yield *vector fields* on M_V. The map $\mathbf{j}_{V^*}$ takes an element Y of $V^*[-1]$ to the constant vector field on M_V associated with Y. The reader can observe that the *Leibniz rule is part of the definition* of the map $\mathbf{j}_{V^*} : V^*[-1] \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ (see Equation (27)). The equation (28) in the definition of $\mathbf{j}_{V^*}$ just says that the map $\mathbf{j}_{V^*}$ takes an element Y of $\mathbf{S}^\bullet(V^*[-1])$ to the constant differential operator on M_V associated with Y.

Note that on $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ we have a natural product given by the composition $\circ$. Let $\circ^{\mathrm{op}}$ denote the product on $\mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\mathrm{op}}$. If $a, b \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ then $a \circ^{\mathrm{op}} b = b \circ a$. We have the following proposition.

Proposition 10. *The composite*

$$\mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1]) \xrightarrow{\mathbf{j}_{V^*} \otimes \mathbf{i}_V} \mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1]) \\ \xrightarrow{\circ^{\mathrm{op}}} \mathrm{End}(\mathbf{S}^\bullet(V[1]))$$

is an isomorphism of graded $\mathbb{K}$ *vector spaces with* 0 *differential.*

Proof. Denote the given composite by G_r.

Since all vector spaces involved in this proposition have 0 differential, it suffices to check that G_r is an isomorphism of graded $\mathbb{K}$-vector spaces. Further, it is easy to check that $\mathbf{i}_V$, $\mathbf{j}_{V^*}$ and $\circ^{\mathrm{op}}$ are degree preserving. It therefore, suffices to check that G_r is an isomorphism of $\mathbb{K}$-vector spaces. All the $\mathbb{K}$-vector spaces involved in this proposition are finite-dimensional. Further, $\mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1])$ and $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ have the same dimension as $\mathbb{K}$-vector spaces. It therefore, suffices to check that G_r injective.

Choose a basis $\{x_1, \dots, x_m\}$ of V. Let $\{y_1, \dots, y_m\}$ be a basis of V^* dual to $\{x_1, \dots, x_m\}$. By convention, we list the elements of any subset of $\{1, \dots, m\}$ in *ascending order.* We can then define an ordering $\prec$ on the set of subsets of $\{1, \dots, m\}$ by setting

$$S \prec T \text{ if } |S| < |T|$$

and $\{i_1, \dots, i_k\} \prec \{j_1, \dots, j_k\}$ if $(i_1, \dots, i_k) \prec (j_1, \dots, j_k)$ in the lexicographic order.

For $S = \{i_1, \ldots, i_k\}$, let x_S denote $x_{i_1} \wedge \cdots \wedge x_{i_k}$. Similarly, y_S will denote $y_{i_1} \wedge \cdots \wedge y_{i_k}$. It is easy to verify that

$$\mathbf{j}_{V^*}(y_S)(x_S) = \pm 1 \tag{29}$$

$$\mathbf{j}_{V^*}(y_S)(x_T) = 0 \text{ if } T \prec S.$$

An element of $\mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1])$ is given by an expression of the form

$$\sum_{S \subset \{1,\ldots,m\}} y_S \otimes a_S,\ a_S \in \mathbf{S}^\bullet(V[1]).$$

Choose $S_0 \subset \{1, \ldots, m\}$ to be the *least* subset (under the ordering $\prec$) of $\{1, \ldots, m\}$ such that $a_{S_0} \neq 0$. Then

$$G_r\left(\sum_{S \subset \{1,\ldots,m\}} y_S \otimes a_S\right)(x_{S_0}) = \pm a_{S_0}$$

by (29). It follows that G_r is injective. □

Remark 6. Let M_V be the supermanifold of whose ring of functions is $\mathbf{S}^\bullet(V[1])$. Under our convention that all operators on $\mathbf{S}^\bullet(V[1])$ act on the right, G_r identifies $\wedge^i T[-i] \otimes \mathbf{S}^\bullet(V[1])$ with the space of principal symbols of differential operators of order i on M_V. Proposition 10 says that every endomorphism of $\mathbf{S}^\bullet(V[1])$ is given by a differential operator on M_V.

Notation. For the rest of this paper, G_r shall denote the isomorphism in Proposition 10 and F_r shall denote its inverse. In addition, if

$$\tau : \mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V^*[-1])$$

denotes the swap map the composite

$$\begin{array}{c} \mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1]) \\ \tau \big\downarrow \\ \mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V^*[-1]) \xrightarrow{\mathbf{i}_V \circ^{\mathrm{op}} \mathbf{j}_{V^*}} \mathrm{End}(\mathbf{S}^\bullet(V[1])) \end{array}$$

shall be denoted by G_l. The inverse of G_l will be denoted by F_l.

Let $\pi_k : \mathbf{S}^\bullet(V[1]) \to \wedge^k V[k]$ denote the natural projection. Note that we have a nondegenerate pairing $\langle , \rangle$ on $\mathbf{S}^\bullet(V[1])$. This is given by the composite

$$\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \xrightarrow{\wedge} \mathbf{S}^\bullet(V[1]) \xrightarrow{\pi_m} \wedge^m V[m]$$

where m is the dimension of V.

Definition 1. The *adjoint* $\Phi^+ \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ of a homogenous element Φ of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ is the unique element of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ satisfying

$$\langle \Phi(a), b \rangle = (-1)^{|\Phi||b|} \langle a, \Phi^+(b) \rangle \ \forall \text{ homogenous } a, b \in \mathbf{S}^\bullet(V[1]).$$

The adjoint of an arbitrary element of $\mathrm{End}(\mathbf{S}^\bullet(V[1])$ is the sum of the adjoints of its homogenous components.

Let

$$\mathrm{ev}_V : \mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1]))$$

be the map

$$W \otimes \Phi \mapsto \Phi(W).$$

Denote the map which takes an element of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ to its adjoint by $\mathbf{A}_V$. Let $\langle\ ,\mathrm{ev}^+\rangle_V$ denote the composite

$$\begin{array}{c}
\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1]) \\
\Big\downarrow {\scriptstyle \mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \otimes \mathbf{A}_V} \\
\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1]) \\
\Big\downarrow {\scriptstyle \mathbf{S}^\bullet(V[1]) \otimes \mathrm{ev}_V} \\
\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \\
\Big\downarrow {\scriptstyle \langle\ ,\ \rangle} \\
\wedge^m V[m].
\end{array}$$

Then the following diagram commutes.

$$\begin{tikzcd}
\mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes \mathbf{S}^\bullet(V[1]) \arrow[r, "\mathrm{ev}_V \otimes \mathbf{S}^\bullet(V[1])"] \arrow[d, "\mathbf{S}^\bullet(V[1]) \otimes \tau"] & \mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \arrow[d, "\langle,\rangle"'] \\
\mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1])) \arrow[r, "\langle\ ,\mathrm{ev}^+\rangle_V"] & \wedge^m V[m].
\end{tikzcd}$$

The map $\tau : \mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ in the above diagram swaps factors.

The following proposition describes the basic properties of the adjoint.

Proposition 11. (1) *Let L be an element of* $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$. *Then*

$$L^{++} = L.$$

(2) *If L_1 and L_2 are homogenous elements of* $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$, *then*

$$(L_1 \circ L_2)^+ = (-1)^{|L_1||L_2|}(L_2)^+ \circ (L_1)^+.$$

(3) *If $Z \in \mathbf{S}^\bullet(V[1])$ is a homogenous element, then*

$$\mathbf{i}_V(Z)^+ = \mathbf{i}_V(Z).$$

(4) *If $Y \in \mathbf{S}^\bullet(V^*[-1])$ is a homogenous element, then*

$$\mathbf{j}_{V^*}(Y)^+ = (-1)^{|Y|}\mathbf{j}_{V^*}(Y).$$

(5) *If $Z \in \mathbf{S}^\bullet(V[1])$ and $Y \in \mathbf{S}^\bullet(V^*[-1])$ are homogenous elements, then*

$$G_l(Y \otimes Z)^+ = (-1)^{|Y|}G_r(Y \otimes Z).$$

Proof. Observe that if $L \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ is homogenous, then $|L^+| = |L|$. Also note that if $a, b \in \mathbf{S}^\bullet(V[1])$ are homogenous, then $\langle a, b\rangle = (-1)^{|a||b|}\langle b, a\rangle$. Also recall that the pairing $\langle , \rangle$ is nondegenerate.

If $a, b \in \mathbf{S}^\bullet(V[1])$ are homogenous elements and if $L \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ is homogenous, note that

$$\begin{aligned}\langle L^{++}(a), b\rangle &= (-1)^{|b||a|+|b||L|}\langle b, L^{++}(a)\rangle = (-1)^{|b||a|+|b||L|+|a||L|}\langle L^+(b), a\rangle \\ &= (-1)^{|b||a|+|b||L|+|a||L|+|a||b|+|a||L|}\langle a, L^+(b)\rangle = \langle L(a), b\rangle.\end{aligned}$$

Part (1) of this proposition now follows immediately from this calculation.

For the rest of this proof a and b shall be homogenous elements of $\mathbf{S}^\bullet(V[1])$.

If L_1 and L_2 are homogenous elements of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$, then

$$\begin{aligned}&(-1)^{|L_1||L_2|}\langle a, (L_2)^+ \circ (L_1)^+(b)\rangle \\ &= (-1)^{|L_1||L_2|+|L_2|(|b|+|L_1|)}\langle L_2(a), L_1^+(b)\rangle \\ &= (-1)^{|L_1||L_2|+|L_2|(|b|+|L_1|)+|L_1||b|}\langle L_1 \circ L_2(a), b\rangle \\ &= (-1)^{|b|(|L_1|+|L_2|)}\langle L_1 \circ L_2(a), b\rangle.\end{aligned}$$

Part (2) of this proposition now follows from the observation that

$$|L_1 \circ L_2| = |L_1| + |L_2|.$$

Part (3) of this proposition is immediate from the relevant definitions and the fact that

$$a \wedge Z \wedge b = (-1)^{|b||Z|} a \wedge b \wedge Z.$$

To verify part (4), choose a basis $\{x_1, \dots, x_m\}$ of V. Let $\{y_1, \dots, y_m\}$ be a basis of V^* dual to $\{x_1, \dots, x_m\}$. For an ordered subset $S = \{i_1, \dots, i_k\}$ of $\{1, \dots, m\}$ let y_S denote $y_{i_1} \wedge \dots \wedge y_{i_k}$ and let x_S denote $x_{i_1} \wedge \dots \wedge x_{i_k}$. If T is disjoint from S, note that

$$\mathbf{j}_{V^*}(y_S)(x_T \wedge x_S) = x_T \wedge \mathbf{j}_{V^*}(y_S)(x_S) = (-1)^{\frac{k(k-1)}{2}} x_T.$$

Let T and T' be subsets of $\{1, \dots, m\}$ disjoint from S. Then,

$$\begin{aligned}&\mathbf{j}_{V^*}(y_S)(x_T \wedge x_S) \wedge (x_{T'} \wedge x_S) \\ &= (-1)^{\frac{k(k-1)}{2}} x_T \wedge x_{T'} \wedge x_S \\ &= (-1)^{\frac{k(k-1)}{2}}(-1)^{|T'||S|} x_T \wedge x_S \wedge x_{T'} \\ &= (-1)^{|T'||S|} x_T \wedge x_S \wedge \mathbf{j}_{V^*}(y_S)(x_{T'} \wedge x_S) \\ &= (-1)^{(|T'|+|S|)|S|+|S|} x_T \wedge x_S \wedge \mathbf{j}_{V^*}(y_S)(x_{T'} \wedge x_S).\end{aligned}$$

Putting $a = x_T \wedge x_S$, $b = x_{T'} \wedge x_S$ we see that $|b| = |T'| + |S|$. Further, $|Y| = -|S|$. Part (4) now follows from the above computation once we recall that $\langle a, b\rangle = \pi_m(a \wedge b)$.

Part (5) follows from part (2), part (3) and part (4).

$$G_l(Y\otimes Z) = (-1)^{|Z||Y|}\mathbf{j}_{V^*}(Y)\circ \mathbf{i}_V(Z).$$

Thus,

$$\begin{aligned} G_l(Y\otimes Z)^+ &= (-1)^{|Z||Y|}(\mathbf{j}_{V^*}(Y)\circ \mathbf{i}_V(Z))^+ \\ &= (-1)^{|Z||Y|}(-1)^{|Z||Y|}(\mathbf{i}_V(Z))^+\circ(\mathbf{j}_{V^*}(Y))^+ \\ &= (-1)^{|Y|}(\mathbf{i}_V(Z))\circ(\mathbf{j}_{V^*}(Y)) = (-1)^{|Y|}G_r(Y\otimes Z). \qquad \square \end{aligned}$$

Recall that $\pi_j : \mathbf{S}^\bullet(V[1]) \to \wedge^j V[j]$ denotes the natural projection. We will denote the projection

$$\mathbf{S}^\bullet(V^*[-1])\otimes \pi_j : \mathbf{S}^\bullet(V^*[-1])\otimes \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V^*[-1])\otimes \wedge^j V[j]$$

by π_j itself. Let I denote the endomorphism of $\mathbf{S}^\bullet(V^*[-1])$ that multiplies $\wedge^i V^*[-i]$ by $(-1)^i$. The following proposition is really a corollary of Proposition 11.

Proposition 12. *If* $L \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$, *then*

$$\pi_0(F_l(L)) = I(\pi_0(F_r(L^+))).$$

Proof. This is almost immediate from part (5) of Proposition 11. Let $Y \in \mathbf{S}^\bullet(V^*[-1])$ and $Z \in \mathbf{S}^\bullet(V[1])$ be homogenous. By part (5) of Proposition 11

$$G_l(Y\otimes Z)^+ = (-1)^{|Y|}G_r(Y\otimes Z).$$

By definition, $F_r(G_r(Y\otimes Z)) = Y\otimes Z$. On the other hand,

$$F_l(G_r(Y\otimes Z)^+) = F_l((-1)^{|Y|}G_l(Y\otimes Z) = (-1)^{|Y|}Y\otimes Z.$$

Since G_r and $\mathbf{A}_V$ are degree preserving isomorphisms of $\mathbb{K}$-vector spaces, any homogenous element in $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ is of the form $G_r(Y\otimes Z)^+$. It follows from the above computation that

$$F_l(M) = (I\otimes \mathbf{S}^\bullet(V[1]))F_r(M^+)$$

for any homogenous $M \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$. Therefore,

$$\pi_j(F_l(L)) = I(\pi_j(F_r(L^+)))$$

for any j. When $j = 0$, we get the desired proposition. $\square$

Convention To simplify notation, we follow the following convention: if $a \in \mathbf{S}^\bullet(V^*[-1])\otimes \mathbf{S}^\bullet(V[1])$, then $G_r(a)$ will be denoted by a itself. *Keeping this convention in mind:*

Proposition 13. *If* $a, b \in \mathbf{S}^\bullet(V^*[-1])\otimes \mathbf{S}^\bullet(V[1])$, *then*

$$\pi_0(F_r(a\circ b)) = \pi_0(F_r(\pi_0(a)\circ b)).$$

Proof. It suffices to check that if $Z \in \wedge^i V[i]$with $i > 0$, then

$$\pi_0(Y \otimes Z \circ Y' \otimes Z') = 0$$

for any $Z' \in \mathbf{S}^\bullet(V[1])$, $Y, Y' \in \mathbf{S}^\bullet(V^*[-1])$. Note that by our convention,

$$Y \otimes Z \circ Y' \otimes Z' = \mathbf{i}_V(Z) \circ \mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z') \circ \mathbf{j}_{V^*}(Y').$$

For subsets S and T of $\{1, \ldots, m\}$, let x_S and y_T be as in the proof of Proposition 11, part (4). By Proposition 10,

$$\mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z') \circ \mathbf{j}_{V^*}(Y') = \sum_{S,T \subset \{1,\ldots,m\}} a_{S,T} y_T \otimes x_S$$

for some $a_{S,T} \in \mathbb{K}$. Then,

$$\mathbf{i}_V(Z) \circ \mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z') \circ \mathbf{j}_{V^*}(Y') = \sum_{S,T \subset \{1,\ldots,m\}} a_{S,T} y_T \otimes x_S \wedge Z.$$

Since $Z \in \wedge^i V[i]$, $x_S \wedge Z \in \oplus_{k \geq i} \wedge^k V[k] \subset \oplus_{k>0} \wedge^k V[k]$. It follows that

$$\pi_0\left(\sum_{S,T \subset \{1,\ldots,m\}} a_{S,T} y_T \otimes x_S \wedge Z\right) = 0.$$

This proves the desired proposition. □

Remark 7. Proposition 10 said that every element of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ can be thought of as a differential operator on M_V. The isomorphism F_r makes this identification. The map $\pi_0 : \mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V^*[-1])$ should be thought of as the map which "takes the constant term" of a differential operator. Proposition 13 says that

$$\text{const. term}(\mathcal{D}_1 \circ \mathcal{D}_2) = \text{const. term}((\text{ const. term } (\mathcal{D}_1)) \circ \mathcal{D}_2)$$

for any two differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on M_V.

Recall that by Proposition 10, F_r identifies $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ with

$$\mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1]).$$

We also remarked (in Remark 6) that the direct summand

$$\wedge^i V^*[-i] \otimes \mathbf{S}^\bullet(V[1])$$

of $\mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1])$ can be thought of as the space of principal symbols of differential operators of order i on $\mathbf{S}^\bullet(V[1])$. Reflecting this understanding, we denote $\wedge^i V^*[-i] \otimes \mathbf{S}^\bullet(V[1])$ by D_i.

Note that the composition $\circ : \mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\otimes 2} \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ equips $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ with the structure of a graded associative $\mathbb{K}$-algebra. This algebra structure induces a Lie superalgebra structure on $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$. If $a, b \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ are homogenous, then

$$[a, b]_V = a \circ b - (-1)^{|a||b|} b \circ a.$$

Also note that the map $\circ^{\mathrm{op}} : \mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\otimes 2} \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ equips $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ with the structure of a graded associative $\mathbb{K}$-algebra. We denote this $\mathbb{K}$-algebra by $\mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\mathrm{op}}$. Its algebra structure induces the structure of a Lie superalgebra on $\mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\mathrm{op}}$. If $a, b \in \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ are homogenous, then

$$[a,b]_V^{\mathrm{op}} = a \circ^{\mathrm{op}} b - (-1)^{|a||b|} b \circ^{\mathrm{op}} a = [b,a]_V.$$

Proposition 14.

$$[D_1, D_1]_V \subset D_1.$$

Proof. Let $H \in \mathbf{S}^\bullet(V[1])$,let $Z \in \mathbf{S}^\bullet(V[1])$ and let $Y \in \mathbf{S}^\bullet(V^*[-1])$. Recall that $(H \bullet Y)$ denotes $\mathbf{j}_{V^*}(Y)(H)$. Note that $HZ := H \wedge Z = \mathbf{i}_V(Z)(H)$. Keep in mind that $Y \otimes Z$ is identified with $G_r(Y \otimes Z)$. Then

$$Y \otimes Z(H) = (H \bullet Y)Z.$$

Let $Z_1, Z_2 \in \mathbf{S}^\bullet(V[1])$ be homogenous and let $y_1, y_2 \in V^*[-1]$. Then, if $H \in \mathbf{S}^\bullet(V[1])$,

$$\begin{aligned}(y_1 \otimes Z_1)&\circ(y_2 \otimes Z_2)(H)\\ &= ((H \bullet y_2)Z_2 \bullet y_1)Z_1\\ &= (H \bullet y_2)(Z_2 \bullet y_1)Z_1 + (-1)^{|Z_2|}((H \bullet y_2) \bullet y_1)Z_2Z_1.\end{aligned}$$

Therefore,

$$(y_1 \otimes Z_1) \circ (y_2 \otimes Z_2) = y_2 \otimes (Z_2 \bullet y_1)Z_1 + (-1)^{|Z_2|} y_2 \wedge y_1 \otimes Z_2Z_1. \tag{30}$$

Similarly,

$$(y_2 \otimes Z_2) \circ (y_1 \otimes Z_1) = y_1 \otimes (Z_1 \bullet y_2)Z_2 + (-1)^{|Z_1|} y_1 \wedge y_2 \otimes Z_1Z_2. \tag{31}$$

If $\mathcal{D}_1 = y_1 \otimes Z_1$ and $\mathcal{D}_2 = y_2 \otimes Z_2$ then $|\mathcal{D}_1| = |Z_1| - 1$ and $|\mathcal{D}_2| = |Z_2| - 1$. It then follows from (30) and (31) that

$$\begin{aligned}\mathcal{D}_1 \circ \mathcal{D}_2 &- (-1)^{|\mathcal{D}_1||\mathcal{D}_2|}\mathcal{D}_2 \circ \mathcal{D}_1\\ &= y_2 \otimes (Z_2 \bullet y_1)Z_1 - (-1)^{|\mathcal{D}_1||\mathcal{D}_2|} y_1 \otimes (Z_1 \bullet y_2)Z_2.\end{aligned} \tag{32}$$

Note that the right-hand side is an element of D_1. The desired proposition now follows immediately. □

We will assume that the Lie superalgebra $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ is equipped with the bracket $[,]_V$ unless we explicitly state otherwise.

Proposition 14 tells us that D_1 is a Lie subalgebra of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$. It also follows immediately from Proposition 14 that

$$[D_1, D_1]_V^{\mathrm{op}} \subset D_1.$$

It follows that D_1 equipped with the bracket $[,]_V^{\mathrm{op}}$ is a Lie subalgebra of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\mathrm{op}}$. D_m can be identified with the top symmetric power of D_1 over $\mathbf{S}^\bullet(V[1])$. In other words,

$$D_m \simeq \mathbf{S}^m_{\mathbf{S}^\bullet(V[1])} D_1.$$

One can then study the right adjoint action of the Lie algebra D_1 (equipped with the bracket $[,]_V^{\text{op}}$) on D_m. For an element L of D_1, let $\text{ad}(L)$ denote the right adjoint action of L on D_m with respect to the bracket $[,]_V^{\text{op}}$. Then

$$\begin{aligned}
&\text{ad}(L)(\mathcal{D}_m\mathcal{D}_{m-1}\dots\mathcal{D}_1)\\
&= \mathcal{D}_m\mathcal{D}_{m-1}\dots[\mathcal{D}_1, L]_V^{\text{op}} + (-1)^{|L||\mathcal{D}_1|}\mathcal{D}_m\mathcal{D}_{m-1}\dots[\mathcal{D}_2, L]_V^{\text{op}}\mathcal{D}_1\\
&\quad + \cdots (-1)^{|L|(|\mathcal{D}_1|+\cdots+|\mathcal{D}_{m-1}|)}[\mathcal{D}_m, L]_V^{\text{op}}\dots\mathcal{D}_1\\
&= \mathcal{D}_m\mathcal{D}_{m-1}\dots[L, \mathcal{D}_1] + (-1)^{|L||\mathcal{D}_1|}\mathcal{D}_m\mathcal{D}_{m-1}\dots[L, \mathcal{D}_2]\mathcal{D}_1\\
&\quad + \cdots (-1)^{|L|(|\mathcal{D}_1|+\cdots+|\mathcal{D}_{m-1}|)}[L, \mathcal{D}_m]\dots\mathcal{D}_1
\end{aligned}$$

for homogenous elements $\mathcal{D}_i \in D_1$. In defining $\text{ad}(L)$, D_m is treated as $\mathbf{S}^m_{\mathbf{S}^\bullet(V[1])}D_1$.

One also has the right adjoint action $\overline{\text{ad}}(L)$ of L on $D_1^{\otimes m}$ (with respect to the bracket $[,]_V^{\text{op}}$ on D_1).

$$\begin{aligned}
&\overline{\text{ad}}(L)(\mathcal{D}_m \otimes \mathcal{D}_{m-1} \otimes \cdots \otimes \mathcal{D}_1)\\
&= \mathcal{D}_m \otimes \mathcal{D}_{m-1} \otimes \cdots \otimes [\mathcal{D}_1, L]_V^{\text{op}}\\
&\quad + (-1)^{|L||\mathcal{D}_1|}\mathcal{D}_m \otimes \mathcal{D}_{m-1} \otimes \cdots \otimes [\mathcal{D}_2, L]_V^{\text{op}} \otimes \mathcal{D}_1\\
&\quad + \cdots (-1)^{|L|(|\mathcal{D}_1|+\cdots+|\mathcal{D}_{m-1}|)}[\mathcal{D}_m, L]_V^{\text{op}} \otimes \cdots \otimes \mathcal{D}_1\\
&= \mathcal{D}_m \otimes \mathcal{D}_{m-1} \otimes \cdots \otimes [L, \mathcal{D}_1] + (-1)^{|L||\mathcal{D}_1|}\mathcal{D}_m \otimes \mathcal{D}_{m-1} \otimes \cdots \otimes [L, \mathcal{D}_2] \otimes \mathcal{D}_1\\
&\quad + \cdots (-1)^{|L|(|\mathcal{D}_1|+\cdots+|\mathcal{D}_{m-1}|)}[L, \mathcal{D}_m] \otimes \cdots \otimes \mathcal{D}_1
\end{aligned}$$

for homogenous elements $\mathcal{D}_i \in D_1$.

Let $p : D_1^{\otimes m} \to D_m$ denote the map

$$\mathcal{D}_m \otimes \mathcal{D}_{m-1} \otimes \cdots \otimes \mathcal{D}_1 \mapsto \mathcal{D}_m\mathcal{D}_{m-1}\dots\mathcal{D}_1.$$

The following proposition is clear from the definitions.

Proposition 15. *The following diagram commutes.*

$$\begin{array}{ccc}
D_1^{\otimes m} & \xrightarrow{p} & D_m \\
\Big\downarrow{\scriptstyle \overline{\text{ad}}(L)} & & \Big\downarrow{\scriptstyle \text{ad}(L)} \\
D_1^{\otimes m} & \xrightarrow{p} & D_m.
\end{array}$$

Choose a basis $\{x_1, \dots, x_m\}$ of V and a basis $\{y_1, \dots, y_m\}$ of V^* dual to $\{x_1, \dots, x_m\}$. Let $\mathbf{1}^m : \mathbb{K} \to \wedge^m V^*[-m] \otimes \wedge^m V[m]$ be the map

$$1 \mapsto y_m \wedge \cdots \wedge y_1 \otimes x_1 \wedge \cdots \wedge x_m.$$

Let $\tau : \mathbf{S}^\bullet(V[1]) \otimes \wedge^m V^*[-m] \to \wedge^m V^*[-m] \otimes \mathbf{S}^\bullet(V[1])$ denote the swap map. Denote the composite map

$$\mathbf{S}^\bullet(V[1]) \xrightarrow{(\tau\otimes\wedge^m V[m])\circ(\mathbf{S}^\bullet(V[1])\otimes\mathbf{1}^m)} \wedge^m V^*[-m] \otimes \mathbf{S}^\bullet(V[1]) \otimes \wedge^m V[m]$$

by $\mathbf{1}^m$ itself.

Lemma 1. *Let $L \in D_1$ be a homogenous element. The following diagram commutes.*

$$\begin{tikzcd} \mathbf{S}^\bullet(V[1]) \arrow[r, "\mathbf{1}^m"] \arrow[d, "-L^+"] & \wedge^m V^*[-m] \otimes \mathbf{S}^\bullet(V[1]) \otimes \wedge^m V[m] \arrow[d, "(-1)^{|L|m}\mathrm{ad}(L)\otimes\wedge^m V[m]"] \\ \mathbf{S}^\bullet(V[1]) \arrow[r, "\mathbf{1}^m"] & \wedge^m V^*[-m] \otimes \mathbf{S}^\bullet(V[1]) \otimes \wedge^m V[m]. \end{tikzcd}$$

Proof. This lemma is proven by a direct computation. We once more recall that if $H, Z \in \mathbf{S}^\bullet(V[1])$ and if $Y \in \mathbf{S}^\bullet(V^*[-1])$ then HZ denotes

$$H \wedge Z = \mathbf{i}_V(Z)(H)$$

and $(H \bullet Y)$ denotes $\mathbf{j}_{V^*}(Y)(H)$.

Choose a basis $\{y_1, \ldots, y_m\}$ of V^* and a basis $\{x_1, \ldots, x_m\}$ of V dual to $\{y_1, \ldots, y_m\}$. We may assume without loss of generality that $L = y_1 \otimes Z$ where $Z \in \mathbf{S}^\bullet(V[1])$ is homogenous. Then, if $H \in \mathbf{S}^\bullet(V[1])$ is homogenous, by Proposition 11, parts (2), (3), and (4),

$$L^+(H) = (-1)^{-|Z|-1}((HZ) \bullet y_1) = (-1)^{|Z|+1}((HZ) \bullet y_1).$$

Thus,

$$\begin{aligned} (33)\quad & \mathbf{1}^m(L^+(H)) \\ & = (-1)^{(|H|+|Z|-1)m}(-1)^{|Z|+1} y_m \wedge \cdots \wedge y_1 \otimes ((HZ) \bullet y_1) \otimes x_1 \wedge \cdots \wedge x_m. \end{aligned}$$

On the other hand, if we treat H as an element of D_0, then

$$(y_1 \otimes Z) \circ H(P) = (PH \bullet y_1)Z = P(H \bullet y_1)Z + (-1)^{|H|}(P \bullet y_1)HZ$$

$$H \circ (y_1 \otimes Z)(P) = (P \bullet y_1)ZH.$$

Note that the degree of the operator $y_1 \otimes Z$ is $|Z| - 1$. It follows that

$$[y_1 \otimes Z, H](P) = P(H \bullet y_1)Z.$$

Thus,

$$[y_1 \otimes Z, H] = (H \bullet y_1)Z.$$

By (32) in the proof of Proposition 14,

$$[y_1 \otimes Z, y_i] = (-1)^{|Z|} y_1 \otimes (Z \bullet y_i).$$

Therefore,

$$y_m \wedge \cdots \wedge [y_1 \otimes Z, y_i] \wedge \cdots \wedge y_1 = 0 \ \forall\ i \neq 1.$$

Thus,

$$\begin{aligned}
&\mathrm{ad}(L) \otimes \wedge^m V[m](y_m \wedge \cdots \wedge y_1 \otimes H \otimes x_1 \wedge \cdots \wedge x_m) \\
&= y_m \wedge \cdots \wedge y_1 \otimes \Big((H \bullet y_1)Z \\
&\quad + (-1)^{|H|(|Z|-1)}(-1)^{|Z|}(Z \bullet y_1)H\Big) \otimes x_1 \wedge \cdots \wedge x_m \\
&= y_m \wedge \cdots \wedge y_1 \otimes \Big((H \bullet y_1)Z \\
&\quad + (-1)^{|H|(|Z|-1)}(-1)^{|Z|}(-1)^{|H|(|Z|-1)}H(Z \bullet y_1)\Big) \otimes x_1 \wedge \cdots \wedge x_m \\
&= y_m \wedge \cdots \wedge y_1 \otimes ((H \bullet y_1)Z + (-1)^{|Z|}H(Z \bullet y_1)) \otimes x_1 \wedge \cdots \wedge x_m.
\end{aligned}$$

Note that

$$(HZ \bullet y_1) = H(Z \bullet y_1) + (-1)^{|Z|}(H \bullet y_1)Z.$$

Further recall that

$$\mathbf{1}^m(H) = (-1)^{|H|m} y_m \wedge \cdots \wedge y_1 \otimes H \otimes x_1 \wedge \cdots \wedge x_m.$$

Therefore,

$$\begin{aligned}
&(-1)^{|L|m}\mathrm{ad}(L) \otimes \wedge^m V[m](\mathbf{1}^m(H)) \\
&= (-1)^{(|H|+|Z|-1)m}\mathrm{ad}(L) \otimes \wedge^m V[m](y_m \wedge \cdots \wedge y_1 \otimes H \otimes x_1 \wedge \cdots \wedge x_m) \\
&= (-1)^{(|H|+|Z|-1)m} y_m \wedge \cdots \wedge y_1 \otimes ((H \bullet y_1)Z + (-1)^{|Z|}H(Z \bullet y_1)) \\
&\qquad \otimes x_1 \wedge \cdots \wedge x_m \\
&= -(-1)^{(|H|+|Z|-1)m}(-1)^{|Z|+1} y_m \wedge \cdots \wedge y_1 \otimes (HZ \bullet y_1) \otimes x_1 \wedge \cdots \wedge x_m \\
&= -\mathbf{1}^m(L^+(H)).
\end{aligned}$$

This proves the desired lemma. □

Remark 8. Lemma 1 seems to be a phenomenon occurring in purely odd supergeometry only. Let M_V be the supermanifold whose ring of functions is $\mathbf{S}^\bullet(V[1])$. The pairing $\langle , \rangle$ on $\mathbf{S}^\bullet(V[1])$ is the pairing

$$\langle f, g \rangle = \int_{M_V} fg \;\forall\; f, g \in \mathbf{S}^\bullet(V[1]).$$

By $\int_{M_V}$, we of course mean a *Berezinian integral*. We can think of the usual geometric analog of $\mathbf{S}^\bullet(V[1])$ to be the ring of compactly supported functions on a smooth oriented manifold M. The analog of the pairing $\langle , \rangle$ on $\mathbf{S}^\bullet(V[1])$ is the pairing

$$(f, g) \mapsto \int_M fg\,d\mu$$

where $d\mu$ is the measure arising out of a volume form on M.On M_V there is a constant top-order differential operator ∂ that is unique upto scalar. Lemma 1 then says that if D is a differential operator on M_V that is purely of first-order, and if f is a function on M_V, the Lie bracket of D with $f\partial$ is $\pm(D^+f)\partial$ where D^+ is the adjoint of D. In the usual geometric setting, the analog of ∂ would be a global, nowhere vanishing section of the top wedge power of the tangent bundle of M. Even if such a section exists on M, the analog of Lemma 1 does not hold even in the 1-dimensional case. For example, if $M = \mathbb{R}$, then the adjoint of the operator $f\frac{d}{dx}$ is the operator $-\left(\frac{df}{dx} + f\frac{d}{dx}\right)$. This follows from the standard integration by parts. But

$$\left[f\frac{d}{dx}, g\frac{d}{dx}\right] = f\frac{dg}{dx} - g\frac{df}{dx} \neq \pm\left(-g\frac{df}{dx} - f\frac{dg}{dx}\right).$$

We get back to proving more propositions in linear algebra that we require for future use.

Choose a basis $\{x_1, \ldots, x_m\}$ of V and a basis $\{y_1, \ldots, y_m\}$ of V^* dual to $\{x_1, \ldots, x_m\}$. Define $\mathbf{k}_V : \mathbf{S}^\bullet(V[1]) \to \mathrm{End}(\mathbf{S}^\bullet(V^*[-1]))$ by the formulae

$$\mathbf{k}_V(x_i)(y_j) = \delta_{ij}$$

$$\mathbf{k}_V(x_i)(1) = 0$$

$$\mathbf{k}_V(x_i)(Y_1 \wedge Y_2) = \mathbf{k}_V(x_i)(Y_1) + (-1)^{|Y_1|}Y_1 \wedge \mathbf{k}_V(x_i)(Y_2)$$

$$\mathbf{k}_V(X_1 \wedge X_2)(Y) = \mathbf{k}_V(X_1)(\mathbf{k}_V(X_2)(Y)).$$

To simplify notation, we shall denote $\mathbf{k}_V(Z)(Y)$ by $(Z|Y)$.

Remark 9. The map $\mathbf{j}_{V^*}$ identifies an element Y of $\mathbf{S}^\bullet(V^*[-1])$ with the operation "differentiation on the right by Y". The map $\mathbf{k}_V$ identifies an element Z of $\mathbf{S}^\bullet(V[1])$ with "differentiation on the left by Z".

We have the following proposition.

Proposition 16. *Let $Y \in \mathbf{S}^\bullet(V^*[-1])$ and let $Z \in \mathbf{S}^\bullet(V[1])$. Then,*

$$\pi_0(F_r(\mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z))) = (Z|Y).$$

Proof. This is again proven by a direct computation. We may assume without loss of generality that Y and Z are homogenous.

Choose a basis $\{x_1, \ldots, x_m\}$ of V and a basis $\{y_1, \ldots, y_m\}$ of V^* dual to $\{x_1, \ldots, x_m\}$. Let $H \in \mathbf{S}^\bullet(V[1])$. Let $Z = x_S$ and let $Y = y_T$ for some $S, T \subset \{1, \ldots, m\}$. If S and T are disjoint and if S is nonempty, $\mathbf{j}_{V^*}(Y)$ and $\mathbf{i}_V(Z)$ commute upto sign. It follows that $\pi_0(F_r(\mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z))) = 0$. Also, $(Z|Y) = 0$ if S and T are disjoint and S is nonempty. If S is empty, $x_S = 1$ by convention. Therefore, $\pi_0(F_r(\mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z))) = Y$ and $(Z|Y) = Y$. We therefore, have to prove this proposition for the case when S and T are not disjoint.

Let S and T be arbitrary subsets of $\{1, \ldots, m\}$. Suppose that $j \notin S \cup T$.

$$\begin{aligned}\mathbf{j}_{V^*}(y_j \wedge Y) \circ \mathbf{i}_V(Z \wedge x_j)(H) &= (HZ \wedge x_j \bullet y_j \wedge Y)\\ &= (HZ \bullet Y) - ((HZ \bullet y_j) \wedge x_j \bullet Y).\end{aligned}$$

Since j is not in T. It follows that

$$Z' \wedge x_j \bullet Y = \pm(Z' \bullet Y) \wedge x_j.$$

Therefore

$$\mathbf{j}_{V^*}(y_j \wedge Y) \circ \mathbf{i}_V(Z \wedge x_j)(H) = (HZ \bullet Y) \pm ((HZ \bullet y_j) \bullet Y) \wedge x_j.$$

It follows that

$$\pi_0(F_r(\mathbf{j}_{V^*}(y_j \wedge Y) \circ \mathbf{i}_V(Z \wedge x_j))) = \pi_0(F_r(\mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z))).$$

Note that

$$(Z \wedge x_j | y_j \wedge Y) = (Z|Y)$$

since $j \notin S \cup T$. The desired proposition follows for homogenous $Z = x_S$ and $Y = y_T$ by induction on $|S \cap T|$. For general Y and Z the proposition follows from the fact that the maps

$$\mathbf{S}^\bullet(V^*[-1]) \times \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V^*[-1])$$

$$(Y, Z) \mapsto \pi_0(F_r(\mathbf{j}_{V^*}(Y) \circ \mathbf{i}_V(Z)))$$

and

$$\mathbf{S}^\bullet(V^*[-1]) \times \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V^*[-1])$$

$$(Y, Z) \mapsto (Z|Y)$$

are both $\mathbb{K}$-bilinear. $\square$

Let $(\text{ - } || \text{ - })_V : \mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V^*[-1]) \to \mathbb{K}$ denote map

$$Z \otimes Y \mapsto p_0(Z|Y)$$

where $p_0 : \mathbf{S}^\bullet(V^*[-1]) \to \mathbb{K}$ denotes the projection to the degree 0 direct summand.

Let $\gamma_V : \mathbf{S}^\bullet(V[1]) \to \mathbf{S}^\bullet(V^*[-1]) \otimes \wedge^m V[m]$ be the isomorphism such that

$$\pi_m(\text{ - } \wedge \text{ - })_V = [(\text{ - } || \text{ - })_V \otimes \wedge^m V[m]] \circ (\mathbf{S}^\bullet(V[1]) \otimes \gamma_V).$$

Let ζ_V denote γ_V^{-1}.

Proposition 17. *If* $Z, W \in \mathbf{S}^\bullet(V[1])$, *then*

$$\zeta_V(\{(Z| \text{ - }) \otimes \wedge^m V[m]\}(\gamma(W))) = Z \wedge W.$$

Proof. This proposition is verified by a direct computation. Choose a basis $\{x_1, \dots, x_m\}$ of V and a basis $\{y_1, \dots, y_m\}$ of V^* dual to $\{x_1, \dots, x_m\}$. Without loss of generality, $Z = x_1 \wedge \cdots \wedge x_k$ and $W = x_{l+1} \wedge \cdots \wedge x_m$.

Then

$$\gamma_V(W) = y_l \wedge \cdots \wedge y_1 \otimes x_1 \wedge \cdots \wedge x_m$$

$$((Z| \text{ - }) \otimes \wedge^m V[m])(\gamma_V(W)) = (x_1 \wedge \cdots \wedge x_k | y_l \wedge \cdots \wedge y_1) \otimes x_1 \wedge \cdots \wedge x_m.$$

If $k > l$ then $(x_1 \wedge \cdots \wedge x_k | y_1 \wedge \cdots \wedge y_l) = 0$ and $x_1 \wedge \cdots \wedge x_k \wedge x_{l+1} \wedge \cdots \wedge x_m = 0$, proving this proposition. We may thus assume that $k \leq l$. Then

$$\begin{aligned} ((Z|\text{ - }) \otimes \wedge^m V[m])(\gamma_V(W)) &= (x_1 \wedge \cdots \wedge x_k | y_l \wedge \cdots \wedge y_1) \otimes x_1 \wedge \cdots \wedge x_m \\ &= (-1)^{k(l-k)} y_l \wedge \cdots \wedge y_{k+1} \otimes x_1 \wedge \cdots \wedge x_m \\ &= \gamma_V(x_1 \wedge \cdots \wedge x_k \wedge x_{l+1} \wedge \cdots \wedge x_m). \end{aligned}$$

This proves the desired proposition. $\square$

Let J_V denote the endomorphism of $\mathbf{S}^\bullet(V[1])$ multiplying $\wedge^i V[i]$ by $(-1)^i$. We also have the following proposition. Let

$$\tau : \wedge^m V[m] \otimes \mathbf{S}^\bullet(V^*[-1]) \otimes \wedge^m V[m] \to \mathbf{S}^\bullet(V^*[-1]) \otimes \wedge^m V[m] \otimes \wedge^m V[m]$$

denote the map that interchanges $\wedge^m V[m]$ and $\mathbf{S}^\bullet(V^*[-1]) \otimes \wedge^m V[m]$. Let

$$\begin{aligned} \simeq : \wedge^m V[m] \otimes \wedge^m V^*[-m] &\to \mathbb{K} \\ x_1 \wedge \cdots \wedge x_m \otimes y_m \wedge \cdots \wedge y_1 &\mapsto 1. \end{aligned}$$

Proposition 18. *The following diagram commutes.*

$$\begin{array}{ccc} \wedge^m V[m] \otimes \mathbf{S}^\bullet(V^*[-1]) \otimes \wedge^m V[m] \otimes \wedge^m V^*[-m] & \xrightarrow{(\text{ - } \bullet \text{ - }) \otimes \simeq} & \mathbf{S}^\bullet(V[1]) \\ \big\downarrow{\scriptstyle \tau \otimes \wedge^m V^*[-m]} & & {\scriptstyle J_V}\big\downarrow \\ \mathbf{S}^\bullet(V^*[-1]) \otimes \wedge^m V[m] \otimes \wedge^m V[m] \otimes \wedge^m V^*[m] & \xrightarrow{\zeta_V \otimes \simeq} & \mathbf{S}^\bullet(V[1]). \end{array}$$

Proof. This is verified by a direct computation as well. Let $\{x_i\}$, $\{y_i\}$ be as in the proof of the previous proposition.

$$(x_1 \wedge \cdots \wedge x_m \bullet y_m \wedge \cdots \wedge y_{k+1}) = x_1 \wedge \cdots \wedge x_k$$

implies that

$$\begin{aligned} ((\text{ - } \bullet \text{ - }) \otimes \simeq)(x_1 \wedge \cdots \wedge x_m \otimes y_m \wedge \cdots \wedge y_{k+1} \otimes x_1 \wedge \cdots \wedge x_m \otimes y_m \wedge \cdots \wedge y_1) \\ = x_1 \wedge \cdots \wedge x_k. \end{aligned}$$

Also,

$$\begin{aligned} &\tau(x_1 \wedge \cdots \wedge x_m \otimes y_m \wedge \cdots \wedge y_{k+1} \otimes x_1 \wedge \cdots \wedge x_m) \\ &\qquad = (-1)^{mk}(y_m \wedge \cdots \wedge y_{k+1} \otimes x_1 \wedge \cdots \wedge x_m \otimes x_1 \wedge \cdots \wedge x_m) \end{aligned}$$

and

$$\zeta_V(y_m \wedge \cdots \wedge y_{k+1} \otimes x_1 \wedge \cdots \wedge x_m) = (-1)^{k(m-k)} x_1 \wedge \cdots \wedge x_k$$

imply that $(\zeta_V \otimes \simeq) \circ (\tau \otimes \wedge^m V^*[-m])$ takes

$$(x_1 \wedge \cdots \wedge x_m \otimes y_m \wedge \cdots \wedge y_{k+1} \otimes x_1 \wedge \cdots \wedge x_m \otimes y_m \wedge \cdots \wedge y_1)$$

to

$$(-1)^{-k^2} x_1 \wedge \cdots \wedge x_k = J_V(x_1 \wedge \cdots \wedge x_k).$$

This proves the desired proposition. $\square$

Let $\mathbf{C}_V$ denote the comultiplication on $\mathbf{S}^\bullet(V[1])$. Think of $\mathbf{C}_V$ as an element of $\mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes \mathbf{S}^\bullet(V[1])$.

Proposition 19. *If* $Y \in \mathbf{S}^\bullet(V^*[-1])$ *then*

$$(\mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes (\text{ - }||Y) \circ \mathbf{C}_V) = \mathbf{j}_{V^*}(Y).$$

Proof. This is yet another proposition that is verified by a direct computation. Let $\{x_i\}, \{y_i\}$ be as in the proof of the previous proposition. Without loss of generality, $Y = y_1 \wedge \cdots \wedge y_k$. Then

$$\mathbf{C}_V(x_l \wedge \cdots \wedge x_1) = x_l \wedge \cdots \wedge x_{k+1} \otimes x_k \wedge \cdots \wedge x_1 + \sum_{S \neq \{l,\dots,k+1\}} \pm x_S \otimes x_{\overline{S}}$$

implies that

$$\begin{aligned}(\mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes (\text{ - }||Y) \circ \mathbf{C}_V)(x_l \wedge \cdots \wedge x_1) &= x_l \otimes \cdots \otimes x_{k+1} \\ &= \mathbf{j}_{V^*}(Y)(x_l \wedge \cdots \wedge x_1).\end{aligned}$$

The desired proposition follows immediately. □

3.2. Applying the linear algebra to $\mathbf{S}^\bullet(\boldsymbol{\Omega}[1])$. Let $\mathcal{M}$ be a locally free coherent $\mathcal{O}_X$-module. We begin with the remark that every proposition in the previous subsection holds in $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$ (and hence in $\mathrm{D}^b(X)$) with V replaced by $\mathcal{M}$, V^* replaced by $\mathcal{M}^* = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$. We are interested in the case when $\mathcal{M} = \Omega$. All graded $\mathcal{O}_X$-modules in this subsection are to be thought of as complexes of $\mathcal{O}_X$-modules with 0-differential.

All $\mathbb{K}$-vector spaces in this section are finite-dimensional. In this subsection and in future sections, n shall denote the dimension of X.

3.2.1. An important point for the reader to note. Each proposition or lemma in this subsection is proven by proving it for an arbitrary open subscheme U of X such that Ω is trivial over U. Then, $\Omega|_U = \mathcal{O}_U \otimes_{\mathbb{K}} V$ and $T|_U = \mathcal{O}_U \otimes_{\mathbb{K}} V^*$ for some finite-dimensional $\mathbb{K}$-vector space V. After observing that every map involved in the proposition/lemma is $\mathcal{O}_U$-linear, proving the proposition/lemma reduces to proving the corresponding proposition/lemma in Section 3.1. The proposition/lemma in Section 3.1 corresponding to each proposition here can be thought of as the "local computation" required to prove the corresponding proposition/lemma in this subsection.

With the above announcement we can just state the propositions and lemma that we wish to state.

Note that there is a morphism

$$\mathbf{i}_\Omega : \mathbf{S}^\bullet(\Omega[1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$$

such that whenever U is an open subscheme on X such that $\Omega \simeq \mathcal{O}_U \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-vector space V, then

$$\mathbf{i}_\Omega|_U = \mathbf{i}_V \otimes_{\mathbb{K}} \mathcal{O}_U.$$

We denote $\mathbf{i}_\Omega$ by $\mathbf{i}$ to simplify notation.

Similarly, we have a morphism

$$\mathbf{j}_T : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$$

such that whenever U is an open subscheme on X such that $\Omega \simeq \mathcal{O}_U \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-vector space V, then

$$\mathbf{j}_T|_U = \mathbf{j}_{V^*} \otimes_{\mathbb{K}} \mathcal{O}_U.$$

We denote $\mathbf{j}_T$ by $\mathbf{j}$ to simplify notation.

The following proposition corresponds to Proposition 10.

Proposition 20. *The composite*

$$\mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\mathbf{j}\otimes\mathbf{i}} \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\circ^{\mathrm{op}}} \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$$

is an isomorphism in $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$.

Notation. G_r shall denote the isomorphism in Proposition 20 and F_r shall denote its inverse. In addition, if

$$\tau : \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1])$$

denotes the swap map the composite

$$\mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\quad\tau\quad} \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \xrightarrow{\mathbf{i}\circ^{\mathrm{op}}\mathbf{j}} \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$$

shall be denoted by G_l. The inverse of G_l will be denoted by F_l.

Let $\pi_k : \mathbf{S}^\bullet(\Omega[1]) \to \Omega^k[k]$ denote the natural projection. Note that we have a nondegenerate pairing $\langle , \rangle$ on $\mathbf{S}^\bullet(\Omega[1])$. This is given by the composite

$$\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \xrightarrow{(\text{ - } \wedge \text{ - })} \mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\quad\pi_n\quad} \Omega^n[n].$$

Recall the definition of the adjoint of an element of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ from Section 3.1. Let $\mathbf{A}_V : \mathrm{End}(\mathbf{S}^\bullet(V[1])) \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ denote the map taking an element of $\mathrm{End}(\mathbf{S}^\bullet(V[1]))$ to its adjoint. We have a morphism $\mathbf{A}_\Omega : \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ in $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$ such that if U is an open subscheme of X such that $\Omega \simeq \mathcal{O}_U \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-vector space V, then

$$\mathbf{A}_\Omega = \mathbf{A}_V \otimes_{\mathbb{K}} \mathcal{O}_U.$$

Similarly, the map

$$\mathrm{ev}_V : \mathbf{S}^\bullet(V[1]) \otimes \mathrm{End}(\mathbf{S}^\bullet(V[1]))$$
$$Z \otimes \Phi \mapsto \Phi(Z)$$

yields a map

$$\mathrm{ev} : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \to \mathbf{S}^\bullet(\Omega[1])$$

such that

$$\mathrm{ev}|_U = \mathrm{ev}_V \otimes_{\mathbb{K}} \mathcal{O}_U$$

on any open subscheme U of X such that $\Omega|_U = V \otimes_{\mathbb{K}} \mathcal{O}_U$ for some $\mathbb{K}$-vector space V.

We denote $\mathbf{A}_\Omega$ by $\mathbf{A}$ to simplify notation. Let $\langle\ ,\mathrm{ev}^+\rangle$ denote the composite

$$\begin{array}{c}
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]) \\
\Big\downarrow{\scriptstyle \mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{A}} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]) \\
\Big\downarrow{\scriptstyle \mathbf{S}^\bullet(\Omega[1])\otimes\mathrm{ev}} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \\
\Big\downarrow{\scriptstyle \langle\ ,\ \rangle} \\
\Omega^n[n].
\end{array}$$

By the corresponding fact for a finite-dimensional $\mathbb{K}$-vector space V, the following diagram commutes in $\mathrm{Ch}^b(\mathcal{O}_X-\mathrm{mod})$ (and hence in $\mathrm{D}^b(X)$).

$$\begin{tikzcd}
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\mathrm{ev}\otimes\mathbf{S}^\bullet(\Omega[1])"] \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\tau"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \arrow[d, "\langle,\rangle"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \arrow[r, "\langle\ ,\mathrm{ev}^+\rangle"] & \Omega^n[n].
\end{tikzcd} \tag{34}$$

The map $\tau : \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ in the above diagram swaps factors.

The following proposition corresponds to Proposition 11 in Section 3.1.

Proposition 21. (1) *To begin with,*

$$\mathbf{A} \circ \mathbf{A} = \mathbf{1}_{\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))}.$$

(2) *If* $\tau : \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ *is the swap map, then the following diagram in* $\mathrm{Ch}^b(\mathcal{O}_X-\mathrm{mod})$ *commutes.*

$$\begin{tikzcd}
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \arrow[r, "(\mathbf{A}\otimes\mathbf{A})\circ\tau"] \arrow[d, "\circ"] & \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \arrow[d, "\circ"'] \\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \arrow[r, "\mathbf{A}"] & \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])).
\end{tikzcd}$$

(3) *Also,*

$$\mathbf{A} \circ \mathbf{i} = \mathbf{i}.$$

(4) *If* $I : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1])$ *denotes the endomorphism multiplying* $\wedge^i T[-i]$ *by* $(-1)^i$ *then*

$$\mathbf{A} \circ \mathbf{j} = \mathbf{j} \circ I.$$

(5) *Finally,*

$$\mathbf{A} \circ G_l = G_r \circ (I \otimes \mathbf{S}^\bullet(\Omega[1])).$$

Recall that $\pi_j : \mathbf{S}^\bullet(\Omega[1]) \to \Omega^j[j]$ denotes the natural projection. We will denote the projection

$$\mathbf{S}^\bullet(T[-1]) \otimes \pi_j : \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(T[-1]) \otimes \Omega^j[j]$$

by π_j itself. Let I be as in part (4), Proposition 21. The following proposition corresponds to Proposition 12 of Section 3.1.

Proposition 22.

$$\pi_0 \circ F_l = I \circ \pi_0 \circ F_r \circ \mathbf{A}.$$

Denote the composite

$$\begin{array}{ccc} (\mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]))^{\otimes 2} & & \\ {\scriptstyle G_r \circ G_r}\Big\downarrow & & \\ \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) & \xrightarrow{F_r} & \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \end{array}$$

by m to simplify notation. The proposition below corresponds to Proposition 13 of Section 3.1.

Proposition 23. *As morphisms in* $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$,

$$\pi_0 \circ \mathrm{m} = \pi_0 \circ \mathrm{m} \circ (\pi_0 \otimes \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1])).$$

Denote the direct summand $\wedge^i T[-i] \otimes \mathbf{S}^\bullet(\Omega[1])$ of $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ by D_i. Note that the map $[,]_V : \mathrm{End}(\mathbf{S}^\bullet(V[1]))^{\otimes 2} \to \mathrm{End}(\mathbf{S}^\bullet(V[1]))$ extends to a morphism $[,] : \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))^{\otimes 2} \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$. If U is an open subscheme on X such that $\Omega \simeq \mathcal{O}_U \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-vector space V, then

$$[,]|_U = [,]_V \otimes_{\mathbb{K}} \mathcal{O}_U.$$

The following proposition corresponds to Proposition 14 of Section 3.1.

Proposition 24. *The composite*

$$D_1 \otimes D_1 \longrightarrow \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))^{\otimes 2} \xrightarrow{[,]} \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$$

factors through the inclusion of D_1 *in* $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$.

Let V be a $\mathbb{K}$-vector space of dimension n. Denote the direct summands $V^*[-1] \otimes \mathbf{S}^\bullet(V[1])$ and $\wedge^n V^*[-n] \otimes \mathbf{S}^\bullet(V[1])$ of $\mathbf{S}^\bullet(V^*[-1]) \otimes \mathbf{S}^\bullet(V[1])$ by $\overline{D}_1$ and $\overline{D}_n$ respectively, *unlike* in Section 3.1. Recall that in Section 3.1 we defined maps

$$\mathrm{ad}(L) : \wedge^n V^*[-n] \otimes \mathbf{S}^\bullet(V[1]) \to \wedge^n V^*[-n] \otimes \mathbf{S}^\bullet(V[1])$$
$$\overline{\mathrm{ad}}(L) : \overline{D}_1^{\otimes n} \to \overline{D}_1^{\otimes n}$$

for any $L \in \overline{D}_1$. These yield maps

$$\begin{gathered} \mathrm{ad}_V : \wedge^n V^*[-n] \otimes \mathbf{S}^\bullet(V[1]) \otimes \overline{D}_1 \to \wedge^n V^*[-n] \otimes \mathbf{S}^\bullet(V[1]) \\ Y \otimes L \mapsto \mathrm{ad}(L)(Y) \\ \overline{\mathrm{ad}}_V : \overline{D}_1^{\otimes n} \otimes \overline{D}_1 \to \overline{D}_1^{\otimes n} \\ W \otimes L \mapsto \overline{\mathrm{ad}}(L)(W). \end{gathered}$$

These yield morphisms

$$\begin{gathered} \mathrm{ad} : \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes D_1 \to \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \\ \overline{\mathrm{ad}} : D_1^{\otimes n} \otimes D_1 \to D_1^{\otimes n} \end{gathered}$$

in $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$. If U is an open subscheme on X such that $\Omega \simeq \mathcal{O}_U \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-vector space V, then

$$\begin{gathered} \mathrm{ad}|_U = \mathrm{ad}_V \otimes_{\mathbb{K}} \mathcal{O}_U \\ \overline{\mathrm{ad}}|_U = \overline{\mathrm{ad}}_V \otimes_{\mathbb{K}} \mathcal{O}_U. \end{gathered}$$

The map $p_V : \overline{D}_1^{\otimes n} \to \overline{D}_n$ also yields a map

$$p : D_1^{\otimes n} \to \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1])$$

such that if $\Omega \simeq V \otimes_{\mathbb{K}} \mathcal{O}_U$ for some open subscheme U of X and some $\mathbb{K}$-vector space V, then,

$$p = p_V \otimes_{\mathbb{K}} \mathcal{O}_U.$$

Note that the dimension n of X is the rank of Ω as well. Let

$$\mathbf{1}^n : \mathcal{O}_X \to \wedge^n T[-n] \otimes \Omega^n[n]$$

denote the map dual to the evaluation map. There are maps

$$\tau : \wedge^n T[-n] \otimes \Omega^n[n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes D_1 \to \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes D_1 \otimes \Omega^n[n]$$

and

$$\tau' : \wedge^n T[-n] \otimes \Omega^n[n] \otimes \mathbf{S}^\bullet(\Omega[1]) \to \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n].$$

τ is obtained by swapping $\Omega^n[n]$ and $\mathbf{S}^\bullet(\Omega[1]) \otimes D_1$. τ' is obtained by swapping $\Omega^n[n]$ and $\mathbf{S}^\bullet(\Omega[1])$. Denote the composites

$$\tau \circ (\mathbf{1}^n \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes D_1)$$

and

$$\tau' \circ (\mathbf{1}^n \otimes \mathbf{S}^\bullet(\Omega[1]))$$

by $\mathbf{1}^n$. The following proposition corresponds to Proposition 15 of Section 3.1.

Proposition 25. *The following diagram commutes in* $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{tikzcd} D_1^{\otimes n} \otimes D_1 \arrow[r, "p\otimes D_1"] \arrow[d, "\overline{\mathrm{ad}}"] & D_n \otimes D_1 \arrow[d, "\mathrm{ad}"'] \\ D_1^{\otimes n} \arrow[r, "p"] & D_n. \end{tikzcd}$$

The following lemma corresponds to Lemma 1 of Section 3.1. In the following lemma, $\mathbf{A}$ denotes the composite

$$D_1 \longrightarrow \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \xrightarrow{\mathbf{A}} \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])).$$

Lemma 2. *The following diagram commutes in* $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{tikzcd} \mathbf{S}^\bullet(\Omega[1]) \otimes D_1 \arrow[r, "\mathbf{1}^n"] \arrow[d, "-\mathrm{ev}\circ(\mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{A})"] & \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes D_1 \otimes \Omega^n[n] \arrow[d, "\mathrm{ad}\otimes\Omega^n[n]"'] \\ \mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\mathbf{1}^n"] & \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n]. \end{tikzcd}$$

The map

$$(\text{ - } | \text{ - })_V : \mathbf{S}^\bullet(V[1]) \otimes \mathbf{S}^\bullet(V^*[-1]) \to \mathbf{S}^\bullet(V^*[-1])$$

$$Z \otimes Y \mapsto (Z|Y)$$

yields a map

$$(\text{ - } | \text{ - }) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

As usual, if U is an open subscheme on X such that $\Omega \simeq \mathcal{O}_U \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-vector space V, then

$$(\text{ - } | \text{ - }) = (\text{ - } | \text{ - })_V \otimes_{\mathbb{K}} \mathcal{O}_U.$$

Recall that the composition product on $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ was denoted by $\circ$. Let $\circ^{\mathrm{op}}$ denote the product on $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))^{\mathrm{op}}$. The following proposition corresponds to Proposition 16 of Section 3.1.

Proposition 26.

$$\pi_0 \circ F_r \circ (\mathbf{i} \circ^{\mathrm{op}} \mathbf{j}) = (\text{ - } | \text{ - }) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

Let $(\text{ - } || \text{ - }) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \to \mathcal{O}_X$ denote the map such that for any open $U \subset X$ such that $\Omega \simeq V \otimes_{\mathbb{K}} \mathcal{O}_U$ for some $\mathbb{K}$-vector space V,

$$(\text{ - } || \text{ - }) = (\text{ - } || \text{ - })_V \otimes_{\mathbb{K}} \mathcal{O}_U.$$

Let $\gamma : \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(T[-1]) \otimes S_X$ be the isomorphism such that

$$\pi_n \circ (\text{ - } \wedge \text{ - }) = ((\text{ - } || \text{ - }) \otimes S_X) \circ (\mathbf{S}^\bullet(\Omega[1]) \otimes \gamma).$$

Let ζ denote the inverse of γ.

The following proposition corresponds to Proposition 17 of Section 3.1.

Proposition 27.

$$\zeta([(\text{ - }|\text{ - })\otimes S_X]\circ[\mathbf{S}^\bullet(\Omega[1])\otimes\gamma]) = (\text{ - }\wedge\text{ - }) : \mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]).$$

Let $\tau : S_X\otimes\mathbf{S}^\bullet(T[-1])\otimes S_X \to \mathbf{S}^\bullet(T[-1])\otimes S_X\otimes S_X$ be the map swapping S_X with $\mathbf{S}^\bullet(T[-1]) \otimes S_X$. Let $\simeq$ be the identification of $S_X \otimes S_X^{-1}$ with $\mathcal{O}_X$. The following proposition corresponds to Proposition 18 of Section 3.1. Recall that J is the endomorphism of $\mathbf{S}^\bullet(\Omega[1])$ that multiplies $\Omega^j[j]$ by $(-1)^j$.

Proposition 28. *The following diagram commutes in* $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$.

$$\begin{tikzcd}
S_X \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[r, "\tau\otimes S_X^{-1}"] \arrow[d, "(\text{ - }\bullet\text{ - })\otimes\simeq"] & \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X \otimes S_X^{-1} \arrow[d, "\zeta\otimes\simeq"'] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "J"] & \mathbf{S}^\bullet(\Omega[1]).
\end{tikzcd}$$

Let $\mathbf{C}_\Omega$ denote the coproduct on $\mathbf{S}^\bullet(\Omega[1])$. Think of $\mathbf{C}_\Omega$ as a morphism in $\mathrm{D}^b(X)$ from $\mathcal{O}_X$ to $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1])$. The following proposition corresponds to Proposition 19 of Section 3.1:

Proposition 29.

$$\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes (\text{ - }||\text{ - }) \circ \mathbf{C}_\Omega = \mathbf{j} : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])).$$

4. The adjoint of Φ_R

This section is a continuation of Section 2. All maps in this section are in $\mathrm{D}^b(X)$ unless explicitly stated otherwise. Both in this section and the next, we use results from Section 3.2. All morphisms in $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$ described in Section 3.2 induce morphisms in $\mathrm{D}^b(X)$. The diagrams that were shown to commute in $\mathrm{Ch}^b(\mathcal{O}_X - \mathrm{mod})$ in Section 3.2 also commute in $\mathrm{D}^b(X)$.

4.1. Stating the main lemma of this section. Let Φ_L and Φ_R be as in Section 2. By Theorem 2′, the following diagrams commute in $\mathrm{D}^b(X)$.

$$\begin{tikzcd}
\widehat{C}^\bullet(X) \arrow[r, "\alpha_L"] \arrow[d, "\mathrm{I_{HKR}}"] & \widehat{C}^\bullet(X) \otimes \Omega[1] \arrow[d, "\mathrm{I_{HKR}}\otimes\Omega[1]"'] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\Phi_L"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]
\end{tikzcd}$$

$$\begin{tikzcd}
\widehat{C}^\bullet(X) \arrow[r, "\alpha_R"] \arrow[d, "\mathrm{I_{HKR}}"] & \widehat{C}^\bullet(X) \otimes \Omega[1] \arrow[d, "\mathrm{I_{HKR}}\otimes\Omega[1]"'] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\Phi_R"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1].
\end{tikzcd}$$

Φ_R can be thought of as lying in $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \Omega[1])$. Recall the endomorphism $\mathbf{A}$ of $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ (defined in Section 3.2) which takes a section of $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ to its adjoint.

Definition 2. The adjoint Φ_R^+ of Φ_R is the element $(\mathbf{A}\otimes\Omega[1])\circ\Phi_R$ of $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X,\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\Omega[1])$.

This section is devoted to finding an explicit formula for Φ_R^+.

Note that the Atiyah class of the tangent bundle T of X yields a morphism $\mathrm{At}_T:\Omega[1]\to\Omega[1]\otimes\Omega[1]$. Let $p:\Omega[1]^{\otimes i}\to\Omega^i[i]$ denote the standard projection. Let

$$\mathrm{At}_T^i:\Omega[1]\to\mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1]$$

denote the composite

$$(p\otimes\Omega[1])\circ(\Omega[1]^{\otimes(i-1)}\otimes\mathrm{At}_T)\circ\cdots\circ\mathrm{At}_T.$$

Then if $\frac{z}{\mathrm{e}^z-1}=1+\sum_i c_iZ^i$ the map

$$\frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}:=\mathbf{1}+\sum_i c_i\mathrm{At}_T^i:\Omega[1]\to\mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1]$$

makes sense. Denote the element

$$\det\left(\frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}\right)\in\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X,\mathbf{S}^\bullet(\Omega[1]))$$

by $\mathbf{f}$.

Note that $\mathbf{i}(\mathbf{f})$ and $\mathbf{i}(\mathbf{f}^{-1})$ are elements of $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X,\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])))$. Let $\mathbf{i}(\mathbf{f})\circ\Phi_R\circ\mathbf{i}(\mathbf{f}^{-1})$ denote the composite

$$\begin{array}{c}
\mathcal{O}_X\\
\Big\downarrow{\scriptstyle\Phi_R}\\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\Omega[1]\simeq\mathcal{O}_X\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathcal{O}_X\otimes\Omega[1]\\
\Big\downarrow{\scriptstyle\mathbf{i}(\mathbf{f})\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathbf{i}(\mathbf{f}^{-1})\otimes\Omega[1]}\\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\Omega[1]\\
\Big\downarrow{\scriptstyle\circ\otimes\Omega[1]}\\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\Omega[1].
\end{array}$$

The following lemma is the main lemma of this section.

Lemma 3.

$$\Phi_R^+=\mathbf{i}(\mathbf{f})\circ\Phi_R\circ\mathbf{i}(\mathbf{f}^{-1}).$$

The proof of this lemma requires further preparation. The following subsection is devoted to a key lemma (Lemma 4) used to prove Lemma 3. The proof of Lemma 3 itself is at the end of this section (in Section 4.4).

4.2. Comparing two "sections" of $D_n \otimes \Omega^n[n]$. Let

$$p_k : \mathbf{S}^\bullet(T[-1]) \to \wedge^k T[-k]$$

be the standard projection. View Φ_L as an element of

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \Omega[1]).$$

We have the following proposition.

Proposition 30.

$$(p_k \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]) \circ (F_r \otimes \Omega[1])(\Phi_L) = 0 \ \forall \ k \neq 1.$$

Remark 10. The above proposition just states that Φ_L can be thought of as an element of $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, D_1 \otimes \Omega[1])$. Recall that D_1 is like the space of "purely first-order differential operators on $\mathbf{S}^\bullet(\Omega[1])$". In principle, this proposition should follow from Proposition 7 and Theorem $2'$. I however, give a concrete proof below that uses the definition of Φ_L from Section 2, as I cant see how the above proposition follows immediately from Proposition 7.

Proof of Proposition 30. Let

$$\widetilde{\mathbf{C}} : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(T[-1])$$

denote coproduct on $\mathbf{S}^\bullet(T[-1]$. Denote the composite $(\mathbf{S}^\bullet(T[-1]) \otimes p_1) \circ \widetilde{\mathbf{C}}$ by $\hat{\mathbf{C}}$. Denote the wedge product on $\mathbf{S}^\bullet(\Omega[1])$ by μ *in this proof only.* Note that the dual of the map

$$\overline{\omega} : \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]$$

defined in Section 2 is the composite

$$(\mathbf{S}^\bullet(T[-1]) \otimes \mathrm{At}_T) \circ (\hat{\mathbf{C}} \otimes T[-1]).$$

It follows that

$$\overline{\omega} = (\mu \otimes \Omega[1]) \circ (\mathbf{S}^\bullet(\Omega[1]) \otimes \mathrm{At}_T).$$

In the latter composite, At_T is thought of as a morphism in $\mathrm{D}^b(X)$ from $\Omega[1]$ to $\Omega[1] \otimes \Omega[1]$. Therefore,

$$\overline{\omega}^i = (\mu \otimes \Omega[1]) \circ (\mathbf{S}^\bullet(\Omega[1]) \otimes \mathrm{At}_T^i) \tag{35}$$

where At_T^i is as in the previous subsection.

It follows from (35) that $(p_k \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]) \circ (F_r \otimes \Omega[1])(\overline{\omega}^i \circ \overline{\mathbf{C}})$ is given by the composite

$$\begin{array}{c} \mathcal{O}_X \\ \Big\downarrow{\scriptstyle \overline{\mathbf{C}}} \\ \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \Omega[1] \\ \Big\downarrow{\scriptstyle ((p_k \otimes \mathbf{S}^\bullet(\Omega[1])) \circ F_r) \otimes \mathrm{At}_T^i} \\ \wedge^k T[-k] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^i[i] \otimes \Omega[1] \\ \Big\downarrow{\scriptstyle \wedge^k T[-k] \otimes \mu \otimes \Omega[1]} \\ \wedge^k T[-k] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]. \end{array}$$

Note that from the above description, proving that

$$(((p_k \otimes \mathbf{S}^\bullet(\Omega[1])) \circ F_r) \otimes \Omega[1]) \circ \overline{\mathbf{C}} = 0$$

will imply that

$$(p_k \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]) \circ (F_r \otimes \Omega[1])(\overline{\omega}^i \circ \overline{\mathbf{C}}) = 0.$$

The desired proposition will then follow the fact that

$$\Phi_L = \sum_i c_i \overline{\omega}^i \circ \overline{\mathbf{C}}$$

where $\sum_i c_i z^i = \frac{z}{e^z - 1}$.

Proving this proposition has therefore been reduced to proving that

$$(((p_k \otimes \mathbf{S}^\bullet(\Omega[1])) \circ F_r) \otimes \Omega[1]) \circ \overline{\mathbf{C}} = 0 \ \forall k \neq 1. \tag{36}$$

Let U be an affine open subscheme of X such that $\Omega \simeq V \otimes_K \mathcal{O}_U$ for some K-vector space V. Proving (36) reduces to proving that

$$(((p_k \otimes \mathbf{S}^\bullet(V[1])) \circ F_r) \otimes V[1]) \circ \mathbf{C}_V = 0 \ \forall k \neq 1 \tag{37}$$

where $\mathbf{C}_V \in \mathrm{End}(\mathbf{S}^\bullet(V[1])) \otimes V[1] \simeq \mathrm{Hom}_{\mathbb{K}}(\mathbf{S}^\bullet(V[1]), \mathbf{S}^\bullet(V[1]) \otimes V[1])$ is the map

$$v_1 \wedge \cdots \wedge v_j \mapsto \sum_i (-1)^{j-i} v_1 \wedge \widehat{\cdots i \cdots} \wedge v_j \otimes v_i,$$

F_r is as in Section 3.1 and $p_k : \mathbf{S}^\bullet(V^*[-1]) \to \wedge^k V^*[-k]$ denotes the standard projection. (37) however, follows from the fact that

$$\mathbf{C}_V = \sum_{j=1}^{j=n} \mathbf{j}_V(y_j) \otimes x_j \implies (F_r \otimes V[1]) \circ \mathbf{C}_V = \sum_{j=1}^{j=n} y_j \otimes x_j$$

for any bases $\{x_1, \ldots, x_n\}$ of V and $\{y_1, \ldots, y_n\}$ of V^* dual to each other. $\square$

The proof of Proposition 30 also helps us understand the map Φ_L better: let

$$\mathrm{id} : \mathcal{O}_X \to T[-1] \otimes \Omega[1]$$

denote the dual of the evaluation map from $\Omega[1] \otimes T[-1]$ to $\mathcal{O}_X$.

Proposition 31. *As an element of* $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, D_1 \otimes \Omega[1])$, Φ_L *is given by the composite*

$$\mathcal{O}_X \xrightarrow{\mathrm{id}} T[-1] \otimes \Omega[1] \xrightarrow{T[-1] \otimes \frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}} T[-1] \otimes \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1].$$

Proof. By the proof of Proposition 30, $\overline{\mathbf{C}} : \mathbf{S}^{\bullet}(\Omega[1]) \to \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1]$ is given by the composite

$$\begin{tikzcd}
\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathcal{O}_X \arrow[d, "\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathrm{id}"'] & \\
\mathbf{S}^{\bullet}(\Omega[1]) \otimes T[-1] \otimes \Omega[1] \arrow[r, "(\,\text{-} \bullet \text{-}\,) \otimes \Omega[1]"] & \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1].
\end{tikzcd}$$

Note that by definition, $\Phi_L = \frac{\overline{\omega}}{\mathrm{e}^{\overline{\omega}}-1} \circ \overline{\mathbf{C}}$. Also, Equation (35) in the proof of Proposition 30 says that $\overline{\omega}^i$ is given by the composite

$$\begin{tikzcd}
\mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \arrow[d, "\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathrm{At}_T^i"'] & \\
\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \arrow[r, "(\,\text{-} \wedge \text{-}\,) \otimes \Omega[1]"] & \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1].
\end{tikzcd}$$

Therefore $\frac{\overline{\omega}}{\mathrm{e}^{\overline{\omega}}-1}$ is given by the composite

$$\begin{tikzcd}
\mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \arrow[d, "\mathbf{S}^{\bullet}(\Omega[1]) \otimes \frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}"'] & \\
\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \arrow[r, "(\,\text{-} \wedge \text{-}\,) \otimes \Omega[1]"] & \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1].
\end{tikzcd}$$

It follows that as a morphism in $\mathrm{D}^b(X)$ from $\mathbf{S}^{\bullet}(\Omega[1])$ to $\mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1]$, Φ_L is given by the composite

$$\begin{tikzcd}
\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathcal{O}_X \arrow[d, "\mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathrm{id}"] & \\
\mathbf{S}^{\bullet}(\Omega[1]) \otimes T[-1] \otimes \Omega[1] \arrow[d, "\mathbf{S}^{\bullet}(\Omega[1]) \otimes T[-1] \otimes \frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}"] & \\
\mathbf{S}^{\bullet}(\Omega[1]) \otimes T[-1] \otimes \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \arrow[r, "((\,\text{-} \bullet \text{-}\,) \wedge \text{-}\,) \otimes \Omega[1]"] & \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1].
\end{tikzcd}$$

This proves the desired proposition. $\square$

By Proposition 30, Φ_L can be thought of as an element of

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, D_1 \otimes \Omega[1]) \simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, T[-1] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]).$$

Let Φ_L^n denote the composite

$$\begin{array}{c} \mathcal{O}_X \\ \Big\downarrow{\scriptstyle \Phi_L^{\otimes n}} \\ (T[-1] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1])^{\otimes n} \\ \Big\downarrow{\scriptstyle \tau} \\ T^{\otimes n}[-n] \otimes \mathbf{S}^\bullet(\Omega[1])^{\otimes n} \otimes \Omega^{\otimes n}[n] \xrightarrow{p' \otimes \mathrm{m} \otimes p} \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n]. \end{array}$$

The map τ in the above diagram is a rearrangement of factors. The map $\mathrm{m} : \mathbf{S}^\bullet(\Omega[1])^{\otimes n} \to \mathbf{S}^\bullet(\Omega[1])$ is n-fold multiplication. p' is the standard projection from $T^{\otimes n}[-n]$ to $\wedge^n T[-n]$ and p is the projection from $\Omega^{\otimes n}[n]$ to $\Omega^n[n]$. Note that Φ_L^n is an element of $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, D_n \otimes \Omega^n[n])$. Let $\mathbf{1}^n : \mathbf{S}^\bullet(\Omega[1]) \to D_n \otimes \Omega^n[n]$ be as in Section 3.2. Then:

Lemma 4. *The following diagram commutes in* $\mathrm{D}^b(X)$.

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\mathbf{1}} & \mathcal{O}_X \\ \Big\downarrow{\scriptstyle \mathbf{f}} & & \Big\downarrow{\scriptstyle \Phi_L^n} \\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\mathbf{1}^n} & D_n \otimes \Omega^n[n]. \end{array}$$

Proof. *Step* 1: *The inverse of* $\mathbf{1}^n$*.* Let $\mathbf{e} : \mathcal{O}_X \to \Omega^n[n] \otimes \wedge^n T[-n]$ denote the natural isomorphism dual to the evaluation map from $\wedge^n T[-n] \otimes \Omega^n[n]$ to $\mathcal{O}_X$. Denote the evaluation map from $\Omega^n[n] \otimes \wedge^n T[-n]$ to $\mathcal{O}_X$ by $\mathbf{b}$. We claim that the inverse to $\mathbf{1}^n$ is given by the following composite.

$$\begin{array}{c} \mathcal{O}_X \otimes D_n \otimes \Omega^n[n] \\ \Big\downarrow{\scriptstyle \mathbf{e} \otimes D_n \otimes \Omega^n[n]} \\ \Omega^n[n] \otimes \wedge^n T[-n] \otimes D_n \otimes \Omega^n[n] \\ \Big\downarrow{\scriptstyle \tau} \\ \Omega^n[n] \otimes D_n \otimes \Omega^n[n] \otimes \wedge^n T[-n] \\ \Big\downarrow{\scriptstyle \simeq} \\ \Omega^n[n] \otimes \wedge^n T[-n] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n] \otimes \wedge^n T[-n] \\ \Big\downarrow{\scriptstyle \mathbf{b} \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{b}} \\ \mathbf{S}^\bullet(\Omega[1]). \end{array} \tag{38}$$

The map τ in (38) swaps $\wedge^n T[-n]$ and $D_n \otimes \Omega^n[n]$.

To see this, note that if U is an open subscheme of X such that

$$\Omega|_U \simeq V \otimes_{\mathbb{K}} \mathcal{O}_U$$

for some n-dimensional $\mathbb{K}$-vector space V, the maps involved in (38) can be described explicitly. Choose a basis $\{x_1, \dots, x_n\}$ of V and let $\{y_1, \dots, y_n\}$ be the dual basis of V^*. Let

$$\begin{aligned} \mathbf{e}_V : \mathbb{K} &\to \wedge^n V[n] \otimes \wedge^n V^*[-n] \\ 1 &\mapsto x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1. \end{aligned}$$

Let

$$\begin{aligned} \mathbf{b}_V : \wedge^n V[n] \otimes \wedge^n V^*[-n] &\to \mathbb{K} \\ x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1 &\mapsto 1. \end{aligned}$$

Let $\mathbf{1}^n_V$ be as in Section 3.1. The map denoted by $\mathbf{1}^n_V$ was denoted by $\mathbf{1}^m$ in Section 3.1. Then, if $H \in \mathbf{S}^\bullet(V[1])$ is a homogenous element,

$$\mathbf{1}^n(H) = (-1)^{n|H|} y_n \wedge \cdots \wedge y_1 \otimes H \otimes x_1 \wedge \cdots \wedge x_n.$$

$$\begin{aligned} &\big(\mathbf{e}_V \otimes \wedge^n V^*[-n] \otimes \mathbf{S}^\bullet(V[1]) \otimes \wedge^n V[n]\big) \circ \mathbf{1}^n(H) \\ &= (-1)^{n|H|} x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1 \otimes y_n \wedge \cdots \wedge y_1 \otimes H \otimes x_1 \wedge \cdots \wedge x_n. \end{aligned}$$

$$\begin{aligned} &\tau\big((-1)^{n|H|} x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1 \otimes y_n \wedge \cdots \wedge y_1 \otimes H \otimes x_1 \wedge \cdots \wedge x_n\big) \\ &= (-1)^{n|H|}(-1)^{n|H|} x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1 \\ &\qquad \otimes H \otimes x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1. \end{aligned}$$

$$\begin{aligned} (\mathbf{b}_V \otimes \mathbf{S}^\bullet(V[1]) \otimes \mathbf{b}_V)\Big((-1)^{n|H|}(-1)^{n|H|} x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1 \\ \otimes H \otimes x_1 \wedge \cdots \wedge x_n \otimes y_n \wedge \cdots \wedge y_1\Big) = H. \end{aligned}$$

The fact that the composite given in (38) is the inverse to $\mathbf{1}^n$ follows from the facts that $\mathbf{b}|_U = \mathbf{b}_V \otimes_{\mathbb{K}} \mathcal{O}_U$, $\mathbf{e}|_U = \mathbf{e}_V \otimes_{\mathbb{K}} \mathcal{O}_U$ and $\mathbf{1}^n|_U = \mathbf{1}^n_V \otimes_{\mathbb{K}} \mathcal{O}_U$.

Step 2. Let us look at the composition of the composite map in (38) with Φ^n_L. Φ^n_L can also be identified with the composite

$$(\mathbf{b} \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n]) \circ (\Omega^n[n] \otimes \Phi^n_L) : \Omega^n[n] \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n].$$

Thinking of Φ^n_L as the above composite, it follows from (38) that the inverse of $\mathbf{1}^n$ composed with $\Phi^n_L : \mathcal{O}_X \to D_n \otimes \Omega^n[n]$ is given by the composite

$$
\begin{array}{ll}
(39) & \begin{array}{c}
\mathcal{O}_X \\
\Big\downarrow{\scriptstyle \mathbf{e}} \\
\Omega^n[n] \otimes \wedge^n T[-n] \\
\Big\downarrow{\scriptstyle \Phi_L^n \otimes \wedge^n T[-n]} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n] \otimes \wedge^n T[-n] \\
\Big\downarrow{\scriptstyle \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{b}} \\
\mathbf{S}^\bullet(\Omega[1]).
\end{array}
\end{array}
$$

Let $\mathbf{b}_1 : \Omega[1] \otimes T[-1] \to \mathcal{O}_X$ denote the evaluation map. Let

$$
\mathrm{m} : \mathbf{S}^\bullet(\Omega[1])^{\otimes n} \to \mathbf{S}^\bullet(\Omega[1])
$$

denote the n-fold product. Let $p : \Omega[1]^{\otimes n} \to \Omega^n[n]$ denote the natural projection. Note that $\Phi_L^n : \Omega^n[n] \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n]$ is also given by the following composite.

$$
\begin{array}{ll}
(40) & \begin{array}{c}
\Omega^n[n] \\
\Big\downarrow \\
(\Omega[1] \otimes \mathcal{O}_X)^{\otimes n} \\
\Big\downarrow{\scriptstyle (\Omega[1] \otimes \Phi_L)^{\otimes n}} \\
(\Omega[1] \otimes T[-1] \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1])^{\otimes n} \\
\Big\downarrow{\scriptstyle (\mathbf{b}_1 \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1])^{\otimes n}} \\
(\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1])^{\otimes n} \\
\Big\downarrow{\scriptstyle \tau} \\
\mathbf{S}^\bullet(\Omega[1])^{\otimes n} \otimes \Omega[1]^{\otimes n} \\
\Big\downarrow{\scriptstyle \mathrm{m} \otimes p} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n].
\end{array}
\end{array}
$$

The topmost vertical arrow in the above diagram is induced by the natural inclusion from Ω^n to $\Omega^{\otimes n}$. The map τ in the above diagram is a rearrangement of factors.

It follows from Proposition 31 and that the composite

$$
(\mathbf{b}_1 \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]) \circ (\Omega[1] \otimes \Phi_L) : \Omega[1] \to \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]
$$

is precisely $\frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}$.

Therefore the composite in (40) is equal to the composite

$$\begin{array}{l}\Omega^n[n] \\ \downarrow \\ \Omega[1]^{\otimes n} \\ \Big\downarrow \left(\frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}\right)^{\otimes n} \\ (\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1])^{\otimes n} \xrightarrow{\tau} \mathbf{S}^\bullet(\Omega[1])^{\otimes n} \otimes \Omega[1]^{\otimes n} \xrightarrow{\mathrm{m}\otimes p} \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n].\end{array}$$

It follows from Equation (39) that the inverse of $\mathbf{1}^n$ composed with Φ_L^n is given by the composite

$$\begin{array}{c}\mathcal{O}_X \\ \Big\downarrow \mathbf{e} \\ \Omega^n[n] \otimes \wedge^n T[-n] \\ \Big\downarrow \\ \Omega[1]^{\otimes n} \otimes \wedge^n T[-n] \\ \Big\downarrow \left(\frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}\right)^{\otimes n} \otimes \wedge^n T[-n] \\ (\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1])^{\otimes n} \otimes \wedge^n T[-n] \\ \Big\downarrow \tau \\ \mathbf{S}^\bullet(\Omega[1])^{\otimes n} \otimes \Omega[1]^{\otimes n} \otimes \wedge^n T[-n] \\ \Big\downarrow \mathrm{m}\otimes p \otimes \wedge^n T[-n] \\ \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^n[n] \otimes \wedge^n T[-n] \xrightarrow{\mathbf{S}^\bullet(\Omega[1])\otimes \mathbf{b}} \mathbf{S}^\bullet(\Omega[1]).\end{array}$$

The above composite is $\det\left(\frac{\mathrm{At}_T}{\exp(\mathrm{At}_T)-1}\right)$ by the definition of det. $\square$

4.3. Another long remark — Lemma 4 and our dictionary. Lemma 4 is the root reason for the Todd genus, an expression having a form similar to the Jacobian of the differential $d(\exp^{-1})$, showing up in the Riemann–Roch theorem. Markarian [6] remarks that a lemma in [6] similar to Lemma 3 in this paper is like pulling back the canonical volume form on a Lie group via the exponential map. He makes a remark in [5] that a formula analogous to that describing the pullback of the canonical (left-invariant) volume form on a Lie group via the exponential map is responsible for the Todd genus showing up in the Riemann–Roch theorem. Lemma 4 is precisely where something like this happens. We will attempt to make the parallel between Lemma 4 and "pulling back the canonical left-invariant volume form by the map $\overline{\exp}$" more transparent. In Lemma 4 of this paper, it is $\overline{\exp}$ rather than the exponential map itself that is involved.

We warn the reader that ad and $\overline{\mathrm{ad}}$ have the same meaning in this subsection as in Section 2.5. Their meaning in this section is therefore, different from their meaning in Section 3, the rest of Section 4 and Section 5.

4.3.1. The classical situation. Keep the dictionary developed up to Section 2.5 in mind. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Choose a basis $\{X_i\}$ of $\mathfrak{g}$ and a basis $\{Y_i\}$ of $\mathfrak{g}^*$ dual to $\{X_i\}$. Let n be the dimension of $\mathfrak{g}$. Let $\mathbf{1}_{\mathfrak{g}}$ denote the element $\sum_{i=1}^n X_i \otimes Y_i$ of $\mathfrak{g} \otimes \mathfrak{g}^*$. Let $C(G)$ and $C(\mathfrak{g})$ be as in Section 2.5. Letting an element of $\mathfrak{g}$ act as a differential operator on $C(G)$ (as in Section 2.5) yields a connection on $C(G)$ for each element of $\mathfrak{g} \otimes \mathfrak{g}^*$. $\mathbf{1}_{\mathfrak{g}}$ yields the canonical connection d_G.

The element $\mathbf{1}^n := Y_n \wedge \cdots \wedge Y_1 \otimes X_1 \wedge \cdots \wedge X_n$ of $\wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^n$ yields a section of the trivial line bundle $\mathcal{V} \times \wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^n$ over the neighborhood $\mathcal{V}$ of 0 in $\mathfrak{g}$. This section yields a map

$$\mathbf{1}^n_{\mathfrak{g}} : C(\mathfrak{g}) \to C(\mathfrak{g}) \otimes \wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^n$$
$$f \mapsto f \otimes Y_n \wedge \cdots \wedge Y_1 \otimes X_1 \wedge \cdots \wedge X_n.$$

The map $\mathbf{1}^n$ in Lemma 4 is analogous to $\mathbf{1}^n_{\mathfrak{g}}$.

On the other hand, if we think of an element of $\mathfrak{g}^*$ as a function on $\mathcal{V}$, we have an $(i+1)$-fold multiplication $\mu_i : C(\mathfrak{g}) \otimes \mathfrak{g}^{*\otimes i} \to C(\mathfrak{g})$. Let $\mathrm{ad}^{\circ i}$ denote the composite

$$(\mathfrak{g}^{*\otimes i-1} \otimes \mathrm{ad}) \circ \cdots \circ \mathrm{ad} : \mathfrak{g}^* \to \mathfrak{g}^{*\otimes i} \otimes \mathfrak{g}^*.$$

Note that $(\mu_i \otimes \mathfrak{g}^*) \circ (C(\mathfrak{g}) \otimes \mathrm{ad}^{\circ i}) = \overline{\mathrm{ad}}^i$ where $\overline{\mathrm{ad}}$ is as in Section 2.5. Given any (convergent) power series $f(z) = \sum_i c_i z^i$ let $f(\overline{\mathrm{ad}})$ denote the map

$$\sum_i c_i \overline{\mathrm{ad}}^i : C(\mathfrak{g}) \otimes \mathfrak{g}^* \to C(\mathfrak{g}) \otimes \mathfrak{g}^*.$$

Let Ψ be as in Section 2.5. Then

$$\Psi = \frac{-\overline{\mathrm{ad}}}{\mathrm{e}^{\overline{\mathrm{ad}}} - 1} \circ d_{\mathfrak{g}}$$

as a map from $C(\mathfrak{g})$ to $C(\mathfrak{g}) \otimes \mathfrak{g}^*$. Therefore, as an element of $C(\mathfrak{g}) \otimes \mathfrak{g} \otimes \mathfrak{g}^*$,

$$\Psi = \left(\mathfrak{g} \otimes \frac{-\overline{\mathrm{ad}}}{\mathrm{e}^{\overline{\mathrm{ad}}} - 1} \right) \circ \mathbf{1}_{\mathfrak{g}}.$$

We can therefore think of the element $\bigwedge^n_{C(\mathfrak{g})}(-\Psi)$ of $C(\mathfrak{g}) \otimes \wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^n$. Denote this by $(-\Psi)^n$. Note that $(-\Psi)^n$ is a section of the trivial line bundle $\mathcal{V} \times \wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^n$ over $\mathcal{V}$. Moreover $\mathbf{1}^n_{\mathfrak{g}}$ is an isomorphism of $\mathbb{R}$-vector spaces. We can therefore ask for the function

$$f_{\mathfrak{g}} := (\mathbf{1}^n_{\mathfrak{g}})^{-1}((-\Psi)^n) \in C(\mathfrak{g}).$$

One can check from the formula for $-\Psi$ that

$$f_{\mathfrak{g}} = \det \left(\frac{\overline{\mathrm{ad}}}{\mathrm{e}^{\overline{\mathrm{ad}}} - 1} \right). \tag{41}$$

At this stage, we remark that the map Φ_L^n in Lemma 4 is analogous to $(-\Psi)^n$. $\det\left(\frac{\overline{\mathrm{ad}}}{\mathrm{e}^{\overline{\mathrm{ad}}}-1}\right)$ is analogous to $\mathbf{f}$ in Lemma 4. Lemma 4 itself is analogous to Equation (41).

4.3.2. Pulling back the canonical left invariant volume form on G via $\overline{\exp}$. Finally, we observe that an element of $C(\mathfrak{g}) \otimes \wedge^n \mathfrak{g}^* \otimes \mathfrak{g}^n$ and a volume form on $\mathcal{V}$ together yield another volume form on $\mathcal{V}$ by letting $\wedge^n \mathfrak{g}^*$ contract with $\mathfrak{g}^n$. In this manner, the canonical volume form $Y_n \wedge \cdots \wedge Y_1$ on $\mathcal{V}$ and $\mathbf{1}_{\mathfrak{g}}^n$ yield the canonical volume form $Y_n \wedge \cdots \wedge Y_1$ on $\mathcal{V}$.

Consider the left invariant volume form ω_G on U_G arising out of the element $Y_n \wedge \cdots \wedge Y_1$ of $\wedge^n \mathfrak{g}^*$. In the same manner the volume form ω yielded by $\overline{\exp}^*(\omega_G)$ and Ψ^n equals $Y_n \wedge \cdots \wedge Y_1$. But on the other hand, ω is also equal to $\det(d(\overline{\exp}^*))(-1)^n f_{\mathfrak{g}}$. It follows that (41) is equivalent to the formula

$$\det(d(\overline{\exp}^*)) = \det\left(\frac{-\overline{\mathrm{ad}}}{\mathrm{e}^{\overline{\mathrm{ad}}} - 1}\right).$$

This in turn is equivalent to the formula for the pullback of a left invariant volume form on G via $\overline{\exp}$.

4.4. Proof of Lemma 3. We are now equipped to prove Lemma 3. Let

$$\mathrm{ad}(\Phi_R) : D_n \to D_n \otimes \Omega[1]$$

denote the composite

$$D_n \otimes \mathcal{O}_X \xrightarrow{D_n \otimes \Phi_R} D_n \otimes D_1 \otimes \Omega[1] \xrightarrow{\mathrm{ad} \otimes \Omega[1]} D_n \otimes \Omega[1]$$

where $\mathrm{ad} : D_n \otimes D_1 \to D_n$ is as in Section 3.2. We begin with the following proposition.

Proposition 32.

$$\Phi_R^+(\mathbf{f}) = 0.$$

Proof. The upper square in the commutative diagram below commutes by Lemma 4. The lower square in the diagram below commutes by Lemma 2.

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\mathbf{1}"] \arrow[d, "\mathbf{f}"] & \mathcal{O}_X \arrow[d, "\Phi_L^n"'] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\mathbf{1}^n"] \arrow[d, "-\Phi_R^+"] & D_n \otimes \Omega^n[n] \arrow[d, "\mathrm{ad}(\Phi_R) \otimes \Omega^n[n]"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \arrow[r, "\mathbf{1}^n \otimes \Omega[1]"] & D_n \otimes \Omega[1] \otimes \Omega^n[n].
\end{tikzcd} \tag{42}$$

The proof of Proposition 30 with Φ_R instead of Φ_L would show that

$$\Phi_R \in \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, D_1 \otimes \Omega[1]).$$

Let $\overline{\mathrm{ad}}$ be as in Section 3.2. Let $\overline{\mathrm{ad}}(\Phi_R)$ denote the composite

$$
\begin{array}{c}
(D_1 \otimes \Omega[1])^{\otimes n} \otimes \mathcal{O}_X \\
\Big\downarrow {\scriptstyle (D_1\otimes\Omega[1])^{\otimes n}\otimes\Phi_R} \\
(D_1 \otimes \Omega[1])^{\otimes n} \otimes D_1 \otimes \Omega[1] \\
\Big\downarrow \\
D_1^{\otimes n} \otimes D_1 \otimes \Omega[1]^{\otimes n} \otimes \Omega[1] \\
\Big\downarrow {\scriptstyle \overline{\mathrm{ad}}\otimes\Omega[1]^{\otimes n}\otimes\Omega[1]} \\
D_1^{\otimes n} \otimes \Omega[1]^{\otimes n} \otimes \Omega[1] \longrightarrow (D_1 \otimes \Omega[1])^{\otimes n} \otimes \Omega[1].
\end{array}
$$

The unlabeled arrows in the above diagram are rearrangements of factors.

Note that by the Proposition 25, the following diagram commutes.

$$
\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\mathbf{1}"] \arrow[d, "\Phi_L^{\otimes n}"] & \mathcal{O}_X \arrow[d, "\Phi_L^n"'] \\
(D_1 \otimes \Omega[1])^{\otimes n} \arrow[r] \arrow[d, "\overline{\mathrm{ad}}(\Phi_R)"] & D_n \otimes \Omega^n[n] \arrow[d, "\mathrm{ad}(\Phi_R)\otimes\Omega^n[n]"'] \\
(D_1 \otimes \Omega[1])^{\otimes n} \otimes \Omega[1] \arrow[r] & D_n \otimes \Omega[1] \otimes \Omega^n[n].
\end{tikzcd}
\tag{43}
$$

By Theorem $2'$ and by Proposition 9, Φ_R and Φ_L are commuting operators on $\mathbf{S}^\bullet(\Omega[1])$. It follows that $\overline{\mathrm{ad}}(\Phi_R)(\Phi_L^{\otimes n})$ is 0. Thus, $\mathrm{ad}(\Phi_R)\otimes\Omega^n[n](\Phi_L^n)$ is 0. The desired proposition now follows from (42) and the fact that $\mathbf{1}^n\otimes\Omega[1]$ is invertible in $\mathrm{D}^b(X)$. $\square$

Proof of Lemma 3. Note that the pairing $\langle , \rangle : \mathbf{S}^\bullet(\Omega[1])^{\otimes 2} \to \Omega^n[n]$ induces a nondegenerate pairing

$$\langle , \rangle : \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^\bullet(\Omega[1]))^{\otimes 2} \to \mathrm{RHom}_X(\mathcal{O}_X, \Omega^n[n]) \simeq \mathbb{K}.$$

$\langle \Phi_R(\text{ - }), \text{ - } \rangle : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \Omega^n[n] \otimes \Omega[1]$ denotes the composite

$$
\begin{array}{c}
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \\
{\scriptstyle \Phi_R\otimes\mathbf{S}^\bullet(\Omega[1])} \Big\downarrow \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \otimes \mathbf{S}^\bullet(\Omega[1]) \\
{\scriptstyle \mathbf{S}^\bullet(\Omega[1])\otimes\tau} \Big\downarrow \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \\
{\scriptstyle \langle,\rangle\otimes\Omega[1]} \Big\downarrow \\
\Omega^n[n] \otimes \Omega[1]
\end{array}
$$

map in $\mathrm{D}^b(X)$. The map τ in the above composition of maps in $\mathrm{D}^b(X)$ swaps $\Omega[1]$ and $\mathbf{S}^{\bullet}(\Omega[1])$.

Similarly, $\langle\ \text{-}\ ,\Phi_R^{+}(\ \text{-}\ )\rangle : \mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathbf{S}^{\bullet}(\Omega[1]) \to \Omega^n[n] \otimes \Omega[1]$ denotes the composite

$$\begin{array}{l} \mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathbf{S}^{\bullet}(\Omega[1]) \\ \quad \Big\downarrow {\scriptstyle \mathbf{S}^{\bullet}(\Omega[1])\otimes\Phi_R^{+}} \\ \mathbf{S}^{\bullet}(\Omega[1]) \otimes \mathbf{S}^{\bullet}(\Omega[1]) \otimes \Omega[1] \xrightarrow{\langle,\rangle\otimes\Omega[1]} \Omega^n[n] \otimes \Omega[1]. \end{array}$$

Let $a, b \in \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^{\bullet}(\Omega[1]))$. Then,

$$\langle ab.\mathbf{f}^{-1}, \Phi_R^{+}(\mathbf{f})\rangle = 0 \tag{44}$$

by Proposition 32. But,

$$\langle ab.\mathbf{f}^{-1}, \Phi_R^{+}(\mathbf{f})\rangle = \langle \Phi_R(ab.\mathbf{f}^{-1}), \mathbf{f}\rangle \tag{45}$$

by the commutative diagram (34).

By Proposition 7, Theorem $2'$ and by the fact that

$$\mathrm{I}_{\mathrm{HKR}} : \widehat{C}^{\bullet}(X) \to \mathbf{S}^{\bullet}(\Omega[1])$$

is a homomorphism of algebra objects in $\mathrm{D}^b(X)$,

$$\Phi_R(ab.\mathbf{f}^{-1}) = \Phi_R(a)b\mathbf{f}^{-1} + a\Phi_R(b\mathbf{f}^{-1}). \tag{46}$$

By (44), (45) and (46)

$$0 = \langle \Phi_R(ab.\mathbf{f}^{-1}), \mathbf{f}\rangle = \langle \Phi_R(a)b\mathbf{f}^{-1}, \mathbf{f}\rangle + \langle a\Phi_R(b\mathbf{f}^{-1}), \mathbf{f}\rangle. \tag{47}$$

But for any elements $u, v, w \in \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^{\bullet}(\Omega[1]))$,

$$\langle uv, w\rangle = \langle u, vw\rangle$$

by the definition of $\langle,\rangle$. It then follows from (47) that

$$0 = \langle \Phi_R(a), b.\mathbf{f}^{-1}\mathbf{f}\rangle + \langle a, \Phi_R(b.\mathbf{f}^{-1})\mathbf{f}\rangle = \langle \Phi_R(a), b\rangle + \langle a, \Phi_R(b.\mathbf{f}^{-1})\mathbf{f}\rangle. \tag{48}$$

It follows from the commutative diagram (34) that

$$\Phi_R^{+}(b) = -\Phi_R(b\mathbf{f}^{-1})\mathbf{f}$$

for any $b \in \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^{\bullet}(\Omega[1]))$. This proves Lemma 3. □

Remark 11. The computation proving Lemma 3 that has been written here can very easily be rewritten in a "canonical" manner without choosing elements of $\mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^{\bullet}(\Omega[1]))$. Though that would be the ideal thing to do from the point of view of rigor, we feel that the computation we have depicted conveys the key idea behind the computation more concretely. Computations of a similar nature that show up in Section 5, have however, been written down in a "canonical" manner.

5. Proof of Theorem 1

5.1. Unwinding some definitions. In this subsection, we shall confine ourselves to unwinding the definition of the duality map D_Δ. This will help us focus more on what exactly we need to compute to prove Theorem 1. Let

$$\kappa : \mathcal{O}_\Delta \to \Delta_* \Delta^* \mathcal{O}_\Delta$$

denote the unit of the adjunction $\Delta^* \dashv \Delta_*$ applied to $\mathcal{O}_\Delta \in \mathrm{D}^b(X \times X)$. Also let

$$\beta : \mathcal{O}_X \to \Delta^* \Delta_! \mathcal{O}_X$$

denote the unit of the adjunction $\Delta_! \dashv \Delta^*$ applied to $\mathcal{O}_X \in \mathrm{D}^b(X)$.

Let $\widetilde{\mathrm{D}^b(X)}$ denote the category whose objects are those of $\mathrm{D}^b(X)$ such that

$$\mathrm{Hom}_{\widetilde{\mathrm{D}^b(X)}}(\mathcal{F}, \mathcal{G}) = \mathrm{RHom}_{\mathrm{D}^b(X)}(\mathcal{F}, \mathcal{G})$$

for any pair of objects $\mathcal{F}$ and $\mathcal{G}$ of $\widetilde{\mathrm{D}^b(X)}$. Note that any diagram that commutes in $\mathrm{D}^b(X)$ also does so in $\widetilde{\mathrm{D}^b(X)}$. We perform a particular calculation in $\widetilde{\mathrm{D}^b(X)}$ instead of $\mathrm{D}^b(X)$ only when absolutely necessary. This enables us to take care of the shifts in grading that occur when an element of $\mathrm{RHom}_X(\mathcal{F}, \mathcal{G})$ shows up instead of an element of $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}, \mathcal{G})$. In [8], $\widetilde{\mathrm{D}^b(X)}$ is called the "extended derived category" of X.

Note that $\Delta_! \mathcal{O}_X \simeq \Delta_* S_X^{-1}$. It follows that $\Delta^* \Delta_! \mathcal{O}_X \simeq \Delta^* \mathcal{O}_\Delta \otimes S_X^{-1}$. Let $\simeq : S_X \otimes S_X^{-1} \to \mathcal{O}_X$ be as in Section 3.2. We now state the following proposition.

Proposition 33. *Let $\phi \in \mathrm{RHom}_X(\Delta^* \mathcal{O}_\Delta, S_X)$. Then, as a morphism in $\widetilde{\mathrm{D}^b(X)}$, $D_\Delta^{-1}(\phi)$ is given by the composite*

$$\begin{array}{c} \mathcal{O}_X \\ \Big\downarrow {\scriptstyle \beta} \\ \Delta^* \mathcal{O}_\Delta \otimes S_X^{-1} \\ \Big\downarrow {\scriptstyle \Delta^*(\kappa) \otimes S_X^{-1}} \\ \Delta^* \mathcal{O}_\Delta \otimes \Delta^* \mathcal{O}_\Delta \otimes S_X^{-1} \\ \Big\downarrow {\scriptstyle \Delta^* \mathcal{O}_\Delta \otimes \phi \otimes S_X^{-1}} \\ \Delta^* \mathcal{O}_\Delta \otimes S_X \otimes S_X^{-1} \\ \Big\downarrow {\scriptstyle \Delta^* \mathcal{O}_\Delta \otimes \simeq} \\ \Delta^* \mathcal{O}_\Delta . \end{array}$$

Proof. By the definition of D_Δ, $D_\Delta^{-1} = \mathcal{I}^{-1} \circ \mathcal{T}^{-1} \circ \mathcal{J}^{-1}$ where $\mathcal{I}, \mathcal{T}$ and $\mathcal{J}$ are as in (2), (3) and (4) respectively.

Now,

$$\mathcal{J}^{-1}(\phi) = \Delta_*\phi \circ \kappa. \tag{49}$$

Further,

$$\mathcal{T}^{-1}(\alpha) = \alpha \otimes p_2^* S_X^{-1} \tag{50}$$

and

$$\mathcal{I}^{-1}(\gamma) = \Delta^*\gamma \circ \beta. \tag{51}$$

Now, $\Delta^*(\alpha \otimes p_2^* S_X^{-1}) = \Delta^*\alpha \otimes S_X^{-1}$ since $p_2 \circ \Delta = \mathrm{id}$. The desired proposition now follows from (49), (50) and (51) and the fact that $\Delta^*\Delta_*\psi = \Delta^*\mathcal{O}_\Delta \otimes \psi$ for any morphism ψ in $\mathrm{D}^b(X)$. $\square$

The following propositions help us understand $\Delta^*(\kappa)$ and β explicitly.

Proposition 34. *The following diagram commutes in* $\mathrm{D}^b(X)$.

$$\begin{array}{ccc}
\Delta^*\mathcal{O}_\Delta & \xrightarrow{\Delta^*(\kappa)} & \Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta \\
{\scriptstyle \Delta^*\mathcal{O}_\Delta}\downarrow & & \downarrow{\scriptstyle \Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta} \\
\Delta^*\mathcal{O}_\Delta & \xrightarrow{\mathbf{C}} & \Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta.
\end{array}$$

Proof. *Step* 1. Note that if $\mathcal{F} \in \mathrm{D}^b(X \times X)$, then

$$\Delta_*\Delta^*\mathcal{F} \simeq \mathcal{O}_\Delta \otimes \mathcal{F}.$$

Denote the canonical quotient map $\mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$ by $\mathbf{h}$.

Also observe that if $\mathcal{G} \in \mathrm{D}^b(X)$, then

$$\Delta^*\Delta_*\mathcal{G} \simeq \Delta^*\mathcal{O}_\Delta \otimes \mathcal{G}.$$

Recall that $\Delta^*\mathcal{O}_\Delta$ is represented by the complex $\widehat{C}^\bullet(X)$. Note that the projection from the graded $\mathcal{O}_X$-module $\widehat{C}^\bullet(X)$ to $\widehat{C}^0(X) = \mathcal{O}_X$ is a map of complexes of $\mathcal{O}_X$-modules. This was denoted by η in Section 2. In this proof, we will denote this projection by $\mathbf{p}$.

We claim that tensoring with $\mathbf{h}$ constitutes the unit of the adjunction $\Delta^* \dashv \Delta_*$ and that tensoring with $\mathbf{p}$ constitutes the counit of the adjunction $\Delta^* \dashv \Delta_*$.

To see this, note that $\Delta^*(\mathbf{h})$ is just the map

$$\epsilon : \mathcal{O}_X \to \widehat{C}^\bullet(X)$$

defined in Section 2. This was the unit of the Hopf algebra object $\widehat{C}^\bullet(X)$ of $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$. It follows that $\mathbf{p} \circ \Delta^*(\mathbf{h}) = \mathcal{O}_X$. Also,

$$\Delta_*(\mathbf{p}) \circ (\mathbf{h} \otimes \mathcal{O}_\Delta) = \mathcal{O}_\Delta$$

since $\mathbf{h} \otimes \mathcal{O}_\Delta$ can be identified with the map $\Delta_*(\epsilon)$. It follows that

$$\kappa = \mathbf{h} \otimes \mathcal{O}_\Delta.$$

Step 2. We now show that $\Delta^*(\kappa)$ and $\mathbf{C}$ yield the same morphism in $\mathrm{D}^b(X)$ from $\Delta^*\mathcal{O}_\Delta$ to $\Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta$.

Note that $\mathcal{O}_\Delta$ is represented by the complex $\widehat{B}^\bullet(X)$ in $\mathrm{D}^b(X \times X)$. It follows that both $\widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta$ and $\mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X)$ represent the object $\mathcal{O}_\Delta \otimes \mathcal{O}_\Delta$ of $\mathrm{D}^b(X \times X)$. Let

$$\nu : \widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta \to \widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta$$

denote the map such that on an open subscheme $U = \operatorname{Spec} R \times \operatorname{Spec} R$ of $X \times X$ before completion,

$$\begin{aligned}&\nu(r_0 \otimes \cdots \otimes r_{k+1} \otimes_{R\otimes R} r')\\ &\quad = \sum_{p+q=k;\, p,q\geq 0} r_0 \otimes \cdots \otimes r_p \otimes 1 \otimes_{R\otimes R} 1 \otimes r_{p+1} \otimes \cdots \otimes r_{k+1} \otimes_{R\otimes R} r'.\end{aligned}$$

ν is easily seen to be a map of complexes of $\mathcal{O}_{X\times X}$-modules. Similarly, let

$$\overline{\nu} : \mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X) \to \mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X)$$

denote the map such that on an open subscheme $U = \operatorname{Spec} R \times \operatorname{Spec} R$ of $X \times X$ before completion,

$$\begin{aligned}&\overline{\nu}(r' \otimes_{R\otimes R} r_0 \otimes \cdots \otimes r_{k+1})\\ &\quad = r' \otimes_{R\otimes R} \sum_{p+q=k;\, p,q\geq 0} r_0 \otimes \cdots \otimes r_p \otimes 1 \otimes_{R\otimes R} 1 \otimes r_{p+1} \otimes \cdots \otimes r_{k+1}.\end{aligned}$$

$\overline{\nu}$ is easily seen to be a map of complexes of $\mathcal{O}_{X\times X}$-modules.

Let $\tau : \mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X) \to \widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta$ denote the map swapping factors. Let $\tau' : \mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X) \to \widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta$ denote the map swapping $\mathcal{O}_\Delta$ and $\widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X)$. The following diagram then commutes.

$$\begin{array}{ccc}\mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X) & \xrightarrow{\overline{\nu}} & \mathcal{O}_\Delta \otimes \widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X)\\ \Big\downarrow{\scriptstyle\tau} & & {\scriptstyle\tau'}\Big\downarrow\\ \widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta & \xrightarrow{\nu} & \widehat{B}^\bullet(X) \otimes \widehat{B}^\bullet(X) \otimes \mathcal{O}_\Delta.\end{array}$$

Note that $\Delta^*(\overline{\nu}) = \Delta^*\mathcal{O}_\Delta \otimes \mathbf{C}$. It follows from this and from the above commutative diagram that

$$\Delta^*(\nu) = \mathbf{C} \otimes \Delta^*\mathcal{O}_\Delta. \tag{52}$$

Step 3. We use (52) to compare the morphisms ν and $\kappa \otimes \mathcal{O}_\Delta$ in $\mathrm{D}^b(X \times X)$. Recall that

$$\begin{aligned}&\operatorname{Hom}_{\mathrm{D}^b(X\times X)}(\mathcal{O}_\Delta \otimes \mathcal{O}_\Delta, \mathcal{O}_\Delta \otimes \mathcal{O}_\Delta \otimes \mathcal{O}_\Delta)\\ &\qquad\qquad \cong \operatorname{Hom}_{\mathrm{D}^b(X)}(\Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta, \Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta)\end{aligned}$$

via

$$\alpha \mapsto (\mathbf{p} \otimes \Delta^*\mathcal{O}_\Delta \otimes \Delta^*\mathcal{O}_\Delta) \circ \Delta^*(\alpha).$$

Since $\mathbf{p}$ is induced by the counit of the Hopf algebra object $\widehat{C}^\bullet(X)$ of $\mathrm{Ch}^-(\mathcal{O}_X-\mathrm{mod})$, and $\mathbf{C}$ is induced by the comultiplication of $\widehat{C}^\bullet(X)$,

$$(\mathbf{p}\otimes\Delta^*\mathcal{O}_\Delta)\circ\mathbf{C}=\mathbf{1}_{\Delta^*\mathcal{O}_\Delta}.$$

Also, since $\Delta^*(\kappa)=\epsilon\otimes\Delta^*\mathcal{O}_\Delta$,

$$(\mathbf{p}\otimes\Delta^*\mathcal{O}_\Delta)\circ\Delta^*(\kappa)=\mathbf{1}_{\Delta^*\mathcal{O}_\Delta}.$$

It follows that

$$\begin{aligned}(\mathbf{p}\otimes\Delta^*\mathcal{O}_\Delta\otimes\Delta^*\mathcal{O}_\Delta)\circ\Delta^*(\nu)&=(\mathbf{p}\otimes\Delta^*\mathcal{O}_\Delta\otimes\Delta^*\mathcal{O}_\Delta)\circ\Delta^*(\kappa\otimes\mathcal{O}_\Delta)\\&=\mathbf{1}_{\Delta^*\mathcal{O}_\Delta\otimes\Delta^*\mathcal{O}_\Delta}.\end{aligned}$$

This proves that ν and $\kappa\otimes\mathcal{O}_\Delta$ represent the same morphism in $\mathrm{D}^b(X\times X)$. Therefore, $\mathbf{C}\otimes\Delta^*\mathcal{O}_\Delta$ and $\Delta^*(\kappa)\otimes\Delta^*\mathcal{O}_\Delta$ represent the same morphism in $\mathrm{D}^b(X)$. The desired proposition now follows from the fact that if $\lambda:\mathcal{G}\to\mathcal{H}$ is a morphism in $\mathrm{D}^b(X)$, λ is equal to the composite

$$\mathcal{G}\xrightarrow{\mathcal{G}\otimes\epsilon}\mathcal{G}\otimes\Delta^*\mathcal{O}_\Delta\xrightarrow{\lambda\otimes\Delta^*\mathcal{O}_\Delta}\mathcal{H}\otimes\Delta^*\mathcal{O}_\Delta\xrightarrow{\mathcal{H}\otimes\mathbf{p}}\mathcal{H}.\qquad\square$$

Recall that $p:\Omega[1]^{\otimes i}\to\Omega^i[i]$ denotes the standard projection. Let $\Phi_R^{\circ i}$ denote the composite

$$\Phi_R\otimes\Omega[1]^{\otimes i-1}\circ\cdots\circ\Phi_R:\mathbf{S}^\bullet(\Omega[1])\to\mathbf{S}^\bullet(\Omega[1])\otimes\Omega[1]^{\otimes i}.$$

Let Φ_R^i denote $(\mathbf{S}^\bullet(\Omega[1])\otimes p)\circ\Phi_R^{\circ i}$. Then $\exp(\Phi_R):=\sum_i\frac{1}{i!}\Phi_R^i$ is a morphism in $\mathrm{D}^b(X)$ from $\mathbf{S}^\bullet(\Omega[1])$ to $\mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{S}^\bullet(\Omega[1])$.

The following proposition follows immediately from Proposition 8 and Theorem 2′.

Proposition 35. *The following diagram commutes in* $\mathrm{D}^b(X)$.

$$\begin{array}{ccc}\Delta^*\mathcal{O}_\Delta & \xrightarrow{\mathbf{C}} & \Delta^*\mathcal{O}_\Delta\otimes\Delta^*\mathcal{O}_\Delta\\ {\scriptstyle\mathrm{I_{HKR}}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{I_{HKR}}\otimes\mathrm{I_{HKR}}}\\ \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\exp(\Phi_R)} & \mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{S}^\bullet(\Omega[1]).\end{array}$$

Let $\iota:S_X\to\mathbf{S}^\bullet(\Omega[1])$ denote the inclusion of $S_X\simeq\Omega^n[n]$ into $\mathbf{S}^\bullet(\Omega[1])$ as a direct summand. Let $\overline{\beta}$ denote the composite

$$\mathcal{O}_X\longrightarrow S_X\otimes S_X^{-1}\xrightarrow{\iota\otimes S_X^{-1}}\mathbf{S}^\bullet(\Omega[1])\otimes S_X^{-1}.$$

Proposition 36. *The following diagram commutes in* $\mathrm{D}^b(X)$.

$$\begin{array}{ccc}\mathcal{O}_X & \xrightarrow{\beta} & \Delta^*\mathcal{O}_\Delta\otimes S_X^{-1}\\ \big\downarrow{\scriptstyle\mathbf{1}} & & {\scriptstyle\mathrm{I_{HKR}}\otimes S_X^{-1}}\big\downarrow\\ \mathcal{O}_X & \xrightarrow{\overline{\beta}} & \mathbf{S}^\bullet(\Omega[1])\otimes S_X^{-1}.\end{array}$$

Proof. *Step* 1. Note that if $\mathcal{F} \in \mathrm{D}^b(X \times X)$, then

$$\Delta_! \Delta^* \mathcal{F} \simeq \Delta_* S_X^{-1} \otimes \mathcal{F}$$

Also, if $\mathcal{G} \in \mathrm{D}^b(X)$, then

$$\Delta^* \Delta_! \mathcal{G} \simeq \Delta^* \mathcal{O}_\Delta \otimes S_X^{-1} \otimes \mathcal{G}.$$

Now, $\Delta_* S_X^{-1}$ is isomorphic in $\mathrm{D}^b(X \times X)$ to $\mathcal{O}_\Delta \otimes p_2^* S_X^{-1}$. We now refer to the statement of the Serre duality theorem in Markarian [6]. By the Serre duality theorem there is a canonical map in $\mathrm{D}^b(X \times X)$ from $\mathcal{O}_\Delta$ to $p_2^* S_X$. We denote this map by $\mathbf{q}$. Tensoring $\mathbf{q}$ with $p_2^* S_X^{-1}$ on the right and making the obvious identifications gives us a morphism from $\mathcal{O}_\Delta \otimes p_2^* S_X^{-1}$ to $\mathcal{O}_{X \times X}$. We denote this morphism by $\mathbf{p}$ in this proof.

Let $\beta : \mathcal{O}_X \to \Delta^* \mathcal{O}_\Delta \otimes S_X^{-1}$ be the morphism in $\mathrm{D}^b(X)$ such that the diagram in this proposition commutes. β is well-defined in $\mathrm{D}^b(X)$ since $\mathrm{I}_{\mathrm{HKR}}$ is a quasi-isomorphism.

We claim that tensoring by β and tensoring by $\mathbf{p}$ constitute the unit and counit of the adjunction $\Delta_! \dashv \Delta^*$ respectively. In order to verify this claim, it suffices to verify that

$$\Delta^*(\mathbf{p}) \circ \beta = \mathcal{O}_X \tag{53}$$

as morphisms in $\mathrm{D}^b(X)$ and that

$$(\mathbf{p} \otimes \Delta_! \mathcal{O}_X) \circ \Delta_! \beta = \Delta_! \mathcal{O}_X \tag{54}$$

as morphisms in $\mathrm{D}^b(X \times X)$.

Step 2. We verify (53) on "good" open subschemes of X. We begin by verifying (53) on an open subscheme $U = \operatorname{Spec} R$ of X with local coordinates $y_1, \dots, y_n$. The elements $\{y_i \otimes 1 - 1 \otimes y_i \ i = 1, \dots, m\}$ form a regular sequence generating the ideal I of $R \otimes R$ defining the diagonal on an open affine neighborhood $V = \operatorname{Spec} S$ of the diagonal in $U \times U$. Let

$$z_i = y_i \otimes 1 - 1 \otimes y_i.$$

This regular sequence gives rise to a Koszul complex $\mathcal{K}^\bullet(z_1, \dots, z_n)$. This Koszul complex is a free S-module resolution of $\Delta_* R$.

The third part of the Serre duality theorem as stated in [6] says that the map $\mathbf{q}$ (restricted to V) is equal to the map of complexes

$$\begin{gathered} \mathcal{K}^\bullet(z_1, \dots, z_n) \to p_2^* \Omega^n_{R/\mathbb{K}}[n] \\ z_1 \wedge \cdots \wedge z_n \mapsto dy_1 \wedge \cdots \wedge dy_n \end{gathered}$$

as morphisms in $\mathrm{D}^b(V)$.

Further, let $[z_i]$ denote the class of z_i in $\mathrm{H}^*(R \otimes_S \mathcal{K}^\bullet(z_1, \dots, z_n))$. The R-linear map $dy_i \mapsto [z_i]$ induces an isomorphism of graded algebras between $\mathbf{S}^\bullet(\Omega_{R/\mathbb{K}}[1])$ and $\operatorname{Tor}_*^S(R, R)$ by Proposition 3.4.7 of Loday [4]. Moreover, by the proof of the Hochschild–Kostant–Rosenberg theorem in Section 3.4 of Loday [4], this isomorphism coincides with the isomorphism induced on cohomology by the anti-symmetrization map $\varphi : \mathbf{S}^\bullet(\Omega_{R/\mathbb{K}}[1]) \to \widehat{C}^\bullet(R)$. Before completion,

$$\varphi(r_0 dr_1 \wedge \cdots \wedge dr_k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) r_0 \otimes r_{\sigma(1)} \otimes \cdots \otimes r_{\sigma(k)}.$$

This is immediately seen to be a right inverse of $\mathrm{I}_{\mathrm{HKR}}$. It follows from the facts recalled in this paragraph and the description of $\mathbf{q}$ in the previous paragraph that $\Delta^*(\mathbf{q}) = \pi_n \circ \mathrm{I}_{\mathrm{HKR}}$ as morphisms in $\mathrm{D}^b(U)$.

Therefore, in $\mathrm{D}^b(U)$, $\Delta^*(\mathbf{p})$ is given by the composite

$$\Delta^*\mathcal{O}_\Delta \otimes S_X^{-1}|_U \xrightarrow{\pi_n \circ \mathrm{I}_{\mathrm{HKR}} \otimes S_X^{-1}|_U} S_X|_U \otimes S_X^{-1}|_U \longrightarrow \mathcal{O}_U = R.$$

The unlabeled arrow in the above composite is just the identification of $S_X|_U \otimes S_X^{-1}|_U$ with $\mathcal{O}_U = R$. It follows that $\Delta^*(\mathbf{p}) \circ \beta = \mathcal{O}_U$ as morphisms in $\mathrm{D}^b(U)$.

Step 3. We verify (54) on "good" open subschemes of $X \times X$. Let U and V be as in Step 2 above. Note that

$$\Delta_*\mathrm{I}_{\mathrm{HKR}} \circ (\Delta_!\beta \otimes S_{X\times X}) = \mathbf{1}_{\Delta_* S_X}.$$

Further,

$$\mathbf{p} \otimes \Delta_!\mathcal{O}_X \otimes S_{X\times X} = \mathbf{q} \otimes \mathcal{O}_\Delta.$$

By the discussion in Step 2, as morphisms in $\mathrm{D}^b(V)$,

$$\mathbf{q} \otimes \mathcal{O}_\Delta = \Delta_*\Delta^*\mathbf{q} = \Delta_*(\pi_n \circ \mathrm{I}_{\mathrm{HKR}}).$$

Therefore, as morphisms in $\mathrm{D}^b(V)$,

$$(\mathbf{q} \otimes \mathcal{O}_\Delta) \circ (\Delta_!\beta \otimes S_{X\times X}) = \Delta_*(\pi_n \circ \mathrm{I}_{\mathrm{HKR}}) \circ (\Delta_!\beta \otimes S_{X\times X}) = \mathbf{1}_{\Delta_* S_X}.$$

Tensoring the morphisms involved in the above equation with $S_{X\times X}^{-1}$, we see that

$$(\mathbf{p} \otimes \Delta_!\mathcal{O}_X) \circ \Delta_!\beta = \Delta_!\mathcal{O}_X$$

as morphisms in $\mathrm{D}^b(V)$. This is what we set out to verify.

Step 4. Now observe that

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta \otimes S_X^{-1}) \simeq \oplus_i \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Omega^i \otimes \wedge^n T[i-n]).$$

For $i < n$,

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Omega^i \otimes \wedge^n T[i-n]) = \mathrm{Ext}^{i-n}(\mathcal{O}_X, \Omega^i \otimes \wedge^n T[i-n]) = 0.$$

For $i = n$,

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Omega^i \otimes \wedge^n T[i-n]) &= \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Omega^n \otimes \wedge^n T)\\ &\simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathbb{K}.\end{aligned}$$

It follows that $\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \Delta^*\mathcal{O}_\Delta \otimes S_X^{-1})$ is a 1-dimensional $\mathbb{K}$-vector space. Tensoring with β and tensoring with $\mathbf{p}$ therefore, do indeed form a valid choice of unit and counit of the adjunction $\Delta_! \dashv \Delta^*$ upto some scalar factors. In other words, (53) and (54) are satisfied in $\mathrm{D}^b(X)$ and in $\mathrm{D}^b(X \times X)$ respectively upto some scalar factors. That the scalar factors are indeed 1 follows from the local verifications in Step 2 and Step 3 of this proof.

Step 5. Let $\mathcal{J}$ be as in (4) in Section 1. The last detail to be checked is that this particular choice of unit and counit for the adjunction $\Delta_! \dashv \Delta^*$ satisfies

$$\mathrm{tr}_X(\mathcal{J}(\phi) \circ \beta) = \mathrm{tr}_{X\times X}(\phi) \tag{55}$$

for any element ϕ of $\mathrm{Hom}_{\mathrm{D}^b(X\times X)}(\Delta_* S_X^{-1}, \Delta_* S_X)$. By the arguments in Step 4 of this proof, this "compatibility with traces" is satisfied upto a scalar factor independent of ϕ. We must verify that this scalar factor is 1.

Let $\mathbf{m} : \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$ denote the obvious morphism in this proof. Let $\mathcal{E}$ and $\mathcal{F}$ be objects in $\mathrm{D}^b(X)$. Let $\mathcal{E}^*$ denote the dual $\mathrm{RD}(\mathcal{E})$ of $\mathcal{E}$. We recall the second part of the Serre Duality theorem as stated in [6]. It states that if $f \in \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{E}, \mathcal{F})$, the composite

$$\mathcal{O}_{X\times X} \xrightarrow{(\Delta_* f)\circ \mathbf{m}} p_1^*\mathcal{F} \otimes p_2^*\mathcal{E}^* \otimes \mathcal{O}_\Delta \xrightarrow{p_1^*\mathcal{F}\otimes p_2^*\mathcal{E}^*\otimes \mathbf{q}} p_1^*\mathcal{F} \otimes p_2^*(\mathcal{E}^* \otimes S_X)$$

is exactly the image of the element f_* of

$$\mathrm{Hom}_{\mathbb{K}}(\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}), \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{F}))$$

under the identification

$$\begin{array}{c}
\mathrm{Hom}_{\mathbb{K}}(\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}), \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{F})) \\
\big\downarrow \\
\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{F}) \otimes \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E})^* \\
\big\downarrow \\
\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{F}) \otimes \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}^* \otimes S_X) \\
{\scriptstyle p_1^*\otimes p_2^*}\big\downarrow \\
\mathrm{Hom}_{\mathrm{D}^b(X\times X)}(\mathcal{O}_{X\times X}, p_1^*\mathcal{F} \otimes p_2^*(\mathcal{E} \otimes S_X)).
\end{array}$$

Let $\mathbf{tr} \in \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, S_X)$ be the element satisfying $\mathrm{tr}_X(\mathbf{tr}) = 1$. Applying the above statement with $\mathcal{E} = \mathcal{F} = \mathcal{O}_X$ and $f = \mathcal{O}_X$, we see that the composite

$$\mathcal{O}_{X\times X} \xrightarrow{\mathbf{m}} \mathcal{O}_\Delta \xrightarrow{\mathbf{q}} p_2^* S_X$$

is precisely $p_2^*(\mathbf{tr})$. It follows that the composite $\mathbf{t}$ given by

$$\mathcal{O}_{X\times X} \xrightarrow{\mathbf{m}} \mathcal{O}_{X\times X} \otimes \mathcal{O}_\Delta \xrightarrow{p_1^*(\mathbf{tr})\otimes \mathbf{q}} p_1^* S_X \otimes p_2^* S_X = S_{X\times X}$$

satisfies

$$\mathrm{tr}_{X\times X}(\mathbf{t}) = 1.$$

Let $\mathbf{tr}_\Delta$ denote the following composite:

$$\mathcal{O}_{X\times X} \otimes \mathcal{O}_\Delta \xrightarrow{p_1^*(\mathbf{tr})\otimes \mathbf{q}} S_{X\times X} \otimes \mathcal{O}_{X\times X} \xrightarrow{S_{X\times X}\otimes \mathbf{m}} S_{X\times X} \otimes \mathcal{O}_\Delta.$$

By Lemma 2.2 of Caldararu's paper [1],

$$\mathrm{tr}_{X\times X}(\mathbf{tr}_\Delta) = \mathrm{tr}_{X\times X}(\mathbf{t}) = 1.$$

Therefore,

$$\mathrm{tr}_{X\times X}(\mathbf{tr}_\Delta \otimes p_2^* S_X^{-1}) = 1.$$

To verify (55), it suffices to check that

$$\mathrm{tr}_X(\mathcal{J}(\mathbf{tr}_\Delta \otimes p_2^* S_X^{-1}) \circ \beta) = 1$$

where $\mathcal{J}$ is as in Section 1.

For this, it is enough to check that

$$\mathrm{tr}_X(\mathcal{J}(\mathbf{tr}_\Delta) \circ (\beta \otimes S_X)) = 1. \tag{56}$$

Note that $\Delta^*(\mathbf{tr}_\Delta)$ is given by the composite

$$\mathcal{O}_X \otimes \Delta^*\mathcal{O}_\Delta \xrightarrow{\mathbf{tr}\otimes\Delta^*(\mathbf{q})} S_X \otimes S_X \xrightarrow{S_X\otimes S_X\otimes\epsilon} S_X \otimes S_X \otimes \Delta^*\mathcal{O}_\Delta$$

where ϵ is the map induced in $\mathrm{D}^b(X)$ by the unit $\mathcal{O}_X \to \widehat{C}^\bullet(X)$ of the Hopf algebra object $\widehat{C}^\bullet(X)$ of $\mathrm{Ch}^-(\mathcal{O}_X - \mathrm{mod})$. It follows that $\mathcal{J}(\mathbf{tr}_\Delta)$ is given by the composite

$$\mathcal{O}_X \otimes \Delta^*\mathcal{O}_\Delta \xrightarrow{\mathbf{tr}\otimes\Delta^*(\mathbf{q})} S_X \otimes S_X.$$

To verify (56) it suffices to check that $\Delta^*(\mathbf{q}) \circ (\beta \otimes S_X) = S_X$. This is an immediate consequence of (53). $\square$

5.2. The final (long) computation proving Theorem 1. We first recall some notation. We remind the reader that as in Section 3.2, the product on $\mathbf{S}^\bullet(\Omega[1])$ is denoted by $(\text{-} \wedge \text{-}) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1])$. Let ev be as in Section 3.2. Denote the composite

$$\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \xrightarrow{\mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{j}} \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \xrightarrow{\mathrm{ev}} \mathbf{S}^\bullet(\Omega[1])$$

by $(\text{-} \bullet \text{-})$ as in Section 3.2. Note that the composite

$$\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{i}} \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \xrightarrow{\mathrm{ev}} \mathbf{S}^\bullet(\Omega[1])$$

is $(\text{-} \wedge \text{-})$. Let $(\text{-} || \text{-})$ be as in Section 3.2. We will also denote the isomorphisms $S_X \otimes S_X^{-1} \to \mathcal{O}_X$ and $S_X^{-1} \otimes S_X \to \mathcal{O}_X$ by $\simeq$.

Also recall from Section 1 that

$$\langle \text{-}, \text{-} \rangle = \pi_n \circ (\text{-} \wedge \text{-}) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]).$$

Let $x \in \mathrm{RHom}_X(\mathcal{O}_X, \mathbf{S}^\bullet(\Omega[1]))$.

Outline of the final computation proving Theorem 1. The final computation proving Theorem 1 begins by summarizing the results of the previous subsection to express $\mathrm{I}_{\mathrm{HKR}}(D_{\Delta}^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ as a composite of morphisms in $\widetilde{\mathrm{D}^b(X)}$. This is done in Proposition 37. After Proposition 37, in (59), (60), (61) and (62), the composite yielding $\mathrm{I}_{\mathrm{HKR}}(D_{\Delta}^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ is rewritten to express it in a form that is possible to simplify. Lemma 5 is then used to simplify this composite further, yielding the composite (63). The simplification using Lemma 5 is a crucial step. The composite (63) is further simplified to the composite (64). The fact that the composite (64) of morphisms in $\widetilde{\mathrm{D}^b(X)}$ yields $\mathrm{I}_{\mathrm{HKR}}(D_{\Delta}^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ together with Proposition 28 and Proposition 27 of Section 3.2 yield Theorem 1. Lemma 5 is then proven in Section 5.3.

With the above notation and outline in mind, view Φ_R as an element of

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \Omega[1]).$$

The following proposition begins the final set of steps towards Theorem 1.

Proposition 37. *As a morphism in* $\widetilde{\mathrm{D}^b(X)}$, $\mathrm{I}_{\mathrm{HKR}}(D_{\Delta}^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ *is given by the composite*

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d, "\overline{\beta}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \simeq \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\exp(\Phi_R)\otimes S_X^{-1}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \arrow[d, "\simeq"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\mathbf{S}^\bullet(\Omega[1])\otimes x\otimes S_X^{-1}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes\langle\ -\ ,\ -\ \rangle\otimes S_X^{-1}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes S_X \otimes S_X^{-1} \arrow[d, "\mathrm{ev}\otimes\simeq"'] \\
\mathbf{S}^\bullet(\Omega[1]).
\end{tikzcd}$$

Proof. This proposition just amounts to putting together Propositions 33, 34, 35 and 36 and the definition of $\widehat{\mathrm{I}_{\mathrm{HKR}}}$. $\square$

As in Section 3.2, let $\gamma : \mathbf{S}^\bullet(\Omega[1]) \simeq \mathbf{S}^\bullet(T[-1]) \otimes S_X$ be the isomorphism such that

$$\langle \text{ - }, \text{ - } \rangle = \pi_n \circ (\text{ - } \wedge \text{ - }) = ((\text{ - } \| \text{ - }) \otimes S_X) \circ (\mathbf{S}^\bullet(\Omega[1]) \otimes \gamma). \tag{57}$$

Let ζ denote the inverse of γ.

Also note that by the definitions of F_r and F_l, the following diagrams commute.

(58)

$$\begin{tikzcd}
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \arrow[r, "\mathbf{S}^\bullet(\Omega[1])\otimes\tau(F_l)"] \arrow[d, "\mathrm{ev}"'] & \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \arrow[d, "((\text{ - } \wedge \text{ - })\bullet \text{ - })"] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\mathbf{S}^\bullet(\Omega[1])"] & \mathbf{S}^\bullet(\Omega[1])
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \arrow[r, "\mathbf{S}^\bullet(\Omega[1])\otimes F_r"] \arrow[d, "\mathrm{ev}"'] & \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \arrow[d, "((\text{ - } \bullet \text{ - })\wedge \text{ - })"] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\mathbf{S}^\bullet(\Omega[1])"] & \mathbf{S}^\bullet(\Omega[1]).
\end{tikzcd}$$

In the first diagram in (58),

$$\tau : \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \to \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1])$$

interchanges factors. By Proposition 37, $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ is given by the composite

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d, "\overline{\beta}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \simeq \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\exp(\Phi_R)\otimes x\otimes S_X^{-1}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \arrow[d, "\mathrm{ev}\otimes\langle \text{ - },\text{ - } \rangle\otimes S_X^{-1}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes S_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\simeq"'] \\
\mathbf{S}^\bullet(\Omega[1])
\end{tikzcd} \tag{59}$$

of morphisms in $\widetilde{\mathrm{D}^b(X)}$. Denote the composite

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d, "\exp(\Phi_R)"'] & \\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\tau(F_l)\otimes\mathbf{S}^\bullet(\Omega[1])"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1])
\end{tikzcd}$$

by $\mathbf{P}$.

By (57) and (58) the composite (59) yielding $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ is the same as the composite

$$
\begin{array}{c}
\mathcal{O}_X \\
\Big\downarrow{\scriptstyle \overline{\beta}} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \simeq \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes \mathcal{O}_X \otimes S_X^{-1} \\
\Big\downarrow{\scriptstyle \mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{P}\otimes\gamma(x)\otimes S_X^{-1}} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \\
\Big\downarrow{\scriptstyle ((\,\text{-}\,\wedge\,\text{-}\,)\bullet\,\text{-}\,)\otimes(\,\text{-}\,\|\,\text{-}\,)\otimes\simeq} \\
\mathbf{S}^\bullet(\Omega[1])
\end{array}
\tag{60}
$$

of morphisms in $\widetilde{\mathrm{D}^b(X)}$. Note that the composite

$$
\Omega^j[j] \xrightarrow{\text{“}\overline{\beta}\otimes\Omega^j[j]\text{”}} \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^j[j] \otimes S_X^{-1} \xrightarrow{(\,\text{-}\,\wedge\,\text{-}\,)\otimes S_X^{-1}} \mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1}
$$

is 0. (“$\overline{\beta} \otimes \Omega^j[j]$” is a rearrangment of factors composed with $\overline{\beta} \otimes \Omega^j[j]$). It follows that the composite (60) yielding $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ is equal to the composite

$$
\begin{array}{c}
\mathcal{O}_X \\
\Big\downarrow{\scriptstyle \overline{\beta}} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \simeq \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes \mathcal{O}_X \otimes S_X^{-1} \\
\Big\downarrow{\scriptstyle \mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{Q}\otimes\gamma(x)\otimes S_X^{-1}} \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \\
\Big\downarrow{\scriptstyle (\,\text{-}\,\bullet\,\text{-}\,)\otimes(\,\text{-}\,\|\,\text{-}\,)\otimes\simeq} \\
\mathbf{S}^\bullet(\Omega[1])
\end{array}
\tag{61}
$$

of morphisms in $\widetilde{\mathrm{D}^b(X)}$. In (61), $\mathbf{Q}$ in turn denotes the composite

$$
\begin{array}{lcl}
\mathcal{O}_X & & \\
\Big\downarrow{\scriptstyle \exp(\Phi_R)} & & \\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) & \xrightarrow{\pi_0\circ F_l\otimes\mathbf{S}^\bullet(\Omega[1])} & \mathbf{S}^\bullet(T[-1]) \otimes \mathbf{S}^\bullet(\Omega[1]).
\end{array}
$$

Let $(\exp(\Phi_R)||\text{ - }) : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ denote the composite

$$\begin{tikzcd}
\mathcal{O}_X \otimes \mathbf{S}^\bullet(T[-1]) \arrow[d, "\exp(\Phi_R)\otimes \mathbf{S}^\bullet(T[-1])"'] & \\
\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \arrow[r, "\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))\otimes(\text{ - }||\text{ - })"] & \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])).
\end{tikzcd}$$

The composite in the diagram (61) yielding $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ can be rewritten as

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d, "\overline{\beta}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes S_X^{-1} \simeq \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\mathbf{R}\otimes S_X^{-1}"'] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[d, "(\text{ - }\bullet\text{ - })\otimes\simeq"'] \\
\mathbf{S}^\bullet(\Omega[1])
\end{tikzcd} \tag{62}$$

where

$$\mathbf{R} = [\pi_0(F_l(\exp(\Phi_R)||\text{ - })) \otimes S_X] \circ \gamma(x) : \mathcal{O}_X \to \mathbf{S}^\bullet(T[-1]) \otimes S_X$$

in $\widetilde{\mathrm{D}^b(X)}$.

We remind the reader that since $(\exp(\Phi_R)||\text{ - })$ is a morphism in $\mathrm{D}^b(X)$ from $\mathbf{S}^\bullet(T[-1])$ to $\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$, $\pi_0(F_l(\exp(\Phi_R)||\text{ - }))$ is a morphism in $\mathrm{D}^b(X)$ from $\mathbf{S}^\bullet(T[-1])$ to $\mathbf{S}^\bullet(T[-1])$. It follows from this and the fact that $\gamma(x) \in \mathrm{Hom}_{\widetilde{\mathrm{D}^b(X)}}(\mathcal{O}_X, \mathbf{S}^\bullet(T[-1]) \otimes S_X)$ that $\mathbf{R}$ is indeed an element of $\mathrm{Hom}_{\widetilde{\mathrm{D}^b(X)}}(\mathcal{O}_X, \mathbf{S}^\bullet(T[-1]) \otimes S_X)$ as mentioned above. The following lemma simplifies the computation of $\mathbf{R}$. Let

$$(\text{ - }|\text{ - }) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1])$$

be as in Section 3.2. Let $(\mathrm{td}_X^{-1}|\text{ - })$ denote the composite

$$(\text{ - }|\text{ - }) \circ (\mathrm{td}_X^{-1} \otimes \mathbf{S}^\bullet(T[-1])).$$

This is a morphism in $\mathrm{D}^b(X)$ from $\mathbf{S}^\bullet(T[-1])$ to itself.

Lemma 5.

$$\pi_0(F_l(\exp(\Phi_R)||\text{ - })) = (\mathrm{td}_X^{-1}|\text{ - }) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

We postpone the proof of this lemma to Section 5.3.

It follows from (62) and Lemma 5 that $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ is given by the composite

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d, "\overline{\beta}"] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\gamma(x)\otimes S_X^{-1}"] \\
\mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[r, "(\,\text{-}\,\bullet(\mathrm{td}_X^{-1}|\,\text{-}\,))\otimes\simeq"] & \mathbf{S}^\bullet(\Omega[1])
\end{tikzcd} \tag{63}$$

of morphisms in $\widetilde{\mathrm{D}^b(X)}$.

Now note that the following diagram commutes in $\widetilde{\mathrm{D}^b(X)}$.

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\mathbf{1}_{\mathcal{O}_X}"] \arrow[d] & \mathcal{O}_X \arrow[d, "\overline{\beta}"] \\
S_X \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[r, "\iota\otimes\mathcal{O}_X\otimes S_X^{-1}"] \arrow[d, "S_X\otimes\gamma(x)\otimes S_X^{-1}"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "\mathbf{S}^\bullet(\Omega[1])\otimes\gamma(x)\otimes S_X^{-1}"] \\
S_X \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[r, "f"] \arrow[d, "(\,\text{-}\,\bullet(\mathrm{td}_X^{-1}|\,\text{-}\,))\otimes\simeq"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[d, "(\,\text{-}\,\bullet(\mathrm{td}_X^{-1}|\,\text{-}\,))\otimes\simeq"] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\mathbf{1}_{\mathbf{S}^\bullet(\Omega[1])}"] & \mathbf{S}^\bullet(\Omega[1]),
\end{tikzcd}$$

where $f = \iota \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1}$.

The topmost square in the above diagram commutes by the definition of $\overline{\beta}$. That the remaining squares in the above diagram commute is obvious. The reader should recall that $S_X = \Omega^n[n]$ to make sense out of the map $(\,\text{-}\,\bullet(\mathrm{td}_X^{-1}|\,\text{-}\,)) : S_X \otimes \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(\Omega[1])$ in the above diagram.

It follows from the above diagram and (63) that $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$ is given by the composite

$$\begin{tikzcd}
\mathcal{O}_X \arrow[d] \\
S_X \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[d, "S_X\otimes\gamma(x)\otimes S_X^{-1}"] \\
S_X \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[d, "(\,\text{-}\,\bullet(\mathrm{td}_X^{-1}|\,\text{-}\,))\otimes\simeq"] \\
\mathbf{S}^\bullet(\Omega[1])
\end{tikzcd} \tag{64}$$

of morphisms in $\widetilde{\mathrm{D}^b(X)}$.

Let $\tau : S_X \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \to \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X$ denote the map swapping S_X and $\mathbf{S}^\bullet(T[-1]) \otimes S_X$. Let $\tau' : S_X \otimes \mathcal{O}_X \to \mathcal{O}_X \otimes S_X$ swap factors. We now have the following proposition.

Proposition 38. *The following diagram commutes in* $\widetilde{\mathrm{D}^b(X)}$.

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\mathbf{1}"] \arrow[d] & \mathcal{O}_X \arrow[d] \\
S_X \otimes \mathcal{O}_X \otimes S_X^{-1} \arrow[r, "\tau' \otimes S_X^{-1}"] \arrow[d, "S_X \otimes \gamma(x) \otimes S_X^{-1}"'] & \mathcal{O}_X \otimes S_X \otimes S_X^{-1} \arrow[d, "\gamma(x) \otimes S_X \otimes S_X^{-1}"] \\
S_X \otimes \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X^{-1} \arrow[r, "\tau \otimes S_X^{-1}"] \arrow[d, "(- \bullet(\mathrm{td}_X^{-1} \rfloor -)) \otimes \simeq"'] & \mathbf{S}^\bullet(T[-1]) \otimes S_X \otimes S_X \otimes S_X^{-1} \arrow[d, "\zeta((\mathrm{td}_X^{-1} \rfloor -) \otimes S_X) \otimes \simeq"] \\
\mathbf{S}^\bullet(\Omega[1]) \arrow[r, "J"] & \mathbf{S}^\bullet(\Omega[1]).
\end{tikzcd}$$

Proof. The fact that the first two squares in the above diagram commute is clear. The third square commutes by Proposition 28 of Section 3.2. $\square$

Recall that $\gamma : \mathbf{S}^\bullet(\Omega[1]) \simeq \mathbf{S}^\bullet(T[-1]) \otimes S_X$ and ζ denotes the inverse of γ. By Proposition 27 of Section 3.2,

$$\zeta(((\mathrm{td}_X^{-1} \rfloor -) \otimes S_X)(\gamma(x)) = \mathrm{td}_X^{-1} \wedge x.$$

Therefore, by Proposition 38, the fact that the composite (64) yields $\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x)))$, and the fact that $J^2 = \mathbf{1}_{\mathbf{S}^\bullet(\Omega[1])}$,

$$\mathrm{I}_{\mathrm{HKR}}(D_\Delta^{-1}(\widehat{\mathrm{I}_{\mathrm{HKR}}}(x))) = J(\mathrm{td}_X^{-1} \wedge x).$$

It follows that

$$D_\Delta(\mathrm{I}_{\mathrm{HKR}}^{-1}(y)) = \widehat{\mathrm{I}_{\mathrm{HKR}}}(\mathrm{td}_X \wedge Jy).$$

This proves Theorem 1. $\square$

5.3. Proving Lemma 5. Lemma 5 is the only thing left to be proven. The proof of Lemma 5 uses Lemma 3. We first state and prove the following proposition. All statements in this subsection hold in $\mathrm{D}^b(X)$ and hence in $\widetilde{\mathrm{D}^b(X)}$.

Proposition 39.

$$\pi_0(F_r(\exp(\Phi_R) \rfloor\rfloor -)) = \mathbf{1} : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

Proof. The notation used here is that used while proving Proposition 30. While proving Proposition 30, we noted that $\overline{\varpi}^i \circ \overline{\mathbf{C}}$ is given by the composite

$$\begin{tikzcd} \mathbf{S}^\bullet(\Omega[1]) \arrow[r, "\overline{\mathbf{C}}"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1] \arrow[r, "\mathbf{S}^\bullet(\Omega[1])\otimes \mathrm{At}_T^i"] & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega^i[i] \otimes \Omega[1] \arrow[d, "(\,\text{-}\wedge\text{-}\,)\otimes\Omega[1]"'] \\ & & \mathbf{S}^\bullet(\Omega[1]) \otimes \Omega[1]. \end{tikzcd}$$

It follows by the definition of $\pi_0 \circ F_r$ that

$$(\pi_0 \circ F_r \otimes \Omega[1])(\overline{\varpi}^i \circ \overline{\mathbf{C}}) = 0$$

if $i > 0$. It follows from the fact that $\frac{z}{1-\mathrm{e}^{-z}} = 1 + \sum_{i \geq 1} c_i z^i$ that

$$(\pi_0 \circ F_r \otimes \Omega[1])(\Phi_R) = (\pi_0 \circ F_r \otimes \Omega[1])(\overline{\mathbf{C}}).$$

Therefore, by Proposition 23,

$$(\pi_0 \circ F_r \otimes \mathbf{S}^\bullet(\Omega[1]))(\exp(\Phi_R)) = (\pi_0 \circ F_r \otimes \mathbf{S}^\bullet(\Omega[1]))(\exp(\overline{\mathbf{C}})).$$

Note that

$$\exp(\overline{\mathbf{C}}) = \mathbf{C}_\Omega$$

where $\mathbf{C}_\Omega$ is the comultiplication on $\mathbf{S}^\bullet(\Omega[1])$ treated as an element of

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{O}_X, \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathbf{S}^\bullet(\Omega[1])).$$

For the rest of this proof let $(\mathbf{C}_\Omega || \text{ - })$ denote

$$[\mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes (\text{ - } || \text{ - })] \circ \mathbf{C}_\Omega.$$

Therefore,

$$\pi_0(F_r(\exp(\Phi_R) || \text{ - })) = \pi_0(F_r(\mathbf{C}_\Omega || \text{ - })) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

But $(\mathbf{C}_\Omega || \text{ - }) = \mathbf{j} : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(T[-1])$ by Proposition 29. This proves the desired proposition. □

Proof of Lemma 5. Let $(\exp(\Phi_R) || \text{ - })^+ : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ denote the map $\mathbf{A} \circ (\exp(\Phi_R) || \text{ - }) : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]))$ where $\mathbf{A}$ is as in Section 3.2.

By Proposition 22,

$$\pi_0(F_l(\exp(\Phi_R) || \text{ - })) = I(\pi_0(F_r(\exp(\Phi_R) || \text{ - })^+)) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

By part (2) of Proposition 21,

$$(\exp(\Phi_R) || \text{ - })^+ = (\exp(\Phi_R^+) || \text{ - }) : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])).$$

By Lemma 3,

$$\begin{aligned} &(\exp(\Phi_R^+) || \text{ - }) \\ &= (\exp(-\mathbf{i}(\mathbf{f}) \circ \Phi_R \circ \mathbf{i}(\mathbf{f})^{-1}) || \text{ - }) \\ &= \mathbf{i}(\mathbf{f}) \circ (\exp(-\Phi_R) || \text{ - }) \circ \mathbf{i}(\mathbf{f}^{-1}) : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])). \end{aligned}$$

We remark that $\mathbf{i}(\mathbf{f}) \circ (\exp(-\Phi_R)||\text{ - }) \circ \mathbf{i}(\mathbf{f}^{-1})$ is precisely the composite

$$\begin{array}{c} \mathcal{O}_X \otimes \mathbf{S}^\bullet(T[-1]) \otimes \mathcal{O}_X \\ \Big\downarrow {\scriptstyle \mathbf{i}(\mathbf{f}) \otimes (\exp(-\Phi_R)||\text{ - }) \otimes \mathbf{i}(\mathbf{f}^{-1})} \\ \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])) \otimes \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1]) \xrightarrow{\circ} \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])). \end{array}$$

Thus,

$$\begin{aligned} (65) \quad & \pi_0(F_l(\exp(\Phi_R)||\text{ - })) \\ &= I(\pi_0(F_r(\mathbf{i}(\mathbf{f}) \circ (\exp(-\Phi_R)||\text{ - }) \circ \mathbf{i}(\mathbf{f}^{-1})))) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]). \end{aligned}$$

Note that $\pi_0(F_r(\mathbf{i}(\mathbf{f}))) = 1$ since $\mathbf{f} = \det(1 + \sum_{i>0} c_i \mathrm{At}_T^i)$. It follows from Proposition 23 that

$$\begin{aligned} (66) \quad & \pi_0(F_l(\exp(\Phi_R)||\text{ - })) \\ &= I(\pi_0(F_r((\exp(-\Phi_R)||\text{ - }) \circ \mathbf{i}(\mathbf{f}^{-1})))) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]). \end{aligned}$$

Another point to note is that

$$(\exp(-\Phi_R)||\text{ - }) = (\exp(\Phi_R)||\text{ - }) \circ I : \mathbf{S}^\bullet(T[-1]) \to \mathcal{E}\mathrm{nd}(\mathbf{S}^\bullet(\Omega[1])).$$

It follows from this observation and (66) that

$$\begin{aligned} & \pi_0(F_l(\exp(\Phi_R)||\text{ - })) \\ &= I(\pi_0(F_r((\exp(\Phi_R)||\text{ - }) \circ \mathbf{i}(\mathbf{f}^{-1})))) \circ I \\ &= I(\pi_0(F_r(\mathbf{j}(\pi_0(F_r(\exp(\Phi_R)||\text{ - }))) \circ \mathbf{i}(\mathbf{f}^{-1})))) \circ I \text{ by Proposition 23} \\ &= I(\pi_0(F_r(\mathbf{j}(I(\text{ - })) \circ \mathbf{i}(\mathbf{f}^{-1})))) \text{ by Proposition 39} \\ &= I(\mathbf{f}^{-1}|I(\text{ - })) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]) \text{ by Proposition 26} \end{aligned}$$

where $(\text{ - }|\text{ - }) : \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1])$ is as in Section 3.2.

We remind the reader that $(\mathbf{f}^{-1}|I(\text{ - })) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1])$ is the composite

$$\begin{array}{c} \mathbf{S}^\bullet(T[-1]) \\ \Big\downarrow {\scriptstyle I} \\ \mathcal{O}_X \otimes \mathbf{S}^\bullet(T[-1]) \\ \Big\downarrow {\scriptstyle \mathbf{f}^{-1} \otimes \mathbf{S}^\bullet(T[-1])} \\ \mathbf{S}^\bullet(\Omega[1]) \otimes \mathbf{S}^\bullet(T[-1]) \xrightarrow{(\text{ - }|\text{ - })} \mathbf{S}^\bullet(T[-1]) \end{array}$$

of morphisms in $\mathrm{D}^b(X)$.

Note that

$$I(\mathbf{f}^{-1}|I(\text{ - })) = (J(\mathbf{f}^{-1})|\text{ - }) : \mathbf{S}^\bullet(T[-1]) \to \mathbf{S}^\bullet(T[-1]).$$

Also observe that $J(\mathbf{f}^{-1}) = \mathrm{td}_X^{-1}$. This proves Lemma 5. □

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