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Small time heat kernel behavior on Riemannian complexes

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Abstract. We study how bounds on the local geometry of a Riemannian polyhedral complex yield uniform local Poincaré inequalities. These inequalities have a variety of applications, including bounds on the heat kernel and a uniform local Harnack inequality. We additionally consider the example of a complex, X, which has a finitely generated group of isomorphisms, G, such that $X/G = Y$ is a complex consisting of a finite number of polytopes. We show that when this group, G, has polynomial volume growth, there is a uniform global Poincaré inequality on the complex, X.

Contents

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1. Introduction

It has been observed by several authors, see, e.g., [20, 33], that a theory called first-order calculus can be developed under some assumptions on metric measure spaces. The heat equation and its associated Markov process, Brownian motion, require some additional structure. For instance, it is well understood that the structure of complete Riemannian manifolds induces a well-defined heat equation and Brownian motion. Similarly, it is natural to ask if the structure of Riemannian polyhedral complexes yields a well-defined notion of heat equation and Brownian motion. For one-dimensional complexes (i.e., locally finite metric graphs), this has been studied by probabilists under the name of Walsh Brownian motion. Strictly speaking, Walsh Brownian motion is defined on a (perhaps finite) collection of semiaxes with the same origin. See, e.g., [5, 4]. A construction in this spirit on 2-dimensional Euclidean simplicial complexes is given in [8, 14]. For more general complexes, a completely different approach is considered in [7]. Except in dimension 1, the question of the unicity of the constructed objects has not been thoroughly studied and presents some difficulties. In this paper, we define the heat equation (and, implicitly, the associated process) using the Dirichlet form approach as in [13]. Indeed, just as a Riemannian manifold carries a natural Dirichlet form, so does a Riemannian polyhedral complex. Under certain assumptions, we prove basic estimates for the associated heat kernel.

Riemannian polyhedral complexes are formed by taking a collection of n-dimensional convex polyhedra and joining them along $n-1$ dimensional faces. Within each polyhedron, we will have the same metric structure as a Riemannian manifold. When we join them, we will glue the faces of two polyhedra together so that points on one face are identified with points on the other face, and the metrics on those faces are preserved. We will require that these structures have an upper bound on the number of n-dimensional polyhedra that share an $n-1$-dimensional face and a lower bound on the interior angles and distances between nonadjacent faces. Additionally we assume a bound on the ellipticity of the manifolds. The complex formed by looking at k-dimensional faces is called the k-skeleton. For instance, the 0-skeleton is set of vertices. A 1-skeleton is a graph where the space includes both vertices and points on the edges; sometimes this is called a metric graph [27]. Note that we can triangulate any convex polyhedron to obtain a collection of simplices, and so when the metrics are all Euclidean, this structure is essentially equivalent to looking at a simplicial complex. In Section 1.1 we define these structures as well as some restrictions on their geometry. In Section 1.2 we define and describe the Dirichlet form and its domain.

In Section 2, we show a Poincaré inequality on such a complex, X: For any fixed R_0, there exists a constant, P_0, so that for any $r < R_0$, $z \in X$, and $f \in W^{1,p}(B(z,r))$ we have

$$\int_{B(z,r)} |f(x) - f_{B(z,r)}|^p d\mu(x) \leq P_0\, r^p \int_{B(z,r)} |\nabla f(x)|^p d\mu(x).$$

This inequality allows us to describe small time heat kernel behavior; we do so in Section 3.1. Many important properties follow. Theorems in Sturm [38] can be applied directly to these complexes to show that on any compact subset of the k-skeleton, the heat kernel is locally like the one on $\mathbb{R}^k$, with constants that depend on the choice of compact subset. The essential improvement in our theorem is that the constants are uniform throughout the entire complex. Results where there are only a finite number of glued spaces can be found in Paulik [29] who additionally studies sets whose overlap has positive measure. In Section 3.2 we consider a complex, X, which has a finitely generated group of isomorphisms, G, such that X/G is a complex consisting of a finite number of polytopes. We show that for a group, G, with polynomial volume growth, there is a uniform global Poincaré inequality on the complex, X. In this case, the heat kernel asymptotics apply with global constants.

1.1. Definitions. We begin by defining the complex and its geometric structure, as well as some restrictions on this structure. For a thorough introduction to analysis on polyhedral complexes, see [13].

Definition 1.1. A polyhedral complex X is the union of a collection of convex polyhedra which are joined along lower-dimensional faces. By this we mean that for any two distinct polyhedra P_1, P_2 in the collection, $P_1 \cap P_2$ is a polyhedron whose dimension satisfies $\dim(P_1 \cap P_2) < \max(\dim(P_1), \dim(P_2))$ and $P_1 \cap P_2$ is a face of both P_1 and P_2. We allow this face to be the empty set.

Note that this definition implies $P_1 \cap P_2$ is a connected set. This rules out expressing a circle as two edges whose ends are joined, but it allows us to write it as a triangle of three edges. This definition is not very restrictive, as we can triangulate the polyhedra in order to form a complex which avoids the overlap.

Simplicial complexes are an example of a polyhedral complex; the difference here is that we allow greater numbers of sides. Note that we allow unbounded polyhedra, not just bounded polytopes. We do not define or require an embedding of the complex into Euclidean space; however, each individual polytope or polyhedron can be viewed locally as a subset of Euclidean space.

Definition 1.2. Define a k-skeleton, $X^{(k)}$, for $0 \leq k \leq \dim(X)$ to be the union of all faces of dimension k or smaller. Note that this is also a polyhedral complex.

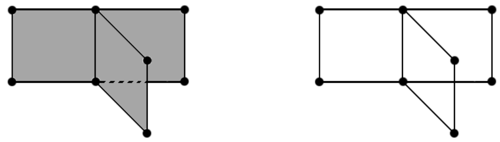

FIGURE 1. Example of a two-dimensional Euclidean complex (left) and its 1-skeleton (right).

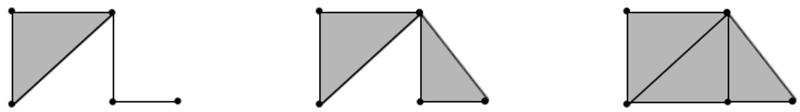

FIGURE 2. Examples of a complex which is not dimensionally homogeneous (left), one which is not 1-chainable (center), and one which is admissible (right).

Definition 1.3. A maximal polyhedron is a polyhedron that is not a proper face of any other polyhedron. We say X is dimensionally homogeneous if all of its maximal polyhedra have dimension n. Note that in combinatorics literature this is called pure. We denote the set of maximal polyhedra by $\mathcal{M}$.

Definition 1.4. X is locally $(n-1)$-chainable if for every connected open set $U \subset X$, $U - X^{(n-2)}$ is also connected. For a dimensionally homogeneous complex X this is equivalent to the property that any two n-dimensional polyhedra that share a lower-dimensional face can be joined by a chain of contiguous $(n-1)$ or n-dimensional polyhedra containing the lower-dimensional face.

Definition 1.5. We call X admissible if it is dimensionally homogeneous and in some triangulation X is locally $(n-1)$-chainable. (See Figure 2.)

We will be working with connected admissible complexes. On each maximal polyhedron, P, we have a covariant bounded measurable Riemannian metric tensor $\mathcal{G}$ which satisfies an ellipticity condition. There exists a constant, Λ_P, so that for any $\zeta \in \mathbb{R}^n$ we have

$$\Lambda_P^{-2} \sum (\zeta^i)^2 \leq \mathcal{G}_{ij}\zeta^i\zeta^j \leq \Lambda_P^2 \sum (\zeta^i)^2.$$

We require the metric to be continuous in the sense that $\mathcal{G}$ is continuous up to the boundary, and the metrics on two neighboring polyhedra induce the same Riemannian metric on their shared face. We will also require the ellipticity to be uniform; that is,

$$\Lambda = \sup_{P \in \mathcal{M}} \Lambda_P < \infty.$$

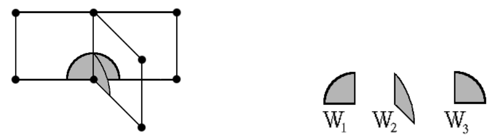

FIGURE 3. Complex with shaded ball B (left); the three wedges for B (right).

Distance is defined as in [13] as the infimum over a set of lengths of paths. Note that this is an intrinsic distance, so X is a length space.

Let $X = \cup_i P_i$, where the P_i are the maximal polyhedra. We will set the measure of A, a Borel subset of X, to be $\mu(A) = \sum_i \mu_i(A \cap P_i)$ where μ_i is the measure on P_i. We will use the notation μ_e and μ_g whenever we need to distinguish between the Euclidean and Riemannian measures.

Definition 1.6. An admissible polyhedral complex, X, equipped with a uniformly elliptic Riemannian metric tensor $\mathcal{G}$ on each polyhedron is called a Riemannian polyhedral complex. When $\Lambda = 1$, it is a Euclidean polyhedral complex. For brevity, we will often call this a Riemannian (respectively Euclidean) complex.

The uniform ellipticity condition forces each Riemannian complex to have a corresponding Euclidean complex with comparable distance:

$$\Lambda^{-1} d_g \leq d_e \leq \Lambda d_g.$$

Similarly, for a complex whose maximal polyhedra have dimension n, the measures are comparable: $\Lambda^{-n}\mu_g \leq \mu_e \leq \Lambda^n \mu_g$.

Definition 1.7. Let X be an admissible polyhedral complex of dimension n such that for every $k \leq n$ the distance between any two nonintersecting k-dimensional faces is bounded below. Let B be a Euclidean ball of radius r whose center is on a D-dimensional face with the property that B intersects no other D-dimensional faces. We define wedges W_j of B to be the closures of each of the connected components of $B - X^{(n-1)}$.

Note that for any z in such an X, a ball $B(z, r)$ satisfying the above criteria exists: for each D, we can take any point $z \in X^{(D)} - X^{(D-1)}$ and any $r < d(z, X^{(D-1)})$ and create $B = B(z, r) \subset X$. Then B is a ball of radius r whose center is on a D-dimensional face, and B intersects no other D-dimensional faces. In essence, the wedges, W_j, are formed when the $(n-1)$- skeleton slices the ball B into pieces. Each W_j has diameter at most $2r$, as each of the points in W_j is within distance r of z, and z is included in W_j.

Example 1.8. In Figure 3 we have an example of a 2-dimensional complex with a shaded ball centered at a vertex. This ball has three wedges; one for each of the two-dimensional faces that share the vertex. Each wedge is a fraction of a sphere.

Definition 1.9. We say X has solid angle bound α if, with respect to Euclidean distance and volume, for $z \in X^{(D)} - X^{(D-1)}$ and $r < d(z, X^{(D-1)})$ the wedges of the ball $B(z, r)$ satisfy

$$\alpha \leq \frac{\mu(W_j)}{\mu(r^n S^{(n-1)})} \leq 1.$$

Note that the right-hand side of the inequality reflects the fact that each of the W_j is a subset of a Euclidean ball.

Assumptions 1.10. We require X to satisfy the following geometric assumptions:

(1) X is a connected admissible complex with n-dimensional maximal polyhedra.
(2) X is uniformly elliptic with constant Λ.
(3) For every k, the maximal number of faces in $X^{(k)}$ that can share a lower-dimensional face is bounded above by M.
(4) For every k, the distance between any two nonintersecting k-dimensional faces is bounded below by ℓ.
(5) X has solid angle bound α.

Note that assumption (3) implies every vertex has degree at most M. Similarly, assumption (4) implies edge lengths are bounded below by ℓ as vertices are 0-dimensional faces.

These assumptions imply each closed ball of finite radius will intersect only finitely many polyhedra. As each of these intersections forms a closed bounded subset of a polyhedron, and each of these is complete, X is complete.

Under Assumptions 1.10, volume doubling occurs locally with a uniform constant. For any R, there exists a constant c so that for any $x \in X$ and $r < R$,

$$\mu(B(x, 2r)) \leq c\mu(B(x, r)).$$

Note that volume doubling will not necessarily hold globally.

Notation 1.11. L^p norms restricted to a subset $A \subset X$ are written as $||f||_{p,A} = \left(\int_A |f(x)|^p d\mu\right)^{1/p}$.

1.2. The Dirichlet form. Now that we have defined the space geometrically, we will define a Dirichlet form whose core consists of compactly supported Lipschitz functions. We denote the space of Lipschitz functions by $\mathrm{Lip}(X)$ and the space of compactly supported Lipschitz functions by $\mathcal{C}_0^{\mathrm{Lip}}(X)$. Note that Lipschitz functions are continuous and differentiable almost everywhere. By Theorem 4 in Section 5.8 of [15], for each $B(x, \epsilon) \subset X - X^{(n-1)}$ and $f \in \mathcal{C}_0^{\mathrm{Lip}}(X)$, f restricted to $B(x, \epsilon)$ is in the Sobolev space $W^{1,\infty}(B(x, \epsilon))$. This tells us that f has a gradient almost everywhere in $X - X^{(n-1)}$. Since $\mu(X^{(n-1)}) = 0$, f has a gradient for almost every x in X.

We would like an energy form that acts like $E(u,v) = \int_X \langle \nabla u, \nabla v \rangle d\mu$ for u, v in its domain, $\mathrm{Dom}(E)$, to define the operator Δ with domain $\mathrm{Dom}(\Delta)$. We can define E in a very general manner which does not depend on the local structure by following a paper of Sturm [39]. We can also define it in a more straightforward manner which uses the geometry of X. We do both, and then show that they coincide.

Sturm assumes that the space (X, d) is a locally compact separable metric space, μ is a Radon measure on X, and that $\mu(U) > 0$ for every nonempty open set $U \subset X$. By 1.10, these assumptions hold both in X and on the skeleta, $X^{(k)}$.

Definition 1.12. We define E^r as

$$E^r(u,v) = \int_X \int_{B(x,r)-\{x\}} \frac{(u(x)-u(y))(v(x)-v(y))}{d^2(x,y)} \frac{2n d\mu(y) d\mu(x)}{\mu(B(x,r)) + \mu(B(y,r))}$$

for $u, v \in \mathrm{Lip}(X)$ where n is the local dimension.

Each E^r with domain $\mathcal{C}_0^{\mathrm{Lip}}(X)$ is closable and symmetric on $L^2(X)$, and its closure has core $\mathcal{C}_0^{\mathrm{Lip}}(X)$. See Lemma 3.1 in [39]. One can take limits of these Dirichlet forms in the following way. The Γ-limit of the E^{r_n} is defined to be the limit that occurs when the following lim sup and lim inf are equal for all $u \in L^2(X)$. See Dal Maso [11] for a thorough treatment.

$$\Gamma - \limsup_{n\to\infty} E^{r_n}(u,u) := \lim_{\alpha\to 0} \limsup_{n\to\infty} \inf_{\substack{v\in L^2(X)\\ ||u-v||\le\alpha}} E^{r_n}(v,v)$$

$$\Gamma - \liminf_{n\to\infty} E^{r_n}(u,u) := \lim_{\alpha\to 0} \liminf_{n\to\infty} \inf_{\substack{v\in L^2(X)\\ ||u-v||\le\alpha}} E^{r_n}(v,v).$$

For any sequence $\{E^{r_n}\}$ with $r_n \to 0$, there is a subsequence $\{r_{n'}\}$ so that the Γ-limit of $E^{r'_n}$ exists by Lemma 4.4 in [39]. These lemmas are put together in Theorem 5.5 in [39] which tells us that this limit, E^0, with domain $\mathcal{C}_0^{\mathrm{Lip}}(X)$ is a closable and symmetric form, and its closure, $(E, \overline{\mathcal{C}_0^{\mathrm{Lip}}(X)})$, is a strongly local regular Dirichlet form on $L^2(X)$ with core $\mathcal{C}_0^{\mathrm{Lip}}(X)$.

Alternatively, we can define the energy form using the structure of the space.

Definition 1.13. For $f \in \mathcal{C}_0^{\mathrm{Lip}}(X)$ we set $\mathcal{E}_0(\cdot,\cdot)$ to the following:

$$\mathcal{E}_0(f,f) = \sum_{P\in\mathcal{M}} \int_P |\nabla f|^2 d\mu.$$

Lemma 1.14. *$(\mathcal{E}_0(\cdot,\cdot), \mathcal{C}_0^{\mathrm{Lip}}(X))$ is a closable form.*

Proof. To show this, we must prove that any sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}_0^{\mathrm{Lip}}(X)$ that converges to 0 in $L^2(X)$ and is Cauchy in $||\cdot||_2 + \mathcal{E}(\cdot,\cdot)$ satisfies $\lim_{n\to\infty} \mathcal{E}(f_n, f_n) = 0$. We will first look at what happens on one fixed polyhedron and then look at what happens on the complex. Let P be a maximal polyhedron. Since $\{f_n\}_{n=1}^{\infty}$ is Cauchy in the norm, we have

$$\lim_{m,n\to\infty} \left(\int_P (f_n - f_m)^2 d\mu\right)^{\frac{1}{2}} + \left(\int_P (\nabla f_n - \nabla f_m)^2 d\mu\right)^{\frac{1}{2}} = 0.$$

This gives us two functions, f and F which are the limits of f_n and ∇f_n respectively. We have $f = 0$ by assumption. On the interior of P, we have the usual Dirichlet form; this implies $F = 0$ on the interior of P and $\lim_{n\to\infty} \int_P \nabla f_n d\mu = 0$. Since $\mu(X - X^{(n-1)}) = 0$, $F = 0$ a.e. on X.

To show L^2 convergence, we need to interchange the limit with the sum over the maximal polyhedra. We can do this for $|\nabla f_n - \nabla f_m|$ by Fatou's Lemma.

$$\begin{aligned}
\lim_{n\to\infty} \sum_{P\in\mathcal{M}} \int_P |\nabla f_n|^2 d\mu &= \lim_{n\to\infty} \sum_{P\in\mathcal{M}} \int_P |\nabla f_n - \lim_{m\to\infty} \nabla f_m|^2 d\mu \\
&= \lim_{n\to\infty} \sum_{P\in\mathcal{M}} \int_P \lim_{m\to\infty} |\nabla f_n - \nabla f_m|^2 d\mu \\
&\le \lim_{n\to\infty} \lim_{m\to\infty} \sum_{P\in\mathcal{M}} \int_P |\nabla f_n - \nabla f_m|^2 d\mu \\
&= 0.
\end{aligned}$$

This tells us that the form is closable. □

We will show that the two energy forms, E and the closure of $\mathcal{E}_0$, are the same. To do this, we show that they are the same on the core $\mathcal{C}_0^{\mathrm{Lip}}(X)$.

Lemma 1.15. *Let* $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ *be the closure of* $(\mathcal{E}_0, \mathcal{C}_0^{\mathrm{Lip}}(X))$. *Then*

$$\mathrm{Dom}(E) = \mathrm{Dom}(\mathcal{E}),$$

and for any $f \in \mathrm{Dom}(E)$ *we have* $E(f,f) = \mathcal{E}(f,f)$.

Proof. We can write X as $(X - X^{(n-1)}) \cup X^{(n-1)}$; this is a collection of maximal polyhedra and a set of measure 0. The interior of each maximal polyhedron is a Riemannian manifold without boundary. X is also a locally compact length space, and so it satisfies the conditions of Example 4G in [38]. This implies it has the strong measure contraction property with an exceptional set. Corollary 5.7 in [38] tells us $E(f,f) = \mathcal{E}(f,f)$ for each $f \in \mathcal{C}_0^{\mathrm{Lip}}(X)$. The equality is shown by approximating the forms using an increasing sequence of open subsets which limit to $X - X^{(n-1)}$. As $\mathcal{C}_0^{\mathrm{Lip}}(X)$ is a core for both E and $\mathcal{E}$, the Dirichlet forms are the same. □

Note that this equality tells us that the forms E^{r_n} have a unique $\Gamma-$limit. Our next result describes what the domain of this form is in more concrete terms.

Definition 1.16. We define

$$W^{1,2}(X) = \{f \in L^2(X) \mid \text{for any maximal polyhedron, } P, f|_P \in W^{1,2}(P), \\ \mathcal{E}(f,f) < \infty, \text{ and } \mathrm{Tr}_i(f) = \mathrm{Tr}_j(f) \text{ on } P_i \cap P_j\}$$

where $\mathrm{Tr}_i : W^{1,2}(P_i) \to L^2(\partial P_i)$ is a trace function on the maximal polyhedron P_i. We write $W_0^{1,2}(X)$ for the set of compactly supported functions in $W^{1,2}(X)$. Similarly, for any domain $\Omega \subset X$ and $1 \leq p < \infty$ we set

$$W^{1,p}(\Omega) \\ = \Bigg\{ f \in L^p(\Omega) \mid \text{for any maximal polyhedron, } P, f|_{P\cap\Omega} \in W^{1,p}(P \cap \Omega), \\ \sum_{P\in\mathcal{M}} \int_{P\cap\Omega} |\nabla f|^p d\mu < \infty, \text{ and } \mathrm{Tr}_i(f) = \mathrm{Tr}_j(f) \text{ on } P_i \cap P_j \cap \Omega \Bigg\}$$

where $\mathrm{Tr}_i : W^{1,p}(P_i) \to L^p(\partial P_i)$ is a trace function on the maximal polyhedron P_i.

Theorem 1.17. $\mathrm{Dom}(\mathcal{E}) = W^{1,2}(X)$.

Proof. We know that $\mathrm{Dom}(\mathcal{E}) = \overline{\mathcal{C}_0^{\mathrm{Lip}}(X)}$, where the closure is taken with respect to the $W^{1,2}(X)$ norm, $||\cdot||_{W^{1,2}} = ||\cdot||_2 + \mathcal{E}(\cdot,\cdot)$. For any $f \in \mathcal{C}_0^{\mathrm{Lip}}(X)$, the support of f is a set with finite measure. For any maximal polyhedron, P, f restricted to P will be in $W^{1,2}(P)$, since

$$||\nabla f||_{2,P} \leq ||\nabla f||_{\infty,P}\mu(P \cap \mathrm{supp}(f)).$$

Similarly, since f has compact support, $\mathcal{E}(f,f) < \infty$.

We can view each maximal polyhedron as a subset of an n dimensional manifold. Then each $f \in \mathrm{Dom}(\mathcal{E})$ has the property that $f|_P$ is in $W^{1,2}(P)$. In particular, polyhedra are Lipschitz domains, and so we can apply Theorem 1.12.2 from Chapter 14 of [13] to these maximal polyhedra. This theorem tells us that for each maximal polyhedron, P_i, a well-defined trace function, $\mathrm{Tr}_i : W^{1,2}(P_i) \to L^2(\partial P_i)$ exists. In particular, the trace is a bounded linear operator, and so for continuous functions, $\mathrm{Tr}_i(f|_{P_i}) = f|_{P_i}$. This gives us $\mathcal{C}_0^{\mathrm{Lip}}(X) \subset W^{1,2}(X)$. When f is the limit of functions $f_m \in \mathcal{C}_0^{\mathrm{Lip}}(X)$, $\mathrm{Tr}_i(f|_{P_i})$ is the limit of $\mathrm{Tr}_i(f_m|_{P_i})$. For every pair of maximal polyhedra, P_i and P_j, we have:

$$\begin{aligned} \mathrm{Tr}_i\, f|_{P_i\cap P_j} &= \lim_{m\to\infty} \mathrm{Tr}_i\, f_m|_{P_i\cap P_j} \\ &= \lim_{m\to\infty} f_m|_{P_i\cap P_j} \\ &= \lim_{m\to\infty} \mathrm{Tr}_j\, f_m|_{P_i\cap P_j} \\ &= \mathrm{Tr}_j\, f|_{P_i\cap P_j}. \end{aligned}$$

Thus for every $f \in \overline{\mathcal{C}_0^{\mathrm{Lip}}(X)}$ we have $\mathrm{Tr}_i = \mathrm{Tr}_j$ on $P_i \cap P_j$. This shows that $\overline{\mathcal{C}_0^{\mathrm{Lip}}(X)} \subset W^{1,2}(X)$. We will now show the reverse containment.

We choose an arbitrary point in X and let B_n represent the ball of radius n centered at that point. Due to the geometric assumptions on X, B_n will have finite volume. Suppose that f is in the set $W^{1,2}(X)$. We can approximate f in the $W^{1,2}(X)$ norm with a sequence of compactly supported functions, f_n, in $W_0^{1,2}(X)$ which have the property that the support of f_n is contained in B_{2n} and $f_n = f$ on B_n.

Theorem 4.1 in Shanmugalingam [34], says that under certain conditions on the space, f_n can be approximated in the $W^{1,2}(B_{2n+1})$ norm by a sequence of locally Lipschitz functions, $h_{n,k}$ with compact support in B_{2n+1}. Assume for the moment that these conditions hold. By a diagonal argument $h_{n,n}$ tends to f in $W^{1,2}(X)$. This shows that $f \in \overline{\mathcal{C}_0^{\mathrm{Lip}}(X)}$.

To complete the proof, we need only show that B_{2n+1} satisfies the conditions of Theorem 4.1 in Shanmugalingam [34]. To do so, we need the following concept.

Definition 1.18. Let u be a real valued function and ρ be a nonnegative Borel measurable function which satisfies the following inequality for all compact rectifiable paths γ with endpoints x and y:

$$|u(x) - u(y)| \leq \int_\gamma \rho ds.$$

The function ρ is called an upper gradient (or very weak gradient) of u. See, e.g., [21] for a discussion of such functions. Note that if $u \in W^{1,1}_{\mathrm{loc}}(X)$, $|\nabla u|$ is an upper gradient for u.

The conditions of Theorem 4.1 in Shanmugalingam [34] are as follows. The first is that volume doubling holds in the ball B_{2n+1}; as noted earlier, this holds. The second condition is that all pairs of measurable functions and their upper gradients (u, ρ) satisfy the following Poincaré style inequality for $\lambda = 1$ and $p = 2$.

$$\fint_B |u - u_B| d\mu \leq C \operatorname{diam}(B) \left(\fint_{\lambda B} \rho^p d\mu \right)^{1/p}.$$

Here B is any ball contained in B_{2n+1}. C does not depend on B, though it does depend on B_{2n+1}.

By Theorem 6.11 in [21], when we consider an individual polyhedral subset of B_{2n+1}, the Poincaré style inequality will hold for all Lipschitz functions, u, and their upper gradients for some $\lambda \geq 1$ and $p = 1$. Since X is chainable, we can form the entire set by gluing together finitely many pieces with sufficient overlap. Theorem 6.15 in [21] says that the glued set satisfies the Poincaré style inequality for all Lipschitz functions as well. In [22], Heinonen and Koskela show that we can replace the condition that u is Lipschitz with the condition that u is measurable. We switch from $\lambda \geq 1$ to $\lambda = 1$ by using Whitney covers; the argument in Section 5.3 of [31] holds in this case. We switch from $p = 1$ to $p = 2$ by Hölder's inequality.

Note that Theorem 4.1 in Shanmugalingam [34] states that Lipschitz functions are dense in $N^{1,2}(B_{2n+1})$, the space of functions and upper gradients, (u, ρ), which have finite norm: $||u||_2 + ||\rho||_2 < \infty$. All gradients are also upper gradients; in particular, $W^{1,2}(B_{2n+1}) \subset N^{1,2}(B_{2n+1})$, and for $u \in W^{1,2}(B_{2n+1})$, the norms coincide. So since $\mathrm{Lip}(B_{2n+1}) \subset W^{1,2}(B_{2n+1})$, we also have density in the $W^{1,2}(B_{2n+1})$ norm. □

Remark 1.19. In the next section, we prove a Poincaré inequality for functions in $W^{1,2}(B)$. This inequality can be used in the proof above to show that $\mathcal{C}_0^{\mathrm{Lip}}(X)$ is dense in $W^{1,2}(X)$. This important and nontrivial density result can be obtained in two rather different ways. One is outlined above and requires a local Poincaré inequality to be valid for functions in $W^{1,2}(X)$. See also [18, 19]. The idea that the (local) volume doubling and Poincaré inequality properties imply the density of Lipschitz functions with compact support in the $W^{1,2}$-norm is useful and important, for instance, in works concerning analysis on domains in $\mathbb{R}^n$ with rough boundary.

Another more specific approach is to show that:

(a) Small neighborhoods of faces of dimension at most $n-2$ can be disregarded because they have small capacity.
(b) Any function $f \in W^{1,2}(X)$ that vanishes in a neighborhood of the faces of dimension at most $n-2$ can be approximated in $W^{1,2}(X)$ by continuous functions that are smooth with bounded derivatives of all order in each open n-face.

The second part of this line of reasoning requires a specific construction (see, e.g.,[6]).

The Dirichlet form $(E, \mathrm{Dom}(E))$ on $L^2(X)$ uniquely determines a positive self-adjoint operator $(\Delta, \mathrm{Dom}(\Delta))$ on $L^2(X)$. Namely, $\mathrm{Dom}(\Delta)$ is defined as the subspace of $\mathrm{Dom}(E)$ of those functions v with the property that there is a constant C such that $E(u, v) \le C\|u\|_2$ for all $u \in \mathrm{Dom}(E)$. This implies that there is a function $w \in L^2(X)$ such that $E(u, v) = \int_X u\, w\, d\mu$ and, by definition, $\Delta v = w$. See, e.g., Fukushima, Ōshima, and Takeda [16]. This sign convention means that when X is the real line, $\Delta f = -f''$.

It is perhaps useful to emphasize that the Laplacian defined above is an operator that is rather mysterious.

Definition 1.20. Let $D_0^\infty(X)$ be the set of all continuous functions with compact support on X such that the restriction to any open n-face is smooth with bounded derivatives of all order. Let $D = D_0^\infty(X) \cap \mathrm{Dom}(\Delta)$.

Note that the space $D_0^\infty(X)$ itself is not contained in $\mathrm{Dom}(\Delta)$. On

$$D = D_0^\infty(X) \cap \mathrm{Dom}(\Delta),$$

Δf is given on each open face by the usual formula in local Euclidean coordinates. However, whether or not the symmetric operator $(\Delta|_D, D)$ is essentially self-adjoint on $L^2(X)$ is not known. Nor is it known that the closure of $(\Delta|_D, D)$ is $(\Delta, \mathrm{Dom}(\Delta))$. These real difficulties are easily overlooked. The set $D = D_0^\infty \cap \mathrm{Dom}(\Delta)$ is easy to describe. For any maximal polyhedron P, let $\vec{n}_P$ be the outward pointing normal unit vector along the boundary of P.

Proposition 1.21. *A function $f \in D_0^\infty(X)$ belongs to* $\mathrm{Dom}(\Delta)$ *if and only if it satisfies*

$$\sum_{P_i : F \subset P_i} \frac{\partial f|_{P_i}}{\partial \vec{n}_{P_i}} = 0 \text{ along } F$$

for any $n-1$-dimensional face F of any maximal polyhedron in X. The set D is dense in $\mathcal{C}_0(X)$ for the uniform norm $||\cdot||_\infty$ and dense in $L^2(X)$.

In this formula, the $n-1$-dimensional face F is fixed, and the sum is over all maximal polyhedra P_i that contain that face (by our assumption, this is a finite sum). The condition is that, along any fixed $n-1$-dimensional face F, the sum of the outward normal derivatives of the restrictions of f to the maximal polyhedra meeting along F is zero.

Proof. Because of Theorem 1.17, this easily follows from using the definition of the Laplacian and Green's formula on each maximal polyhedron. The fact that D is dense in $\mathcal{C}_0(X)$ easily follows from the fact that, for any compact set K and for any fixed small scale, one can construct partitions of unity covering K, (ω_n), $(\sum_n \omega_n)|_K \equiv 1$, whose elementary blocks ω_n are in D with each ω_n supported in a ball of radius ϵ. See [6] for details that easily generalize to the present situation. Density in $L^2(X)$ follows. □

Remark 1.22. Note that this set-up will define a different Laplacian on each of the k-skeleta. To define E^r on a k-skeleton, $X^{(k)}$, set $N = k$, integrate over $X^{(k)}$, and let μ be a k-dimensional measure. This technique will define Δ_k on a dense subset of $L^2(X^{(k)})$.

2. Poincaré inequalities

Definition 2.1. We say that f satisfies a weak local p-Poincaré inequality if there exist constants R_0, κ, and P_0 such that

$$\|f - f_{B(x,r)}\|_{p,B(x,r)} \le P_0 r \|\nabla f\|_{p,B(x,\kappa r)}$$

holds for all $r \le R_0$, where $f_{B(x,r)}$ is the average of f over $B(x,r)$. If $\kappa = 1$, we say that it is strong. If additionally it holds for all $x \in X$ we say that it is uniform. If $R_0 = \infty$, we say that the inequality is global.

We will show that a uniform local Poincaré inequality holds for complexes satisfying the geometric assumptions 1.10. Local Poincaré inequalities have appeared in [21], [42] and [13] for finite complexes or for compact subsets of complexes. In White's article [42], a global Poincaré inequality was shown for Lipschitz functions on an admissible complex made up of a finite number of polyhedra. The constant in White's proof is linear in the number of polyhedra involved, and so it does not extend to an infinite complex. A uniform weak local inequality for Lipschitz functions was also shown on this finite complex. This too differs from our inequality in its dependence on a finite complex.

In Eells and Fuglede's book [13], they show that for any relatively compact subset of an admissible complex, a local Poincaré inequality will hold for locally Lipschitz functions with a constant that depends on the particular choice of compact subset. The larger complex itself can be infinite, but the constant in the inequality depends on the particular choice of compact subset.

In Heinonen and Koskela's article [21], Poincaré inequalities with respect to the L^q norm are shown for Lipschitz functions with upper gradients on finite simplicial complexes of pure dimension q with the property that the link of each vertex is connected. Their approach uses Loewner spaces.

Under the assumptions 1.10 on the geometry of X, we will show there are constants $R_0, P_0 \in (0,\infty)$ such that, for any ball $B = B(z,r)$, $r < R_0$, and any $f \in W^{1,p}(B)$,

$$\|f - f_B\|_{p,B} \leq pP_0 r \|\nabla f\|_{p,B}.$$

The constants R_0 and P_0 are constants depending on the space X. Ultimately, for any fixed $R_0 < \infty$, there exists a P_0 such that the Poincaré inequality above holds.

We begin by proving a local Poincaré inequality for an admissible Euclidean polyhedral complex. The Poincaré inequality on a convex subset of Euclidean space is a well-known statement. We will show it first in a convex space, and then we will generalize it to our locally nonconvex space.

Notation 2.2. We write the average integral of f over a set A by

$$f_A = \fint_A f dx.$$

Lemma 2.3. *Let Ω be a connected convex set with Euclidean distance and structure and Ω_1, Ω_2 be n-dimensional convex subsets of Ω, $n = \dim(\Omega)$. For $f \in W^{1,1}(\Omega)$, the following holds*:

$$\int_{\Omega_2}\int_{\Omega_1} |f(z) - f(y)| dz dy \leq 2^{n-1}\frac{\mathrm{diam}(\Omega)}{n}(\mu(\Omega_1) + \mu(\Omega_2))\int_{\Omega} |\nabla f(y)| dy.$$

Proof. The type of argument used here is classical and can be found in many books including *Aspects of Sobolev-type inequalities* [31]. Details are included for completeness.

Let γ be a path from z to y. The definition of a gradient gives us:

$$|f(z) - f(y)| \leq \int_\gamma |\nabla f(s)| ds.$$

Note that if we are in a one-dimensional space, a convex subset is a line. The desired inequality follows from expanding γ to Ω, and then noting that integrating over x and y has the effect of multiplying the right-hand side by $\mu(\Omega_1)\mu(\Omega_2) \leq \operatorname{diam}(\Omega)(\mu(\Omega_1)+\mu(\Omega_2))$.

Because z and y are in the same convex region Ω with Euclidean distance, we can let the path γ be a straight line:

$$|f(z)-f(y)| \leq \int_0^{|y-z|} \left|\nabla f\left(z+\rho\frac{y-z}{|y-z|}\right)\right| d\rho.$$

We integrate this over $z \in \Omega_1, y \in \Omega_2$. To get a nice bound, we will use a trick from Korevaar and Schoen [25]. We split the path into two halves. For each half, we switch into and out of polar coordinates in a way that avoids integrating $\frac{1}{s}$ near $s=0$. This allows us to have a bound which depends on the volumes of Ω_1 and Ω_2 rather than Ω.

First, we consider the half of the path which is closer to $y \in \Omega_2$. $I_\Omega(\cdot)$ is the indicator function for Ω.

$$\int_{\Omega_1}\int_{\Omega_2}\int_{\frac{|y-z|}{2}}^{|y-z|} \left|\nabla f\left(z+\rho\frac{y-z}{|y-z|}\right)\right| I_\Omega\left(z+\rho\frac{y-z}{|y-z|}\right) d\rho dy dz.$$

We use a change of variable so that $y-z=s\theta$. That is, $|y-z|=s$ and $\frac{y-z}{|y-z|}=\theta$. Note that $\operatorname{diam}(\Omega)$ is an upper bound on the distance between y and z.

$$\cdots = \int_{\Omega_1}\int_{S^{n-1}}\int_0^{\operatorname{diam}(\Omega)}\int_{s/2}^{s} |\nabla f(z+\rho\theta)| I_\Omega(z+\rho\theta) s^{n-1} d\rho ds d\theta dz.$$

We switch the order of integration. Now, ρ will be between 0 and $\operatorname{diam}(\Omega)$ and s will be between ρ and $\min(2\rho, \operatorname{diam}(\Omega))$. This allows us to integrate with respect to s.

$$\begin{aligned}
&\cdots \\
&= \int_{\Omega_1}\int_{S^{n-1}}\int_0^{\operatorname{diam}(\Omega)}\int_\rho^{\min(2\rho,\operatorname{diam}(\Omega))} |\nabla f(z+\rho\theta)| I_\Omega(z+\rho\theta) s^{n-1} ds d\rho d\theta dz \\
&= \int_{\Omega_1}\int_{S^{n-1}}\int_0^{\operatorname{diam}(\Omega)} |\nabla f(z+\rho\theta)| I_\Omega(z+\rho\theta) \\
&\qquad\qquad \cdot \frac{(\min(2\rho,\operatorname{diam}(\Omega)))^n - \rho^n}{n} d\rho d\theta dz.
\end{aligned}$$

Now we reverse the change of variables to set $y=z+\rho\theta$. Since our integral includes an indicator function at $z+\rho\theta$, we have $y \in \Omega$.

$$\cdots = \int_{\Omega_1}\int_\Omega |\nabla f(y)| \frac{(\min(2|y-z|,\operatorname{diam}(\Omega)))^n - |y-z|^n}{n|y-z|^{n-1}} dy dz.$$

One can show $\frac{(\min(2|y-z|,\operatorname{diam}(\Omega)))^n-|y-z|^n}{n|y-z|^{n-1}} \leq 2^{n-1}\frac{\operatorname{diam}(\Omega)}{n}$. This will remove the z dependence in the integral.

$$\cdots \leq 2^{n-1}\frac{\operatorname{diam}(\Omega)}{n}\mu(\Omega_1)\int_\Omega |\nabla f(y)| dy.$$

FIGURE 4. Complex with shaded ball B (left); the two wedges for B and a region which overlaps both of them (right).

We can apply the same argument to the half of the geodesic closest to $z \in \Omega_1$, after first substituting $\rho' = |y-z| - \rho$ to obtain a similar bound. Combining these with the original inequality, we have

$$\int_{\Omega_2}\int_{\Omega_1} |f(z) - f(y)| dz dy \\ \leq 2^{n-1}\frac{\operatorname{diam}(\Omega)}{n}(\mu(\Omega_1) + \mu(\Omega_2)) \int_{\Omega} |\nabla f(y)| dy. \qquad \square$$

To show a local Poincaré inequality on X, we will split the balls, which are not necessarily convex, up into smaller overlapping convex pieces. We will do this using the wedges. Because X is admissible, we can use a chaining argument in order to move through B from one of the W_k to another. We will say W_k and W_j are adjacent if they share an $n-1$-dimensional face, and we let $N(j)$ be the list of indices of faces adjacent to W_j including j. In order to create paths which we can integrate over, we need an overlapping region between the adjacent faces. For $k \in N(j)$, let $W_{k,j} = W_{j,k}$ be the largest subset of $W_k \cup W_j$ which has the property that $W_k \cup W_{k,j}$ and $W_j \cup W_{k,j}$ are both convex. Then, for each x in $W_{k,j}$ there is a way of describing the rays between x and W_k in a distance preserving manner as one would have in $\mathbb{R}^n$.

Example 2.4. In Figure 4 we have a complex and ball with two adjacent wedges. The union of the wedges, W_1 and W_2, is not convex, so we form the region $W_{1,2}$. In this example, both $W_1 \cup W_{1,2}$ and $W_2 \cup W_{1,2}$ are half circles.

Theorem 2.5. *Let X be a Euclidean polyhedral complex satisfying the geometric assumptions 1.10. For each $x_0 \in X$ and $0 < r < R(x_0)$ let $B = B(x_0, r)$, and let its corresponding wedges be labeled $W_{i,j}$. The following holds for $f \in W^{1,2}(X) \cap L^1(B)$:*

$$||f - f_B||_{1,B} \leq 2M \max_{k,j\in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2\right) \frac{2^n r(\mu(W_k) + \mu(W_{j,k}))}{n\mu(W_{j,k})} ||\nabla f||_{1,B}.$$

Here $R(x_0) = d(x_0, X^{(D-1)})$, where D satisfies $x_0 \in X^{(D)} - X^{(D-1)}$. When $x_0 \in X^{(0)}$, $R(x_0) = \inf_{v\in X^{(0)}, v\neq x_0} d(x_0, v)$.

Proof. Note that when x_0 is located on a D, but not $D-1$, dimensional face the restriction $0 < r < R(x_0)$ forces $B(x_0, r)$ to avoid all faces of dimension $D-1$ and lower.

For x in B we have by Jensen:

$$|f(x) - f_B| \leq \frac{1}{\mu(B)} \int_B |f(x) - f(y)| dy.$$

We would like to apply Lemma 2.3 to this; however, B is not necessarily convex. We will construct a path from x to y using a finite number of straight lines, where each of the line segments is contained in a convex region. For simplicity, we consider $x \in W_i$ and $y \in W_k$. X is locally $(n-1)$-chainable, so there is a chain in $B - \{x_0\}$ starting at W_i and ending at W_k indexed by the sequence $\sigma(1) = i, \ldots, \sigma(m) = k$, so that for each j, $W_{\sigma(j)}$ and $W_{\sigma(j+1)}$ are adjacent, and none of the indices repeat. The path alternates between wedges and overlapping regions, moving from a point in $W_{\sigma(j)}$ into a connecting point in $W_{\sigma(j),\sigma(j+1)}$, and then from that connecting point in $W_{\sigma(j),\sigma(j+1)}$ into a point in $W_{\sigma(j+1)}$. We can take points in these regions: $z_1 \in W_{\sigma(1)}, z_2 \in W_{\sigma(1),\sigma(2)}, \ldots, z_{2j-1} \in W_{\sigma(j)}$ and $z_{2j} \in W_{\sigma(j),\sigma(j+1)}$. Note that each pair in this sequence is located in a convex region: either

$$W_{\sigma(j)} \cup W_{\sigma(j),\sigma(j+1)} \quad \text{or} \quad W_{\sigma(j+1)} \cup W_{\sigma(j),\sigma(j+1)}.$$

The line segments between these points define our path γ from x to y.

$$\begin{aligned} |f(x) - f(y)| \leq |f(x) - f(z_1)| & \\ & + \sum_{j=1}^{l-1} \left(|f(z_{2j}) - f(z_{2j-1})| + |f(z_{2j}) - f(z_{2j+1})| \right) \\ & + |f(z_{2l}) - f(y)|. \end{aligned}$$

Since this holds for any z_i in its corresponding wedge, we can average the pieces over all of the possible z's.

$$\begin{aligned} |f(x) - f(y)| \leq & \fint_{W_{i,\sigma(1)}} |f(x) - f(z_1)| dz_1 \\ & + \sum_{j=1}^{l-1} \Bigg(\fint_{W_{\sigma(j)}} \fint_{W_{\sigma(j),\sigma(j+1)}} |f(z_{2j}) - f(z_{2j-1})| dz_{2j} dz_{2j-1} \\ & + \fint_{W_{\sigma(j+1)}} \fint_{W_{\sigma(j),\sigma(j+1)}} |f(z_{2j}) - f(z_{2j+1})| dz_{2j} dz_{2j+1} \Bigg) \\ & + \fint_{W_{\sigma(l),k}} |f(z_{2l}) - f(y)| dz_{2l}. \end{aligned}$$

We will not keep track of the exact path between every pair of regions, although in specific examples one may want to do that in order to achieve a tighter bound. Rather, we integrate over all pairs of neighboring wedges.

$$\dots \le \sum_{l\in N(i)} \fint_{W_{i,l}} |f(x)-f(z)|dz + \sum_j \sum_{l\neq i, l\in N(j)} \fint_{W_l}\fint_{W_{j,l}} |f(z)-f(w)|dzdw + \sum_{j\in N(k)} \fint_{W_{j,k}} |f(z)-f(y)|dz.$$

This new inequality will hold for x and y in any pair of W_i and W_k with $k\neq i$. If we expand our notation so that $W_{i,i} = W_i$, then this will hold when x and y are in the same set $W_k = W_i$. To integrate over all $y \in B$, we can split the integral into two parts; one where x and y are both in W_i and the second where y is in one of the $W_k \neq W_i$. Similarly, we can integrate over x in W_i and then sum over i.

$$\begin{aligned}\frac{1}{\mu(B)}\int_B\int_B |f(x)-f(y)|dydx \\ \le \frac{1}{\mu(B)}\Bigg(&\sum_{i,k}\sum_{l\in N(i)}\int_{W_i}\int_{W_k}\fint_{W_{i,l}} |f(x)-f(z)|dzdydx \\ &+\sum_{i,k,j}\sum_{l\in N(j)}\int_{W_i}\int_{W_k}\fint_{W_{j,l}}\fint_{W_l} |f(z)-f(w)|dwdzdydx \\ &+\sum_{i,k}\sum_{j\in N(k)}\int_{W_i}\int_{W_k}\fint_{W_{j,k}} |f(z)-f(y)|dzdydx\Bigg).\end{aligned}$$

We first integrate to reduce these to double integrals. We then combine them into one double sum by setting $x = w$ and $y = w$ as well as reindexing so that $i = j$ and $l = k$.

$$\dots \le \sum_k \sum_{j\in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2\right)\int_{W_k}\fint_{W_{j,k}} |f(z)-f(w)|dzdw.$$

Applying Lemma 2.3 with $\Omega = W_k \cup W_{j,k}$, $\Omega_1 = W_{j,k}$, $\Omega_2 = W_k$, and $\operatorname{diam}(\Omega) \le 2r$ to each of the pieces we find:

$$\dots \le \sum_k\sum_{j\in N(k)}\left(\frac{\mu(B)}{\mu(W_k)}+2\right)\frac{2^n r(\mu(W_k)+\mu(W_{j,k}))}{n\mu(W_{j,k})}\int_{W_k\cup W_{j,k}} |\nabla f(y)|dy.$$

Note that points in the sets $W_k \cup W_{j,k}$ are counted at most $2M$ times, since each of the W_k has at most M neighbors. This allows us to combine the sums to find:

$$\begin{aligned}&\frac{1}{\mu(B)}\int_B\int_B |f(x)-f(y)|dydx \\ &\le 2M \max_{k,j\in N(k)}\left(\frac{\mu(B)}{\mu(W_k)}+2\right)\frac{2^n r(\mu(W_k)+\mu(W_{j,k}))}{n\mu(W_{j,k})}\int_B |\nabla f(y)|dy.\end{aligned}$$

This is the desired result. □

We use the uniform bound on the solid angles formed to simplify the constant in Theorem 2.5.

Corollary 2.6. *Let X be a Euclidean polyhedral complex satisfying the geometric assumptions 1.10. For each $f \in W^{1,2}(X) \cap L^1(B)$, $z \in X$, and $0 < r < R(z)$ we have*

$$||f - f_B||_{1,B} \leq C_X r ||\nabla f||_{1,B}$$

where $B = B(z, r)$ and the constant $C_X = \frac{2^{3n+3} M^2}{\alpha n}$ depends only on the space X. Here $R(z) = d(z, X^{(D-1)})$, where D is the dimension such that $z \in X^{(D)} - X^{(D-1)}$. When $z \in X^{(0)}$, $R(z) = \inf_{v \in X^{(0)}, v \neq z} d(z, v)$.

Proof. We need to bound $\max_{k,j \in N(k)} \left(\frac{\mu(B)}{\mu(W_k)} + 2 \right) \frac{\mu(W_k) + \mu(W_{j,k})}{\mu(W_{j,k})}$ from Theorem 2.5. Since we want to bound $\mu(W_{j,k})$, we need a way to compare it to $\mu(W_k)$. We can subdivide the space initially by cutting each piece in half in each of the n dimensions, so that there are at most $M' = 2^n M$ pieces. When W'_k and W'_j are adjacent, this tells us that $W'_{j,k}$ has a volume which is larger than $\min(\mu(W'_j), \mu(W'_k))$. Thus $\frac{\mu(W'_k) + \mu(W'_{j,k})}{\mu(W'_{j,k})} \leq 2$.

To bound $\frac{\mu(B)}{\mu(W_k)}$ we will need the solid angle bound. Combining the solid angle bound with the factor of 2^{-n} decrease in wedge size gives us the modified inequality:

$$\mu(W'_k) \leq 2^{-n} \mu(r^n S^{(n-1)}) \leq \frac{\mu(W'_k)}{\alpha}.$$

Summing the left-hand side of the inequality over k tells us that

$$\mu(B) \leq M 2^n 2^{-n} \mu(r^n S^{(n-1)}).$$

If we multiply the right-hand side of the inequality by $M 2^n$, we have

$$M \mu(r^n S^{(n-1)}) \leq \frac{M 2^n \mu(W'_k)}{\alpha}.$$

Combining these two inequalities, we find that:

$$\frac{\mu(B)}{\mu(W'_k)} \leq \frac{M 2^n}{\alpha}.$$

We can substitute these into our constant to get:

$$2M' \max_{k,j \in N(k)} \left(\frac{\mu(B)}{\mu(W'_k)} + 2 \right) \frac{2^n r (\mu(W'_k) + \mu(W'_{j,k}))}{n \mu(W'_{j,k})} \leq 2M 2^n \left(\frac{M 2^n}{\alpha} + 2 \right) \frac{2^{n+1} r}{n}.$$

This combined with Theorem 2.5 gives us:

$$||f - f_B||_{1,B} \leq \frac{2^{3n+3}M^2}{\alpha n} r||\nabla f||_{1,B}. \qquad \square$$

We would like to extend these theorems so that the radius is not dependent on the center of the ball. To do so, we show a weak Poincaré inequality for the Euclidean metric where the bound on the radius is independent of the ball's center. We then transfer it to the Riemannian metric and extend it to a stronger version.

Theorem 2.7. *Let X be a Euclidean polyhedral complex satisfying the geometric assumptions 1.10. The following inequality holds for each $z \in X$, $0 < r < R_0$, and and $f \in W^{1,2}(X) \cap L^1(B)$:*

$$\int_{B(z,r)} |f(x) - f_{B(z,r)}|dx \leq rC_W \int_{B(z,\kappa r)} |\nabla f(x)|dx.$$

Here $C_W = \frac{2^{3n+3}M^3\kappa C_D^{\log_2(\kappa)}}{\alpha n}$, $\kappa = 6\left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right)^n$, C_D is the volume doubling constant for balls of radius less than or equal to ℓ , and $R_0 = \frac{\ell}{\kappa}$.

Proof. Let $z \in X$ and $r \leq \ell 6^{-1}\left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right)^{-n}$. If $d(z, X^{(n-1)}) > r$, then the result follows as a weaker version of Corollary 2.6. Otherwise, we need to find a point v_k which has the property that it is on a k-skeleton, and there are no other faces in the k-skeleton intersected by $B(v_k, d(v_k, z) + r)$. We will do this by descending down the skeleta.

If there is a point within r of z with this property, we will use it.

If not, set $r_0 = 3r$. Then there is a k such that the lowest dimensional skeleton intersected by $B(z, r_0)$ is $X^{(k)}$, and $X^{(k)}$ is intersected by $B(z, r_0)$ in at least two points, v_k and w_k, on two different faces. If these faces did not intersect, then they are at least distance ℓ from one another. This would imply that $\ell \leq 2r_0 = 6r$, a contradiction. Thus those two faces intersect in a smaller j-dimensional face. Call v_j the point on the j-dimensional face which minimizes $\min(d(v_j, v_k), d(v_j, w_k))$. These three points form a triangle with angle $v_kv_jw_k \geq \alpha$, where α is the smallest interior angle in X. Note that this angle is bounded by the assumption $\alpha \leq \frac{\mu(W_k)}{\mu(r^nS^{n-1})}$. The triangle that maximizes $\min(d(v_j, v_k), d(v_j, w_k))$, the minimum distance to this new point, is an isosceles one with angle $v_kv_jw_k = \alpha$. The law of cosines tells us that for the isosceles triangle, $d(v_k, w_k)^2 = 2d(v_k, v_j)^2(1 - \cos(\alpha))$, and so for a general triangle, $d(v_k, v_j) \leq \frac{d(v_k,w_k)}{\sqrt{2(1-\cos(\alpha))}} \leq \frac{2r_0}{\sqrt{2(1-\cos(\alpha))}}$.

If this v_j works, we are done. Otherwise, we have at least two points, v_j and w_j within $\left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right) r_0$ of z. We repeat the process by taking new r's of the form $r_{i+1} = \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right) r_i$ and finding a point on a lower-dimensional skeleton. The worst case scenario will have us repeat this n times until we are left with at least one point on the lowest-dimensional skeleton that intersects $B(z, R)$. The largest radius that we could require is $R = \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right)^n 3r$. Using this R, we can show that $B(v, R)$ does not intersect two such faces. The condition $r \leq \ell 6^{-1} \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right)^{-n}$ tells us that $R \leq \frac{1}{2}\ell$. As two nonintersecting faces cannot be closer than the closest pair, $B(v, R)$ contains at most one.

This construction gives us a center, v, on a k-dimensional face, and a radius, $R \leq \left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right)^n 3r$ so that $B(v, R)$ intersects only the k-dimensional face that v is on. This allows us first to recenter our ball around v and then to apply Corollary 2.6 to f on $B(v, R)$. Then, as

$$\kappa = 6\left(\frac{2}{\sqrt{2(1-\cos(\alpha))}} + 1\right)^n,$$

we find $B(v, R) \subset B(z, \kappa r)$.

$$\begin{aligned}
&\int_{B(z,r)} |f(x) - f_{B(z,r)}| dx \\
&\qquad \leq \frac{1}{\mu(B(z,r))} \int_{B(z,r)} \int_{B(z,r)} |f(x) - f(y)| dx dy \\
&\qquad \leq \frac{\mu(B(v,R))}{\mu(B(z,r))} \fint_{B(v,R)} \int_{B(v,R)} |f(x) - f(y)| dx dy \\
&\qquad \leq \frac{\mu(B(v,R))}{\mu(B(z,r))} \frac{2^{3n+3} M^2}{\alpha n} R \int_{B(v,R)} |\nabla f(x)| dx \\
&\qquad \leq M C_D^{\log_2(\kappa)} \frac{2^{3n+3} M^2}{\alpha n} \kappa r \int_{B(z,\kappa r)} |\nabla f(x)| dx. \qquad \square
\end{aligned}$$

We can extend the Euclidean case to the Riemannian one via the ellipticity bounds using metric comparison arguments similar to those in Theorem 5.1 in [13]. Recall that the subscript $_e$ refers to Euclidean objects and $_g$ refers to Riemannian ones.

Theorem 2.8. *Let X be an admissible n-dimensional Riemannian polyhedral complex with ellipticity bound Λ, Riemannian distance between nonadjacent faces bounded below by $\Lambda\ell$, and Euclidean volume bound*

$$\mu_e(B_e(z, \Lambda r)) \leq M \Lambda^{2n} \mu_e(B_e(z, r/\Lambda))$$

for $r < R_0$. Suppose f satisfies the following weak Euclidean Poincaré inequality for any $r < R_0$:

$$||f - f_{B_e(z,r)}||_{1,B_e(z,r)} \leq rC_W||\nabla_e f||_{1,B_e(z,\kappa r)}.$$

Then f also satisfies the following weak Riemannian Poincaré inequality for any $r < \frac{R_0}{\Lambda}$:

$$||f - f_{B_g(z,r)}||_{1,B_g(z,r)} \leq rC'_W||\nabla_g f||_{1,B_g(z,\kappa' r)}.$$

In the second inequality the norms are with respect to the Riemannian metric, $C'_W = \Lambda^{6n+2}MC_W$, and $\kappa' = \Lambda^2\kappa$.

Proof. We use the comparisons $B_g(r) \subset B_e(\Lambda r)$ and $d\mu_g(x) \leq \Lambda^n dx$ to compare the Riemannian lefthand side with a Euclidean version:

$$\begin{aligned}||f-&f_{B_g(z,r)}||_{1,B_g(z,r)}\\ &\leq \frac{1}{\mu_g(B_g(z,r))}\int_{B_g(z,r)}\int_{B_g(z,r)}|f(x)-f(y)|d\mu_g(x)d\mu_g(y)\\ &\leq \frac{\Lambda^{2n}}{\mu_g(B_g(z,r))}\int_{B_e(z,\Lambda r)}\int_{B_e(z,\Lambda r)}|f(x)-f(y)|dxdy.\end{aligned}$$

We apply the Euclidean weak Poincaré inequality from Theorem 2.7 to get:

$$\begin{aligned}\frac{1}{\mu_e(B_e(z,\Lambda r))}\int_{B_e(z,\Lambda r)}\int_{B_e(z,\Lambda r)}&|f(x)-f(y)|dxdy\\ &\leq \Lambda r C_W\int_{B_e(z,\kappa\Lambda r)}|\nabla_e f(x)|dx.\end{aligned}$$

For the right-hand side, we can use the comparisons $B_e(\kappa\Lambda r) \subset B_g(\kappa\Lambda^2 r)$, $dx \leq \Lambda^n d\mu_g(x)$, and $|\nabla_e f| \leq \Lambda|\nabla_g f|$:

$$\int_{B_e(z,\kappa\Lambda r)}|\nabla_e f(x)|dx \leq \Lambda^{n+1}\int_{B_g(z,\kappa\Lambda^2 r)}|\nabla_g f(x)|d\mu_g(x).$$

We can chain these inequalities together to get:

$$||f - f_{B_g(z,r)}||_{1,B_g(z,r)} \leq \frac{\Lambda^{2n}\mu_e(B_e(z,\Lambda r))}{\mu_g(B_g(z,r))}\Lambda rC_W\Lambda^{n+1}||\nabla f||_{1,B_g(z,\kappa\Lambda^2 r)}.$$

We can use volume doubling to simplify the constant. We know:

$$\mu_e(B_e(z,\Lambda r)) \leq M\Lambda^{2n}\mu_e(B_e(z,r/\Lambda)) \leq M\Lambda^{3n}\mu_g(B_g(z,r)).$$

This gives us:

$$||f - f_{B_g(z,r)}||_{1,B_g(z,r)} \leq rM\Lambda^{6n+2}C_W||\nabla f_g||_{1,B_g(z,\kappa\Lambda^2 r)}. \qquad \square$$

Corollary 2.9. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions 1.10. Let $f \in W^{1,p}(B)$. For all $0 < r < R_0$ and $p \in [1,\infty)$, the strong p-Poincaré inequality holds*:

$$||f - f_{B_g(z,r)}||_{p,B_g(z,r)} \leq rC'_W pC_D||\nabla f||_{p,B_g(z,r)}.$$

Here C_D is a constant which depends only on the volume doubling constant for balls of radius less than $\ell\Lambda$, $R_0 = \ell 6^{-1}\Lambda^{-1}\left(\frac{2}{\sqrt{2(1-\cos(\alpha))}}+1\right)^{-n}$, and C'_W is defined as in Theorem 2.8*.*

Proof. Apply Theorem 2.8 to Theorem 2.7 to get a weak Riemannian 1-Poincaré inequality. Then apply Theorem 2.4 from [37] to get a strong uniform Riemannian 1-Poincaré inequality. Note that the theorem from [37] involves Poincaré inequalities with 2-norms, changing a weak inequality where the right-hand side is integrated over a ball of radius $2r$ into a strong one; however, the proof generalizes to 1-norms and balls of radius Cr for some constant C. The 1-Poincaré inequality then becomes a p-Poincaré inequality by an application of Hölder's inequality; a calculation gives the constants. Note that the function space $W^{1,p}(B)$ is allowed due to the use of the Whitney cover. □

Corollary 2.10. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions* 1.10*. For any R'_0 there exists a constant $C'_W p C'_D$ so that for all $r < R'_0$ and $p \in [1, \infty)$, the strong p-Poincaré inequality holds for all $f \in W^{1,p}(B)$:*

$$||f - f_{B_g(z,r)}||_{p,B_g(z,r)} \leq r C'_W p C'_D ||\nabla f||_{p,B_g(z,r)}.$$

Here C'_D is a constant depending only on the volume doubling constant for balls of radius less than $6\Lambda^2 R'_0\left(\frac{2}{\sqrt{2(1-\cos(\alpha))}}+1\right)^{n}$, and C'_W is defined as in Theorem 2.8*.*

The proof involves successive uses of Whitney covers to expand the radius for a $p = 1$ inequality. This gives the dependence on the local volume doubling constant. This expands to a p-inequality using Hölder. For full details, see [30].

3. Applications

3.1. Heat kernel bounds. The heat kernel, $h_t(x, y)$, is the kernel of the semigroup $e^{-t\Delta}$ as well as the fundamental solution to the heat equation

$$\partial_t u = -\Delta u.$$

Note that our formulation does not have a factor $\frac{1}{2}$, which appears in the probability literature. We include a minus sign as our Laplacian is defined as a nonnegative operator.

In the present context it is important to state precisely what is meant by the solution of the Poisson equation $\Delta u = f$ in an open set Ω or of the heat equation $(\partial_t + \Delta)u = 0$ in $(a, b) \times \Omega$. Observe indeed that there is no established notion of a classical solution in this context. The most useful notion is probably that of a local weak solution. We refer the reader to

Sturm [37] which gives a detailed account. In the easier case of the Poisson equation, a function u is a weak solution of $\Delta u = f$ in Ω (with f a locally integrable function, say) if:

- u is $W^{1,2}_{\rm loc}(\Omega)$.
- $\int_\Omega \nabla u \cdot \nabla \phi d\mu = \int_\Omega f\phi$ for all $\phi \in W^{1,2}_0(\Omega)$.

The definition of weak solution of the heat equation is somewhat more complicated because it involves hypotheses on u and $\partial_t u$. The main hypothesis on $u(t,x)$ is that it belongs (locally) to the Banach space

$$L^2((a,b) \mapsto W^{1,2}(\Omega)).$$

See Sturm [37]. A simple but crucial fact is that for any $f \in L^2(X)$, the function $u(t,x) = e^{-t\Delta} f(x)$ is a weak solution of the heat equation on $(0,\infty)\times X$. When volume doubling and the Poincaré inequality hold true locally, the function $u(t,x) = h_t(x,y)$, $y \in X$ fixed, is a weak solution of the heat equation on $(0,\infty) \times X$, and the time derivatives $u_k(t,x) = \partial_t^k h_t(x,y)$ are also weak solutions of the heat equation on $(0,\infty) \times X$. In this section, we point out some of the many important consequences of the fact that under the geometric assumptions 1.10, $(X, \mu, E, \mathrm{Dom}(E))$ satisfies the uniform local volume doubling property and the uniform local Poincaré inequality (Corollary 2.10).

Remark 3.1. If $X^{(k)}$, the k-skeleton of X, is connected and X satisfies assumptions 1.10, then $X^{(k)}$ will also satisfy 1.10. In that case, all of the results stated for X in this section also apply to $X^{(k)}$ (of course, the dimension of $X^{(k)}$ is k and so n should be replaced by k in all of the results). This occurs, for example, whenever X is comprised solely of polytopes.

Corollary 3.2. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions 1.10. Then E is conservative. In particular, the following are true*:

- *X is stochastically complete: for all $x \in X$ and $t > 0$,*

$$\int_X h_t(x,y) d\mu(y) = 1.$$

- *For some $\alpha > 0$ every nonnegative solution $u \in L^\infty(X, d\mu)$ of*

$$(\Delta + \alpha)u = 0$$

  is identically zero.
- *For all $\alpha > 0$ every nonnegative subsolution $u \in L^\infty(X, d\mu)$ of*

$$(\Delta + \alpha)u = 0$$

  is identically zero.

Proof. Note that our assumptions on the local geometry of the space give us uniform local volume doubling bounds. We use these bounds, but not any Poincaré inequalities, to show the result. The volume doubling bounds imply via covering arguments (see Lemma 5.2.7 in [31]) that volume grows at most as an exponential function. That is, there exist constants C_0, C_1 so that for all $x \in X$ and all $r \geq 1$ the following inequality holds:

$$\mu(B(x,r)) \leq C_0 e^{C_1(1+r)}.$$

For $r \geq 1$, this gives us

$$\frac{r}{\ln(\mu(B(x,r)))} \geq \frac{r}{\ln(C_0) + C_1(1+r)}$$

which implies

$$\int_1^\infty \frac{r}{\ln(\mu(B(x,r)))} dr = \infty.$$

By Theorem 4 in [35] E is conservative. □

The heat equation is parabolic; one way of obtaining information about it is through parabolic Harnack inequalities. Sturm [37] shows that local volume doubling and Poincaré inequalities on a subset of a complete metric space imply a local parabolic Harnack inequality on that subset. He then uses this to find Gaussian estimates on the heat kernel. The equivalence of the parabolic Harnack inequality with Poincaré and volume doubling had previously been done in the Riemannian manifold case by Grigor'yan [17] and Saloff-Coste [32]. We apply Sturm's estimates to the Riemannian complex case, first showing the local Harnack inequality and then the heat kernel bounds.

Corollary 3.3. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions 1.10. For any $R_0 > 0$ there exists a constant*

$$C_H = C_H(X, R_0)$$

so that for any $x \in X$, $r \in (0, R_0)$, and all $t > 0$ we have:

$$\sup_{(s,y)\in Q^-} u(s,y) \leq C_H \inf_{(s,y)\in Q^+} u(s,y)$$

whenever u is a nonnegative local solution of $(\Delta + \partial_t)u = 0$ on the cylinder $Q = (t - 4r^2, t) \times B(x, 2r)$. Here $Q^- = (t - 3r^2, t - 2r^2) \times B(x, r)$ and $Q^+ = (t - r^2, t) \times B(x, r)$.

Proof. This follows from Theorem 3.5 in [37]. The constant depends only on R_0 (and not the choice of x, $r \in (0, R_0)$) due to the uniform local volume doubling and the uniform local Poincaré inequality in Corollary 2.9. □

The uniform Harnack inequality gives us uniform Hölder continuity for local solutions of the heat equation.

Corollary 3.4. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions 1.10. For $R_0 > 0$ there are constants $C = C(X, R_0)$*

and $\alpha \in (0,1)$ *so that for any* $x \in X$, $T \in (-\infty,\infty)$ *and* $r < R_0$ *we have*:

$$|u(s,y) - u(t,z)| \leq C \sup_Q |u| \left(\frac{|s-t|^{1/2} + |y-z|}{r} \right)^\alpha$$

whenever u *is a local solution of* $(\Delta + \partial_t)u = 0$ *on the cylinder*

$$Q = (t - 4r^2, t) \times B(x, 2r),$$

$s, t \in (T - r^2, T)$, *and* $y, z \in B(x,r)$.

For any $t \in (0,\infty)$, *any* $x, y, z \in X$ *with* $z \in B(y, \min(1,\sqrt{t}))$, *the heat kernel and its time derivatives satisfy* :

$$\left| \partial_t^j h_t(x,y) - \partial_t^j h_t(x,z) \right| \leq C \left(\frac{d(y,z)}{\min(1,\sqrt{t})} \right)^\alpha \frac{1}{\min(1,t)^j} h_{2t}(x,y).$$

Proof. Apply Proposition 3.1 in [37], noting that we have a uniform constant for the Harnack inequality (Corollary 3.3). To obtain the heat kernel estimate, use the first statement, analyticity of the heat kernel with respect to the time variable, and the Harnack inequality for the heat kernel itself. □

Definition 3.5. A nonnegative solution u is minimal if for any solution v with the property $0 \leq v \leq u$, there must exist a constant $\lambda \in [0,1]$ so that $v = \lambda u$.

The Harnack inequality allows us to write global minimal solutions of the heat equation on $(-\infty,\infty) \times X$ in terms of minimal solutions to an elliptic equation.

Corollary 3.6. *Let* X *be a Riemannian polyhedral complex satisfying the geometric assumptions* 1.10. *Let* $u \geq 0$ *be a minimal solution to the heat equation on* $(-\infty, T) \times X$, *where* $T < \infty$. *Then there exists a constant* C *and a minimal solution* f *to the equation* $\Delta f = Cf$ *so that* $u(t,x) = e^{Ct} f(x)$.

Proof. Apply the argument of Theorem 2 in Koranyi and Taylor [26] using the local Harnack inequality from Corollary 3.3. □

Theorem 3.7. *Let* X *be a Riemannian polyhedral complex satisfying the geometric assumptions* 1.10. *For any* $R_0 > 0$ *there is a corresponding constant* $C = C(X, R_0)$ *so that*

$$\frac{1}{Ct^{n/2}} \leq h_t(x,x) \leq \frac{C}{t^{n/2}}.$$

for all $x \in X$ *and all* t *such that* $0 < t < R_0^2$.

Proof. See, e.g, Theorems 4.1 and 4.3 in [37]. The stated bound easily follows from the volume estimate $c_1 r^n \leq \mu_g(B(x,r)) \leq C_1 r^n$, $r \in (0, R_0)$, and from the Harnack inequality. See, e.g., [31, Section 5.4.6]. □

We also have off-diagonal bounds on the heat kernel. To state these in the most precise form, let us introduce the bottom λ_0 of the spectrum of Δ, that is,

$$\lambda_0 = \inf\{E(f,f) : f \in \mathrm{Dom}(E), \|f\|_2 = 1\}.$$

Corollary 3.8. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions 1.10. For any R_0 there exists $C = C(X, R_0)$ and $C_j = C_j(X, R_0)$ so that for any $x, y \in X$ and $t > 0$ we have the following bounds*:

$$h_t(x,y) \le \frac{C}{(\min(t, R_0^2))^{n/2}} e^{-\frac{d^2(x,y)}{4t} - \lambda_0 t} \left(1 + \frac{d^2(x,y)}{t}\right)^{N/2}$$

$$h_t(x,y) \ge \frac{1}{C\mu(B(x, \sqrt{\min(t, R_0^2)}))} e^{-C\frac{d^2(x,y)}{t}} e^{-\frac{Ct}{R_0^2}}$$

$$\left|\partial_t^j h_t(x,y)\right| \le \frac{C_j(1+\lambda_0 t)^{1+N/2+j}}{t^j(\min(t, R_0^2))^{n/2}} e^{-\frac{d^2(x,y)}{4t} - \lambda_0 t} \left(1 + \frac{d^2(x,y)}{t}\right)^{N/2+j}.$$

Here, N depends only on the volume doubling constant for balls of radius R_0.

Proof. To get the upper bound, for each $x, y \in X$, apply Theorem 4.1 in [37] to X, taking $Y = B(x, R_0) \cup B(y, R_0)$. Similarly, for each $x, y \in X$ apply Theorem 4.8 from [37] to X with $Y = B(x, 2R_0)$ to get the lower bound. To prove the derivative inequality, we show that a local Sobolev inequality holds. The hypotheses of Theorem 3.7 allow us to use Corollary 2.9 to find uniform constants for local volume doubling and the local Poincaré inequality. These give a local Sobolev inequality by Theorem 2.6 in [37]. In particular, for any $R_0 > 0$ we have both a volume doubling constant and a Sobolev inequality constant that work for all balls of radius R_0 in X. For the third result, apply Theorem 2.6 from [36] to X. □

One of the most interesting applications of the local Harnack inequality provided by the uniform local volume doubling and local Poincaré inequality is a rather complete description of global positive weak solutions which includes, in particular, a strong statement concerning uniqueness for the positive Cauchy problem. This application goes back to the work of Aronson. The generality of the argument is clearly pointed out in [1]. The statement in the present situation is as follows.

Theorem 3.9. *Let X be a Riemannian polyhedral complex satisfying the geometric assumptions 1.10. There exists a constant C such that any nonnegative weak solution u of the heat equation on $(0,T) \times X$ satisfies*

$$\forall\, x, y \in X,\ 0 < s < t < T, \quad u(s,x) \le u(t,y) \exp\left(C\left(1 + \frac{t}{s} + \frac{d(x,y)^2}{t-s}\right)\right).$$

Moreover, uniqueness of the positive Cauchy problem holds on $(0,T) \times X$ for the heat equation. More precisely, if u is a nonnegative weak solution *of the heat equation on* $(0, T) \times X$*, then there exists a unique nonnegative Borel measure* γ *on* X *and* $a > 0$ *such that*

$$\int_X e^{-ad(x_0,x)^2}\gamma(dx) < \infty$$

for some (equivalently, any) $x_0 \in X$ *and*

$$u(t,x) = \int_X h_t(x,y)\gamma(dy), \quad (t,x) \in (0,T) \times X.$$

In particular, if u *is a nonnegative weak solution of the heat equation on* $(0, T) \times X$ *and*

$$\forall\, f \in \mathcal{C}_0^{\mathrm{Lip}}(X), \quad \lim_{t\to 0} \int_X u(t,\cdot) f d\mu = \int_X u_0 f d\mu$$

for some $u_0 \in L^1_{\mathrm{loc}}(X)$*, then* $u(t,x) = \int_X h_t(x,\cdot)u_0 d\mu$*.*

An important consequence of Corollary 3.4 and Corollary 3.8 is the following result.

Theorem 3.10. *Consider the Laplacian* Δ*, restricted to the set*

$$D = D_0^\infty(X) \cap \mathrm{Dom}(\Delta)$$

(*see* Definition 1.20 *and* Proposition 1.21) *as a densely defined unbounded operator on the Banach space* $\mathcal{C}_0(X)$ *equipped with the uniform norm* $\|\cdot\|_\infty$*. Then* (Δ, D) *is a closable operator, and its closure is the infinitesimal generator of the strongly continuous semigroup of operators on* $\mathcal{C}_0(X)$ *defined by*

$$f \mapsto \int_X h_t(\cdot,y) f(y) d\mu(y).$$

Proof. We already noted that D is dense in $\mathcal{C}_0(X)$ (see Proposition 1.21). It is not hard to verify that (Δ, D) satisfies the positive maximal principle. It follows that, if it is closable, it is the generator of a Feller semigroup (a strongly continuous, positivity preserving, semigroup of contraction on $\mathcal{C}_0(X)$). Closability is rather difficult to prove directly, but Corollary 3.4 and Corollary 3.8 show that the heat semigroup, originally defined as an L^2 Markov semigroup, actually produces a strongly continuous semigroup on $\mathcal{C}_0(X)$. The infinitesimal generator of this semigroup on $\mathcal{C}_0(X)$ is a (closed!) extension of (Δ, D). This proves that (Δ, D) is closable. The uniqueness results in [28, 40] show that the semigroup associated with the closure of (Δ, D) on $\mathcal{C}_0(X)$ must coincide with the heat semigroup already constructed. □

When the complex X both satisfies a *global* Poincaré inequality and has a *global* volume doubling bound, stronger results follow that do not hold true, in general, under the weaker hypotheses considered so far. Examples of complexes satisfying these global assumptions are given in the next section.

Corollary 3.11. *Let X be a Riemannian complex which satisfies both a global 2-Poincaré inequality and volume doubling as well as assumptions 1.10. There exist $N = N(X), C = C(X)$ and $C_j = C_j(X)$ so that for any $x, y \in X$ and $t > 0$ we have:*

$$h_t(x,y) \leq \frac{C}{\mu(B(x,\sqrt{t}))} e^{-\frac{d^2(x,y)}{4t}} \left(1 + \frac{d^2(x,y)}{t}\right)^{N/2}$$

$$h_t(x,y) \geq \frac{1}{C\mu(B(x,\sqrt{t}))} e^{-C\frac{d^2(x,y)}{t}}$$

$$\left|\partial_t^j h_t(x,y)\right| \leq \frac{C_j}{\mu(B(x,\sqrt{t}))} t^{-j} e^{-\frac{d^2(x,y)}{4t}} \left(1 + \frac{d^2(x,y)}{t}\right)^{N/2+j}.$$

Proof. The heat kernel bounds follow directly from Corollaries 4.2 and 4.10 in [37]. The derivative bound follows from Corollary 2.7 in [36]. □

Corollary 3.12. *Let X be a volume doubling Riemannian complex satisfying a global 2-Poincaré inequality as well as assumptions 1.10. When we fix an arbitrary point $x \in X$ we find $(E, \mathrm{Dom}(E))$ is recurrent if and only if*

$$\int_1^\infty \frac{r dr}{\mu(B(x,r))} = \infty.$$

In the case where $(E, \mathrm{Dom}(E))$ is transient we have the following estimate for the Green's function :

$$c\int_{d(x,y)}^\infty \frac{r dr}{\mu(B(x,r))} \leq \int_0^\infty h_t(x,y)dt \leq C\int_{d(x,y)}^\infty \frac{r dr}{\mu(B(x,r))}$$

where c, C are constants depending on X.

Proof. The Poincaré inequality, volume doubling, and the fact that E is irreducible allow us to apply Corollary 2.9 in [36]. The Poincaré inequality and volume doubling allow us to apply Corollary 4.11 in [37] to get the bounds on the Green's function. □

3.2. Groups with polynomial growth. A collection of examples can be found by considering metric spaces, X, which are acted upon by a finitely generated group of isometries, G. If $X/G = Y$ and Y can be expressed as a finite Riemannian complex, then X is a Riemannian complex as well. Note that the k-skeleton of Y is $X^{(k)}/G$. We describe the heat kernel behavior for the complex when the group has polynomial volume growth.

We begin by recalling the relevant definitions for finitely generated groups. Then we state a proof that any finitely generated volume doubling group satisfies a Poincaré inequality. We use this to compare functions on the complex with related functions on the group. This will allow us to give the heat kernel behavior for the complex.

Definition 3.13. A finite product of elements from a set S is called a word. If a word is written $s_1 s_2 \ldots s_k$, we say it has length k.

Definition 3.14. A finitely generated group is a group with a generating set, S, where every element in the group can be written as a finite word using elements of S. Although for a given $g \in G$ it can be computationally difficult to determine which word is the smallest one representing g, such a word (or words) exists. If this word has length k, then we write $|g| = k$. When $S = S^{-1}$, we say S is symmetric.

Definition 3.15. We define the volume of a subset of G to be the number of elements of G contained in that subset. We write $|B_r|$ to denote the volume of a ball of radius r, $B_r := \{g \in G : |g| \leq r\}$. For groups, volume is translation invariant, and so we do not lose any generality by calculating the volume of a ball centered at the identity. A group has polynomial volume growth when $|B_r|$ is bounded above by a polynomial as r tends to infinity.

Definition 3.16. For a function f which maps elements of a group to the reals, we define the Dirichlet form on $\ell^2(G)$ to be

$$E(f,f) = \frac{1}{|S|}\sum_{g\in G}\sum_{s\in S}|f(g) - f(gs)|^2.$$

Although we would need to specify directions if we were to define a gradient, we can define an object which behaves like the length of the gradient of f on G. We write this as $|\nabla f(x)| = \sqrt{\frac{1}{|S|}\sum_{s\in S}|f(x) - f(xs)|^2}$. Notationally, this means that $E(f,f) = \sum_{g\in G}|\nabla f(g)|^2$.

One can show a Poincaré type inequality on a volume doubling finitely generated group. The arguments used in this can be found in [9]. We include a proof for completeness.

Lemma 3.17. *Let G be a finitely generated group with symmetric generating set S. For any $f : G \to \mathbb{R}$, the following inequality holds on balls B_r:*

$$\|f - f_{B_r}\|_{1,B_r} \leq \frac{|B_{2r}|}{|B_r|}2r\sqrt{|S|}\|\nabla f\|_{1,B_{2r}}.$$

If the group is volume doubling with constant C_D, this is a weak Poincaré inequality on balls for $p = 1$:

$$\|f - f_{B_r}\|_{1,B_r} \leq 2rC_D\sqrt{|S|}\|\nabla f\|_{1,B_{2r}}.$$

Proof. Let G be a finitely generated group with a symmetric set of generators, S. Let B_r be a ball of radius r; for brevity, we will not explicitly write the center. We can write the norm of f minus its average as follows.

$$\|f - f_{B_r}\|_{1,B_r} \leq \frac{1}{|B_r|}\sum_{x\in B_r}\sum_{y\in B_r}|f(x) - f(y)|.$$

For each $y \in B_r$, there exists a $g \in G$ with $|g| \leq 2r$ such that $y = xg$. We make this substitution and sum over all $g \in G$ with $|g| \leq 2r$ and $xg \in B_r$.

$$\begin{aligned}\frac{1}{|B_r|}\sum_{x\in B_r}\sum_{y\in B_r}|f(x)-f(y)| &\leq \frac{1}{|B_r|}\sum_{x\in B_r}\sum_{g:|g|\leq 2r, xg\in B_r}|f(x)-f(xg)| \\ &= \frac{1}{|B_r|}\sum_{g:|g|\leq 2r}\sum_{x:x,xg\in B_r}|f(x)-f(xg)|.\end{aligned}$$

We will begin with the innermost quantity, and then we simplify the sums. We can write $g = s_1 \dots s_k$ as a reduced word with $k \leq 2r$. We rewrite the difference of f at x and xg by splitting the path between them into pieces.

$$|f(x)-f(xg)| \leq \sum_{i=1}^{|g|}|f(xs_1\dots s_{i-1})-f(xs_1\dots s_i)|.$$

We fix g and sum over all x such that $x, xg \in B_r$.

$$\begin{aligned}\sum_{x:x,xg\in B_r}|f(x)-f(xg)| &\leq \sum_{x:x,xg\in B_r}\sum_{i=1}^{|g|}|f(xs_1\dots s_{i-1})-f(xs_1\dots s_i)| \\ &= \sum_{i=1}^{|g|}\sum_{x:x,xg\in B_r}|f(xs_1\dots s_{i-1})-f(xs_1\dots s_i)|.\end{aligned}$$

We can change variables by letting $z = xs_1 \dots s_{i-1}$. If $i - 1 \leq r$, then $|xs_1 \dots s_{i-1}| \leq 2r$; otherwise, $k + 1 - i \leq r$ and so

$$|xgs_k^{-1}\dots s_i^{-1}| = |xs_1\dots s_k s_k^{-1}\dots s_i^{-1}| = |xs_1\dots s_{i-1}| \leq 2r.$$

Thus $z \in B_{2r}$.

$$\dots \leq \sum_{i=1}^{|g|}\sum_{z\in B_{2r}}|f(z)-f(zs_i)|.$$

Since $s_i \in S$, we can expand the sum to all $s \in S$. To do this, we must account for the multiplicity of the s_i. We could have at most $|g|$ copies of any generator; $|g| \leq 2r$, and so we gain a factor of $2r$.

$$\dots \leq 2r\sum_{s\in S}\sum_{z\in B_{2r}}|f(z)-f(zs)|.$$

Jensen's inequality allows us to rewrite this in terms of the gradient.

$$\begin{aligned}\dots &\leq 2r\sum_{z\in B_{2r}}\sqrt{|S|\frac{1}{|S|}\sum_{s\in S}|f(z)-f(zs)|^2} \\ &= 2r\sum_{z\in B_{2r}}\sqrt{|S|}|\nabla f(z)|.\end{aligned}$$

We use this calculation to get the desired inequality. We take this inequality, divide by $|B_r|$, and sum over the g to get

$$\frac{1}{|B_r|}\sum_{g:|g|\le 2r}\sum_{x\in B_r}|f(x)-f(xg)| \le \frac{1}{|B_r|}\sum_{g:|g|\le 2r}\sum_{z\in B_{2r}} 2r\sqrt{|S|}|\nabla f(z)|.$$

This reduces to

$$\|f-f_{B_r}\|_{1,B_r} \le \frac{|B_{2r}|}{|B_r|}2r\sqrt{|S|}\|\nabla f\|_{1,B_{2r}}.$$

Note that in general, $\frac{|B_{2r}|}{|B_r|}$ will depend on the radius, r. If the group is volume doubling, this gives us a weak Poincaré inequality.

$$\|f-f_{B_r}\|_{1,B_r} \le C_D 2r\sqrt{|S|}\|\nabla f\|_{1,B_{2r}}.$$

□

Consider a complex, X, and a finitely generated group of isomorphisms, G, on the complex such that $X/G = Y$ is an admissible complex consisting of a finite number of polyhedra. One example of this type of complex is the metric version of a Cayley graph; this is the graph where each vertex corresponds to a group element, and two vertices are connected by an edge if they differ by an element of the generating set. In this case, Y is the unit interval.

The local volume doubling and local uniform Poincaré inequality on X lead to the easy transfer from G to X of the global volume doubling and global Poincaré inequality using ideas that go back to Kanai's work [24]. We compare functions defined on the complex, X, with functions defined on the group, G, by transferring a function defined on X to a function defined on G that roughly preserves the norm of both the function and its energy form. As we are frequently switching between X and G, we will use B_X for balls in X and B_G for balls in G.

Definition 3.18. For functions $f : X \to \mathbb{R}$ we define $\widetilde{f} : G \to \mathbb{R}$ by

$$\widetilde{f}(g) = \frac{1}{\mu\left(B_X(g,\delta)\right)}\int_{B_X(g,\delta)} f(x)dx = \fint_{B_X(g,\delta)} f(x)dx.$$

Here, all integration is with respect to the Euclidean norm and $\delta := \mathrm{diam}(Y)$.

It is important to note that the sets $\{g : g \in B_X(r) \cap G\}$ and $B_G(r)$ are potentially different; however, there exist constants such that

$$G \cap B_X\left(\frac{r}{C'}\right) \subset B_G(r) \subset G \cap B_X(C_0 r).$$

The proof of this can be found in [30].

We can compare the norm of f with the norm of $\widetilde{f}$, as well as the norm of ∇f with that of its analogue. Note that given a radius, R_0, Corollary 2.10 tells us that we have a uniform Poincaré inequality for f on balls of radius at most R_0. We use this with $R_0 = 3\,\mathrm{diam}(Y)$ where C_P is the constant associated to the 1-Poincaré inequality on X. The proofs from [10] for these inequalities carry over to this case; careful calculations yield the constants. This type of argument can also be found in Barlow, Bass, and Kumagai [3].

Lemma 3.19. *Let $B_X(r) := B_X(g', r)$ be a ball in X centered at $g' \in G$. For any $c \in R$, $r \geq \delta = \operatorname{diam}(Y)$, $f \in W^{1,1}(X)$ and corresponding $\widetilde{f}$, we have the following comparison*:

$$||f - c||_{1,B_X(r)} \leq C\left(||\nabla f||_{1,B_X(3r)} + ||\widetilde{f} - c||_{1,B_G(2C'r)}\right).$$

$$||\nabla \widetilde{f}||_{1,B_G(r)} \leq C'||\nabla f||_{1,B_X(3C_0 r)}.$$

Here, $C = (2C_P \max_{x\in X} \#\{g \in G\cap X | x \in B_X(g, 2\delta)\} + 2\mu(B_X(e,\delta)))(\delta+1)$, and $C' = \frac{\mu(B_X(e,2\delta))}{\mu(B_X(e,\delta))^2} \max_{x\in X} \#\{g \in G \cap X | x \in B_X(g, 2\delta)\} C_P 2\delta$. Note that the constants C and C' depend on X and Y, but not on r or g'.

These bounds can be used to transfer inequalities between X and G. We can combine them with the weak Poincaré inequality on G to get an inequality on X. Whenever G has polynomial volume growth, X admits a Poincaré inequality with uniform constant at all scales.

Theorem 3.20. *Let X be an admissible Riemannian complex and G be a finitely generated group with with polynomial volume growth such that $X/G = Y$ is a finite polytopal complex satisfying assumptions 1.10. For $1 \leq p < \infty$ there exists a constant $C = C(X)$ so that for all $f \in W^{1,p}(X)$, all $x \in X$, and all $r > 0$ we have*:

$$\inf_c ||f - c||_{p,B_X(x,r)} \leq Cr||\nabla f||_{p,B_X(x,r)}.$$

Note that this implies:

$$||f - f_{B_X(r)}||_{p,B_X(x,r)} \leq 2Cr||\nabla f||_{p,B_X(x,r)}.$$

The constant C does not depend on the center or on the radius of the ball.

Proof. Note that we chose R_0 so that the Poincaré inequality holds for balls of radius up to $3\delta := 3\operatorname{diam}(Y)$. We need to show that it also holds for balls of radius greater than 3δ. Let $r \geq 3\delta$ be given. To start, we assume that the center of $B_X(x, r)$ is in G. By choosing a value of c, we obtain something at least as large as the infimum:

$$\inf_c ||f - c||_{1,B_X(x,r)} \leq ||f - \widetilde{f}_{B_G(x,2C'r)}||_{1,B_X(x,r)}.$$

We can combine Lemma 3.19 with the weak Poincaré inequality on groups (Lemma 3.17) to get:

$$\begin{aligned}\inf_c ||f - c||_{1,B_X(x,r)} &\leq C\left(||\nabla f||_{1,B_X(x,3r)} + ||\widetilde{f} - \widetilde{f}_{B_G(x,2C'r)}||_{1,B_G(x,2C'r)}\right) \\ &\leq C\left(||\nabla f||_{1,B_X(x,3r)} + 2C'rC_D\sqrt{|S|}||\nabla \widetilde{f}||_{1,B_G(x,4C'r)}\right) \\ &\leq C(1 + 2C'C_D\sqrt{|S|}C')r||\nabla f||_{1,B_X(x,12C_0C'r)}.\end{aligned}$$

By using the triangle inequality and Jensen's inequality, it can be shown that

$$||f - f_{B_X(x,r)}||_{1,B_X(x,r)} \leq 2 \inf_c ||f - c||_{1,B_X(x,r)}.$$

If the center, x, were not in G, there is some $g' \in G$ such that the center is within δ of g'. That is, $d_X(x, g') \leq \delta$. By inclusions of balls, we know that:

$$\inf_c ||f - c||_{1,B_X(x,r)} \leq \inf_c ||f - c||_{1,B_X(g',2r)}$$

$$||\nabla f||_{1,B_X(g',24C_0C'r)} \leq ||\nabla f||_{1,B_X(x,25C_0C'r)}.$$

This tells us that X admits a weak 1-Poincaré inequality for the Euclidean structure. Note that the fact that X/G is the finite polytopal complex Y forces X to have the same bounds as Y for the edge lengths, angles, and ellipticity constant. Similarly, we have a bound on the number of edges that share a vertex because Y is finite and the group G is finitely generated. Thus X satisfies the hypotheses of Theorem 2.8. Apply Theorem 2.8 to transfer the weak Euclidean 1-Poincaré inequality to the Riemannian structure; this yields a weak uniform Riemannian 1-Poincaré inequality. It is extended to a strong uniform p-Poincaré inequality via Theorem 2.4 in [37] and Hölder's inequality. □

In [41] Varopolous showed that groups with polynomial growth of degree d have on diagonal behavior $h_{2n}(e, e) \approx n^{-d/2}$. We show that a similar result holds for volume doubling complexes with underlying group structure.

Theorem 3.21. *Let X be a volume doubling Riemannian complex and G be a finitely generated group with with polynomial volume growth such that $X/G = Y$ is a finite polytopal complex satisfying assumptions 1.10. There exists a constant $C = C(X)$ so that for all $x, y \in X$ and $t > 0$, X satisfies the on-diagonal heat kernel estimates*:

$$\frac{1}{C\mu(B(x, \sqrt{t}))} \leq h_t(x, x) \leq \frac{C}{\mu(B(x, \sqrt{t}))}.$$

X additionally satisfies the following off-diagonal estimates where N is the volume doubling constant. There exist $C = C(X)$ and $C_j = C_j(X)$ so that for any $x, y \in X$ and $t > 0$ we have:

$$h_t(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2(x,y)}{4t}} \left(1 + \frac{d^2(x, y)}{t}\right)^{N/2}$$

$$h_t(x, y) \geq \frac{1}{C\mu(B(x, \sqrt{t}))} e^{-C\frac{d^2(x,y)}{t}}$$

$$\left|\partial_t^j h_t(x, y)\right| \leq \frac{C_j}{\mu(B(x, \sqrt{t}))} t^{-j} e^{-\frac{d^2(x,y)}{4t}} \left(1 + \frac{d^2(x, y)}{t}\right)^{N/2+j}.$$

Proof. Apply Theorem 3.7 and Corollary 3.11, noting that by Theorem 3.20 X satisfies both volume doubling and a Poincaré inequality at all scales uniformly. □

3.3. Further remarks. One related question is how do ∇ and $\Delta^{-1/2}$ compare in L^p? When $p = 2$, equality follows from integration by parts. In the case where X is a manifold and G is a group with polynomial volume growth, Dungey [12] showed that the Riesz transform, $\nabla\Delta^{-1/2}$ is bounded in L^p for all $1 < p < \infty$. Ishiwata [23] expanded this to the discrete case for nonbipartite covering graphs whose covering transformation group has polynomial volume growth. Recently Auscher and Coulhon [2] have shown connections between Riesz transforms and Poincaré inequalities on manifolds. It would be interesting to see whether these ideas transfer over to the complexes considered in this section.

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