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Spectral convergence of the discrete Laplacian on models of a metrized graph

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Abstract. A metrized graph is a compact singular 1-manifold endowed with a metric. A given metrized graph can be modelled by a family of weighted combinatorial graphs. If one chooses a sequence of models from this family such that the vertices become uniformly distributed on the metrized graph, then the ith largest eigenvalue of the Laplacian matrices of these combinatorial graphs converges to the ith largest eigenvalue of the continuous Laplacian operator on the metrized graph upon suitable scaling. The eigenvectors of these matrices can be viewed as functions on the metrized graph by linear interpolation. These interpolated functions form a normal family, any convergent subsequence of which limits to an eigenfunction of the continuous Laplacian operator on the metrized graph.

Contents

1. Introduction

Roughly speaking, a metrized graph Γ is a compact singular 1-manifold endowed with a metric. A given metrized graph can be modelled by a family of weighted combinatorial graphs by marking a finite number of points on the metrized graph

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and declaring them to be vertices. On each of these models we have a Laplacian matrix, which acts on functions defined on the vertices of the model. On the metrized graph there is a measure-valued Laplacian operator. The goal of this paper is to prove that the spectra of these two operators are intimately related. Indeed, we will show that if we pick a sequence of models for Γ whose vertices become equidistributed, then the eigenvalues of the discrete Laplacians on the models converge to the eigenvalues of the Laplacian operator on Γ provided we scale them correctly. Moreover, we can show that in a precise sense the eigenvectors of the discrete Laplacian converge uniformly to eigenfunctions of the Laplacian operator on the metrized graph.

This type of approximation result for Laplacian eigenvalues dates back at least to the papers of Fukaya ([KeFu]) and Fujiwara ([KoFu1], [KoFu2]). In [KeFu] the measured Hausdorff topology is defined on closed Riemannian manifolds of a fixed dimension subject to certain curvature hypotheses, and it is shown that convergence of manifolds in this topology implies convergence of eigenvalues for the associated Laplace–Beltrami operators. An analogue of the measured Hausdorff topology for the class of finite weighted graphs is defined in [KoFu1]; a similar convergence result for eigenvalues of the discrete Laplacian operator is obtained. In [KoFu2], Fujiwara approximates a closed Riemannian manifold by embedded finite graphs and proves that the eigenvalues of the Laplace–Beltrami operator can be bounded in terms of the eigenvalues of the associated graph Laplacians. In the present paper we use an approach remarkably similar to the one in [KoFu1]; this is purely a coincidence as the author was unaware of Fujiwara's work until after giving a proof of the main theorem. It would be interesting to see if the methods employed here can improve upon [KoFu2] when applied to the approximation of Riemannian manifolds by graphs.

Without giving too many of the details, let us briefly display some of the content of the main theorem (Theorem 1) in a special case. See Section 2 for precise definitions of all of the objects mentioned here. Let $\Gamma = [0, 1]$ be the interval of length 1. Let G_N be a weighted graph with vertex set $V_N = \{q_1, \ldots, q_N\}$, edge set $E_N = \{q_i q_{i+1} : i = 1, \ldots, N-1\}$, and weight $N-1$ on each edge. The reciprocal of the weight of an edge will be its length. We say that G_N is a model of our metrized graph Γ; see Figure 1.

For each N we can define the Laplacian matrix Q_N associated to the graph G_N. It is an $N \times N$ matrix which contains the weight and incidence data for the graph. Let $\lambda_{1,N}$ be the smallest positive eigenvalue of Q_N. The following table gives the value of $N\lambda_{1,N}$ (the scaled eigenvalue) for several choices of N. In each case, the eigenspace associated to this eigenvalue has dimension 1.

N	$N\lambda_{1,N}$ (approximate)
5	7.6393
10	8.8098
50	9.6690
100	9.7701
200	9.8201
500	9.8498

Now let $f : \Gamma \to \mathbb{C}$ be a continuous function that is smooth away from a finite set of points and that has one-sided derivatives at all of its singular points. The

Laplacian of such a function f is defined to be

$$\Delta f = -f''(x)dx \; - \sum_{\substack{p \text{ a singular} \\ \text{point of } f}} \sigma_p(f)\ \delta_p,$$

where dx is the Lebesgue measure on the interval, δ_p is the point mass at p, and $\sigma_p(f)$ is the sum of the one-sided derivatives of f at the singular point p. We say that a nonzero function f is an eigenfunction for Δ with respect to the measure dx if the following two conditions obtain:

- $\int_\Gamma f(x)dx = 0$.
- There exists an eigenvalue $\lambda > 0$ such that $\Delta f = \lambda f(x)dx$.

The eigenvalues of Δ with respect to the measure dx are $n^2\pi^2$ for $n = 1, 2, 3, \ldots$, and the eigenspace associated to each eigenvalue has dimension 1. Denoting the smallest eigenvalue by $\lambda_1(\Gamma)$, we see that $\lambda_1(\Gamma) = \pi^2 \approx 9.8696$. The values of $N\lambda_{1,N}$ in the above table could reasonably be converging to $\lambda_1(\Gamma)$.

Part (A) of Theorem 1 asserts that the scaled eigenvalues $N\lambda_{1,N}$ do indeed converge to $\lambda_1(\Gamma)$. A similar statement can be made for the second smallest eigenvalues of Q_N and Δ, as well as the third, etc. The fact that the dimensions of the eigenspaces for $\lambda_{1,N}$ and $\lambda_1(\Gamma)$ agree is no coincidence, and a precise statement of this phenomenon constitutes part (B) of Theorem 1. If we choose an ℓ^2-normalized eigenvector h_N of the matrix Q_N for each N, then the sequence $\{h_N\}$ can be viewed as a family of piecewise affine functions on the interval by linear interpolation. Part (C) of the main result states that this family is normal, and any subsequential limit of it will be an L^2-normalized eigenfunction for Δ with respect to the dx measure.

We make all of this precise in the next section after defining some of the necessary notation and conventions. We will follow quite closely the treatment of metrized graphs given in [BR]. Applications of metrized graphs to other areas of mathematics and the physical sciences can be found in [Ku] and [BR]. For a more conversational approach to metrized graphs, see the expository article [BF]. The reader should be aware that metrized graphs also appear in the literature under the names *quantum graphs*, *metric graphs*, and c^2*-networks*.

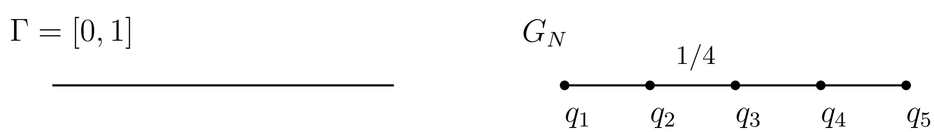

FIGURE 1. Here we see the metrized graph Γ and the graph G_N where $N = 5$. Each edge of G_N has length 1/4 or weight 4. One can view G_N as a discrete approximation to the metric space Γ.

2. Definitions, notation, and statement of the main theorem

A *metrized graph* Γ is a compact connected metric space such that for each point $p \in \Gamma$ there exists a radius $r_p > 0$ and a valence $n_p \in \mathbb{N}$ such that the open ball in Γ of radius r_p about p is isometric to the star-shaped set

$$\{re^{2\pi i m/n_p} : 0 < r < r_p, 1 \leq m \leq n_p\} \subset \mathbb{C}$$

endowed with the path metric. A *vertex set* V for Γ is any finite nonempty subset satisfying the following properties:

(i) V contains all points $p \in \Gamma$ with $n_p \neq 2$.
(ii) For each connected component $U_i \subset \Gamma \setminus V$, the closure $e_i = \overline{U}_i$ is isometric to a closed interval (not a circle).
(iii) The intersection $e_i \cap e_j$ consists of at most one point when $i \neq j$.

The set e_i will be called a *segment of* Γ *with respect to the vertex set* V. Given a vertex set V for Γ, one can associate a combinatorial weighted graph $G = G(V)$ called a *model* for Γ. Indeed, index the vertices of G by the points in V and connect two vertices p, q in G with an edge of weight $1/L$ if Γ has a segment e of length L with endpoints p, q. The hypotheses we have placed on a vertex set ensure that G is a finite connected weighted graph with no multiple or loop edges.

Given a vertex set V for Γ, a segment of length L can be isometrically parametrized by a closed interval $[0, L]$. This parametrization is unique up to a choice of orientation, and so there is a well-defined notion of Lebesgue measure on the segment with total mass L. Defining the Lebesgue measure for each of the finite number of segments gives the Lebesgue measure on Γ, which we denote by dx. Evidently it is independent of the choice of vertex set. For the scope of this paper we will assume that Γ is a fixed metrized graph on which the metric has been scaled so that $\int_\Gamma dx = 1$; i.e., Γ has total length 1.

Choose a signed measure of total mass 1 on Γ of the form

$$\mu = \omega(x)dx + \sum_{j=1}^{n} c_j \delta_{p_j}(x), \tag{1}$$

where ω is a real-valued piecewise continuous function in $L^1(\Gamma)$, $c_1, \ldots, c_n$ are real numbers, and $\delta_p(x)$ denotes the unit point mass at $p \in \Gamma$. We may assume that $X = \{p_1, \ldots, p_n\}$ is a vertex set for Γ containing all points where ω is discontinuous. This particular vertex set X will be fixed for the rest of the article. The choice of measure μ allows some flexibility in applications of the theory (cf. §14 of [BR]).

For each $p \in \Gamma$, we define the set $\mathrm{Vec}(p)$ of *formal unit vectors emanating from* p. This set has n_p elements in it, where n_p is the valence of Γ at p. For $\vec{v} \in \mathrm{Vec}(p)$, we write $p + \varepsilon\vec{v}$ for the point at distance ε from p in the direction $\vec{v}$, a notion which makes sense for ε sufficiently small. Given a function $f : \Gamma \to \mathbb{C}$, a point $p \in \Gamma$, and a direction $\vec{v} \in \mathrm{Vec}(p)$, the *derivative of* f *at* p *in the direction* $\vec{v}$ (or simply *directional derivative*), written $D_{\vec{v}}f(p)$, is given by

$$D_{\vec{v}}f(p) = \lim_{\varepsilon \to 0^+} \frac{f(p + \varepsilon\vec{v}) - f(p)}{\varepsilon},$$

provided this limit exists.

The *Zhang space*, denoted $\mathrm{Zh}(\Gamma)$, is the set of all continuous functions $f : \Gamma \to \mathbb{C}$ for which there exists a vertex set X_f for Γ such that the restriction of f to any connected component of $\Gamma \setminus X_f$ is $\mathcal{C}^2$, and for which $f''(x) \in L^1(\Gamma)$. Directional derivatives of functions in the Zhang space exist for all points of Γ and all directions. The importance of this space of functions will be made clear in just a moment.

The Laplacian $\Delta(f)$ of a function $f \in \mathrm{Zh}(\Gamma)$ is the *measure* defined by

$$\Delta(f) = -f''(x)dx - \sum_{p\in\Gamma}\left\{\sum_{\vec{v}\in\mathrm{Vec}(p)} D_{\vec{v}}f(p)\right\}\delta_p(x).$$

One sees that $\sum_{\vec{v}\in\mathrm{Vec}(p)} D_{\vec{v}}f(p) = 0$ for any point $p \in \Gamma \setminus X_f$ so that the outer sum above is actually finite. For any $f, g \in \mathrm{Zh}(\Gamma)$, the Laplacian operator satisfies the identities

$$\int_\Gamma \overline{g}\; d\Delta(f) = \int_\Gamma f d\overline{\Delta(g)} = \int_\Gamma f'(x)\overline{g'(x)}dx. \tag{2}$$

Letting g be the constant function with value 1 in this last expression shows that the measure $\Delta(f)$ has total mass zero.

Define $\mathrm{Zh}_\mu(\Gamma)$ to be the subspace of the Zhang space consisting of all functions which are orthogonal to the measure μ; i.e., all $f \in \mathrm{Zh}(\Gamma)$ such that $\int_\Gamma f d\mu = 0$. A nonzero function $f \in \mathrm{Zh}_\mu(\Gamma)$ is called an *eigenfunction of the Laplacian* (*with respect to the measure* μ) if there exists an *eigenvalue* $\lambda \in \mathbb{C}$ so that

$$\int_\Gamma \overline{g}\; d\Delta(f) = \lambda \int_\Gamma f(x)\overline{g(x)}dx, \quad \text{for each } g \in \mathrm{Zh}_\mu(\Gamma).$$

The integral on the left side of this equation is called the *Dirichlet inner product* of f and g and is denoted $\langle f, g\rangle_{\mathrm{Dir}}$; the integral on the right side is the usual L^2-inner product and will be denoted $\langle f, g\rangle_{L^2}$. Thus we can restate the defining relation for an eigenfunction as $\langle f, g\rangle_{\mathrm{Dir}} = \lambda \langle f, g\rangle_{L^2}$. Setting $g = f$ and applying the relations in (2) shows that each eigenvalue of the Laplacian Δ must be real and positive. The eigenvalues constitute a sequence tending to infinity; in particular, the dimension of the eigenspace associated to a given eigenvalue is finite. Let $0 < \lambda_1(\Gamma) < \lambda_2(\Gamma) < \lambda_3(\Gamma) < \cdots$ denote the eigenvalues of Δ with respect to the measure μ. The dimension of the eigenspace corresponding to $\lambda_i(\Gamma)$ will be denoted $d_i(\Gamma)$. Even though we have suppressed μ in the notation for the eigenvalues, they do depend heavily on the choice of measure.

The Laplacian operator can be defined on a much larger class of continuous functions than $\mathrm{Zh}(\Gamma)$, and the eigenfunctions of Δ in general need not lie in the Zhang space. However, due to the "smoothness" of our measure μ, every eigenfunction of Δ is indeed $\mathcal{C}^2$ on the complement of our fixed vertex set X. In fact, if $\omega(x)$ (the continuous part of μ) is $\mathcal{C}^m$ on $\Gamma \setminus X$, then each eigenfunction of the Laplacian is $\mathcal{C}^{m+2}$ on the complement of X. All of this is illuminated by Proposition 15.1 of [BR].

Now we turn our attention to the discrete approximation of Γ by its models. For each positive integer $N \geq \#X$, choose a vertex set $X \subset V_N \subset \Gamma$ in such a way that $\#V_N = N$. We will add another hypothesis to this vertex set momentarily. Define the measure dx_N on Γ to be the probability measure with a point mass of weight $1/N$ at each point in V_N. Choose the vertex sets $\{V_N\}$ so that the sequence of measures $\{dx_N\}$ converges weakly to the Lebesgue measure dx. In doing this, we are modelling the metrized graph Γ by finite weighted graphs $G_N = G(V_N)$ whose vertices become equidistributed in Γ. This sequence of models will be of great interest to us.

For each N, we also choose a discrete signed measure μ_N supported on V_N with total mass 1 and finite total variation such that the sequence $\{\mu_N\}$ tends weakly to μ. For example, we could construct such a sequence in the following way. For each $p \in V_N$, consider the set of points of Γ which are closer to p than to any other vertex:

$$A_p^N = \{x \in \Gamma : \operatorname{dist}(x,p) < \operatorname{dist}(x,q) \text{ for all vertices } q \in V_N \setminus \{p\}\},$$

where $\operatorname{dist}(\cdot,\cdot)$ is the metric on Γ. Note that this set is Borel measurable (in fact, it is open). We define the *discretization* of the measure μ associated to V_N by setting $\mu_N(p) = \mu(A_p^N)$ for each $p \in V_N$. The sets A_p^N are pairwise disjoint, so this yields a discrete signed measure on Γ supported on the vertex set V_N. As $X \subset V_N$, we can assert that μ_N has total mass 1 (because then all point masses of μ are picked up by some A_p^N). Recalling the definition of the measure μ in (1), we see that the total variation of μ_N satisfies

$$|\mu_N|(\Gamma) \leq \int_\Gamma |\omega(x)|dx + \sum_{i=1}^n |c_i|.$$

One can check that the measures $\{\mu_N\}$ tend weakly to μ as N tends to infinity.

We now wish to tie together the two notions of functions on the vertices of a model G and functions on the metrized graph Γ. The most natural way to do this is via linear interpolation of the values of a given function on a vertex set. The class of *continuous piecewise affine functions* on Γ, denoted $\mathrm{CPA}(\Gamma)$, is defined to be the set of all continuous functions $f : \Gamma \to \mathbb{C}$ for which there exists a vertex set X_f with the property that f is *affine* on any connected component of the complement of X_f. More precisely, if U is a connected component of $\Gamma \setminus X_f$, then its closure $e = \overline{U}$ admits an isometric parametrization $s_e : [0,L] \to e$. To say that f is affine on e is to say that there are complex constants A, B depending on the segment e so that $f \circ s_e(t) = At + B$. For each vertex set V of Γ, we define $\mathrm{Funct}(V)$ to be the subclass of $\mathrm{CPA}(\Gamma)$ whose values are determined by their values on V. That is, $\mathrm{Funct}(V)$ is any continuous function on Γ which is a linear interpolation of some set of values on the vertex set V. There is a natural identification of $\mathrm{Funct}(V_N)$ with the complex vector space $\mathbb{C}^N$. We also define the ℓ^2-inner product of two elements in $\mathrm{Funct}(V_N)$ to be $\langle f, g\rangle_{\ell^2} = \int_\Gamma f(x)\overline{g(x)}dx_N$. It is important to note which value of N we are considering when computing this inner product.

Each of our preferred models G_N is equipped with a combinatorial weighted Laplacian matrix (or Kirchhoff matrix), denoted by Q_N, which we can view as an abstract linear operator on the space of functions $\mathrm{Funct}(V_N)$. Label the vertices of G_N by $q_1, \ldots, q_N$, and assume that edge $q_iq_j = q_jq_i$ has weight w_{ij} (recall that w_{ij} is the reciprocal of the length of the segment q_iq_j in Γ and $w_{ii} = 0$ for all i). By definition (cf. [Mo] or [Bo]), the entries of Q_N are given by

$$[Q_N]_{ij} = \begin{cases} \sum_k w_{ik}, & \text{if } i = j \\ -w_{ij}, & \text{if } i \neq j \text{ and } q_i \text{ is adjacent to } q_j \\ 0, & \text{if } q_i \text{ is not adjacent to } q_j. \end{cases} \tag{3}$$

For $f \in \mathrm{Funct}(V_N)$, write Q_Nf or $\{Q_Nf\}(x)$ for the unique function in $\mathrm{Funct}(V_N)$ which has value $\sum_j [Q_N]_{ij}f(q_j)$ at the vertex q_i. Evidently the function Q_Nf exhibits the same information one gets from multiplying the matrix Q_N on the right by the column vector with jth entry $f(q_j)$. For functions in $\mathrm{Funct}(V_N)$, the

Laplacian matrix is closely related to the Laplacian operator Δ via the formula

$$\Delta(f) = \sum_{j=1}^{n} \{Q_N f\}(q_j)\ \delta_{q_j}. \tag{4}$$

This is an easy consequence of the definitions of all of the objects involved (cf. [BF], Theorem 4).

In analogy with the setup for the continuous Laplacian, we define the class of functions $\mathrm{Funct}_{\mu_N}(V_N)$ to be the subclass of $\mathrm{Funct}(V_N)$ orthogonal to the measure μ_N (a subspace of complex dimension $N-1$). That is, $f \in \mathrm{Funct}_{\mu_N}(V_N)$ if $\int_\Gamma f d\mu_N = 0$. A nonzero function $f \in \mathrm{Funct}_{\mu_N}(V_N)$ will be called an *eigenfunction for the discrete Laplacian* Q_N (*with respect to the measure* μ_N) if there exists an *eigenvalue* $\lambda \in \mathbb{C}$ so that for all $g \in \mathrm{Funct}_{\mu_N}(V_N)$,

$$\sum_{q \in V_N} \{Q_N f\}(q)\overline{g(q)} = \lambda \sum_{q \in V_N} f(q)\overline{g(q)}. \tag{5}$$

Using (4), we can rewrite this condition as $\langle f, g\rangle_{\mathrm{Dir}} = N\lambda \langle f, g\rangle_{\ell^2}$, which looks a good deal like the defining relation for eigenfunctions of Δ. In §3 we show how the eigenfunctions and eigenvalues of Q_N with respect to the measure μ_N relate to the usual notions of eigenvalues and eigenvectors. We show that all of the eigenvalues of Q_N with respect to the measure μ_N are positive in Proposition 4, and in Corollary 2 we prove that the eigenfunctions of Q_N form a basis for $\mathrm{Funct}_{\mu_N}(V_N)$. Let $0 < \lambda_{1,N} < \lambda_{2,N} < \lambda_{3,N} < \cdots$ denote the eigenvalues of Q_N, and write $d_{i,N}$ for the dimension of the eigenspace corresponding to the eigenvalue $\lambda_{i,N}$.

For each fixed $i \geq 1$ and $N \geq \#X$, let $\mathcal{H}_N(i) \subset \mathrm{Funct}_{\mu_N}(V_N)$ be an ℓ^2-orthonormal basis of the eigenspace of Q_N corresponding to the eigenvalue $\lambda_{i,N}$. If the eigenspace is empty, set $\mathcal{H}_N(i) = \emptyset$ (e.g., if $N < i$). Define $\mathcal{H}(i) = \bigcup_N \mathcal{H}_N(i)$. We think of the family $\mathcal{H}(i)$ as the set of all eigenfunctions of Q_N corresponding to an ith eigenvalue. The family $\mathcal{H}(i)$ is obviously not unique, but we fix a choice of $\mathcal{H}(i)$ for the remainder of the paper.

Theorem 1 (Main Theorem)**.** *Fix $i \geq 1$. With the hypotheses and conventions as above, we have the following conclusions:*

(A) $\lim_{N\to\infty} N\lambda_{i,N} = \lambda_i(\Gamma)$.
(B) *There exists $N_0 = N_0(i)$ so that for all $N > N_0$, the dimension $d_{i,N}$ of the eigenspace for Q_N with respect to the measure μ_N corresponding to the eigenvalue $\lambda_{i,N}$ satisfies $d_{i,N} = d_i(\Gamma)$.*
(C) *The family $\mathcal{H}(i)$ is normal. The subsequential limits of $\mathcal{H}(i)$ lie in $\mathrm{Zh}_\mu(\Gamma)$ and contain an L^2-orthonormal basis for the eigenspace of Δ corresponding to the eigenvalue $\lambda_i(\Gamma)$.*

As remarked above, we can even sharpen the statement of assertion (C) if the continuous part of the measure μ satisfies more smoothness properties. To reiterate, if $\mu = \omega(x)dx + \sum c_j\delta_{p_j}$ and ω is $\mathcal{C}^m$ away from the vertex set X, then the subsequential limits of the family $\mathcal{H}(i)$ are $\mathcal{C}^{m+2}$ on $\Gamma \setminus X$.

The rate of convergence of eigenvalues and eigenfunctions is not studied in this paper. Our methods are purely existential, which is unfortunate due to certain interesting empirical data obtained by the students in the 2003 University of Georgia REU entitled "Analysis on Metrized Graphs". The following two conjectures were made under the hypotheses that the models for Γ can be chosen with all edges having equal weights and that $\mu = dx$:

- There is a positive constant M such that $|\lambda_1(\Gamma) - N\lambda_{1,N}| < M \cdot N^{-3/2}$ for all N. (The example in [KoFu1] might lead one to believe that the error is more like $O(N^{-2})$.)
- The scaled discrete eigenvalues $N\lambda_{i,N}$ increase *monotonically* to the limit. This does not appear to be true for more general measures μ.

For more information about the REU and other data and conjectures resulting from it, see the official site: http://www.math.uga.edu/˜mbaker/REU/REU.html.

In the next section we introduce an integral operator whose spectrum is intimately related to the spectrum of Q_N. We use it to show that $\mathrm{Funct}_{\mu_N}(V_N)$ admits a basis of eigenfunctions of the discrete Laplacian. In §4 we exhibit a reduction of the Main Theorem which is technically simpler to prove. The proof of the Main Theorem will then proceed by induction on the eigenvalue index i (the case $i = 1$ will be identical to all others). We carry out the induction step in Sections 5–7.

3. Integral operators and the spectral theory of Laplacians

To begin, we recall the definition of the function $j_z(x, y)$. It is given by the unique continuous solution of $\Delta_x j_z(x, y) = \delta_y(x) - \delta_z(x)$ subject to the initial condition $j_\zeta(\zeta, y) = 0$ for all $y \in \Gamma$. Here Δ_x denotes the action of the Laplacian with respect to the variable x. As the Laplacian of the j-function has no continuous part, one can show that $j_z(x, y)$ must be piecewise affine in x for fixed values of y, z. It is also true that $j_z(x, y)$ is symmetric in x and y, nonnegative, jointly continuous in all three variable simultaneously, and uniformly bounded by 1. For elementary proofs of these facts see Section 6 of [BF].

For the rest of this section, assume $N \geq \#X$ is a fixed integer. We define $j_{\mu_N} : \Gamma^2 \to \mathbb{C}$ to be

$$j_{\mu_N}(x, y) = \int_\Gamma j_z(x, y) d\mu_N(z) = \sum_{q \in V_N} j_q(x, y)\mu_N(q).$$

Observe that $j_{\mu_N}(x, y)$ is also a piecewise affine function in x for fixed y. We require the following proposition, whose proof is given in [CR] as Lemma 2.16:

Proposition 1. *Let ν be a signed Borel measure on Γ of finite total variation. There is a constant C_ν such that for each $y \in \Gamma$,*

$$\int_\Gamma j_\nu(x, y) d\nu(x) = C_\nu.$$

In light of this proposition, we define the kernel function $g_\nu : \Gamma^2 \to \mathbb{C}$ to be

$$g_\nu(x, y) = j_\nu(x, y) - C_\nu.$$

It follows that $\int_\Gamma g_\nu(x, y) d\nu(y) = 0$. Also, the properties of the j-function mentioned above imply that g_ν is symmetric in its two arguments and continuous on Γ^2. Note that since Γ is compact, this forces g_ν to be uniformly continuous on Γ^2.

Now we recall the fundamental integral transform which effectively inverts the Laplacian on a metrized graph. For $f \in L^2(\Gamma)$, define

$$\varphi_\mu(f) = \int_\Gamma g_\mu(x,y)f(y)dy.$$

It is proved in [BR] that φ_μ is a compact Hermitian operator on $L^2(\Gamma)$ with the property that any element in the image of φ_μ is orthogonal to the measure μ. In fact, we have the following important equivalence:

Theorem 2 ([BR], Theorem 12.1)**.** *A nonzero function $f \in \mathrm{Zh}_\mu(\Gamma)$ is an eigenfunction of φ_μ with eigenvalue $\alpha > 0$ if and only if f is an eigenfunction of Δ with respect to the measure μ with eigenvalue $\lambda = 1/\alpha > 0$.*

By Proposition 15.1 of [BR] we know that any eigenfunction of Δ with respect to the measure μ lies in $\mathrm{Zh}_\mu(\Gamma)$, and the above theorem allows us to conclude that eigenfunctions of φ_μ have the same property.

We now introduce a discrete version of the integral operator whose eigenvalues are related to the eigenvalues of the Laplacian matrix Q_N in much the same way as in the above theorem. Define the *discrete integral operator* $\varphi_N : \mathrm{Funct}(V_N) \to \mathrm{Funct}(V_N)$ by

$$\varphi_N(h)(x) = \int_\Gamma g_{\mu_N}(x,y)h(y)dy_N, \quad \text{where } x \in V_N.$$

The defining equation for $\varphi_N(h)$ only gives its values at the vertices, but recall that any function in $\mathrm{Funct}(V_N)$ is determined by its values on V_N by linear interpolation.

The next lemma shows that the Laplacian matrix is essentially a left inverse for φ_N, up to a scaling factor and a correction term.

Proposition 2. *If $f \in \mathrm{Funct}(V_N)$, then for any $q \in V_N$, we have*

$$\{Q_N\varphi_N(f)\}(q) = \frac{1}{N}f(q) - \left(\int_\Gamma f(x)\ dx_N\right)\mu_N(q).$$

Proof. This is a restatement of Proposition 7.1 of [BR] (setting $\nu = f(x)dx_N$, $\mu = \mu_N$, and using Equation (4) to relate the Laplacian Δ to the discrete Laplacian matrix Q_N). □

Now we exhibit two useful facts about the kernel and image of the operator φ_N.

Lemma 1. $\mathrm{Ker}(\varphi_N)\bigcap \mathrm{Funct}_{\mu_N}(V_N) = 0$.

Proof. Suppose $f \in \mathrm{Ker}(\varphi_N)\bigcap \mathrm{Funct}_{\mu_N}(V_N)$. By Proposition 2, we find for each $q \in V_N$ that

$$0 = \{Q_N\varphi_N(f)\}(q) = \frac{1}{N}f(q) - \left(\int_\Gamma f(x)\ dx_N\right)\mu_N(q).$$

Put $C_f = \int_\Gamma f(x)dx_N$. If $C_f = 0$, then $f \equiv 0$. If $C_f \neq 0$, then the above equality implies $f(q) = NC_f\mu_N(q)$. But we then obtain the following contradiction to the fact that $f \in \mathrm{Funct}_{\mu_N}(V_N)$:

$$\int_\Gamma f(x)d\mu_N(x) = NC_f \sum_{q_i \in V_N} \mu_N(q_i)^2 \neq 0.$$ □

We remark that the previous proof actually shows the kernel of φ_N acting on the space $\mathrm{Funct}(V_N)$ consists of all scalar multiples of the piecewise affine function defined by $q \mapsto \mu_N(q), q \in V_N$.

Lemma 2. *The operator φ_N is ℓ^2-Hermitian with image in $\mathrm{Funct}_{\mu_N}(V_N)$. That is, if $f \in \mathrm{Funct}(V_N)$, then*

$$\int_\Gamma \varphi_N\,(f)\,(x) d\mu_N(x) = 0.$$

Proof. Suppose $f, g \in \mathrm{Funct}(V_N)$. As g_{μ_N} is real and symmetric, we see

$$\begin{aligned} \langle \varphi_N(f), g\rangle_{\ell^2} &= \int_\Gamma \left(\int_\Gamma g_{\mu_N}(x,y) f(y) dy_N \right) \overline{g(x)} dx_N \\ &= \int_\Gamma f(y) \overline{\left(\int_\Gamma g_{\mu_N}(x,y) g(x) dx_N \right)} dy_N \\ &= \langle f, \varphi_N(g)\rangle_{\ell^2} . \end{aligned}$$

Now recall that for any fixed $y \in \Gamma$, the function $g_{\mu_N}(x,y)$ is orthogonal to the measure μ_N. Thus

$$\begin{aligned} \int_\Gamma \varphi_N(f)(x) d\mu_N(x) &= \int_\Gamma \left(\int_\Gamma g_{\mu_N}(x,y) f(y) dy_N \right) d\mu_N(x) \\ &= \int_\Gamma f(y) \left(\int_\Gamma g_{\mu_N}(x,y) d\mu_N(x) \right) dy_N = 0. \end{aligned}$$ □

It is interesting to pause for a moment to see how the eigenvalues and eigenfunctions of Q_N with respect to the measure μ_N relate to the usual notions of eigenvalues and eigenvectors.

Proposition 3. *The function $f \in \mathrm{Funct}_{\mu_N}(V_N)$ is an eigenfunction for Q_N with respect to the measure μ_N if and only if there exists a constant $\lambda \in \mathbb{C}$ such that for all vertices $q \in V_N$,*

$$\{Q_N f\}(q) = \lambda \left\{ f(q) - N \left(\int_\Gamma f(x) dx_N \right) \mu_N(q) \right\}. \tag{6}$$

In particular, f is an eigenfunction for Q_N with respect to the measure dx_N if and only if f is a nonconstant eigenfunction for Q_N in the usual sense of a linear operator.

Proof. The final claim follows from (6) because the integral term vanishes from the result when f is orthogonal to the measure dx_N.

If $f \in \mathrm{Funct}_{\mu_N}(V_N)$ is an eigenfunction for Q_N with respect to the measure μ_N, then there is $\lambda \in \mathbb{C}$ such that for all $g \in \mathrm{Funct}_{\mu_N}(V_N)$,

$$\sum_{q \in V_N} \{Q_N f\}(q) \overline{g(q)} = \lambda \sum_{q \in V_N} f(q) \overline{g(q)}.$$

We can rewrite this relation as $N \langle Q_N f - \lambda f, g\rangle_{\ell^2} = 0$. Setting $F = Q_N f - \lambda f$, we conclude that F is in the ℓ^2-orthogonal complement of the space $\mathrm{Funct}_{\mu_N}(V_N)$ (as a subspace of $\mathrm{Funct}(V_N)$). One easily sees that the function $q \mapsto \mu_N(q)$ for $q \in V_N$ is a basis of the orthogonal complement. Thus $F(q) = M\mu_N(q)$ for some constant M and all $q \in V_N$.

The value $\{Q_N f\}(q)$ can be interpreted as the weight of the point mass of $\Delta(f)$ at the point q using Equation (4). As the Laplacian is always a measure of total mass zero, and the measure μ_N has total mass 1, we may sum the equation $F(q) = M\mu_N(q)$ over all vertices q to see that

$$M = -\lambda \sum_{q \in V_N} f(q) = -N\lambda \int_\Gamma f(x) dx_N.$$

This finishes the proof in one direction. The other direction is an immediate computation. □

Proposition 4. *The eigenvalues of Q_N acting on* Funct(V_N) *are nonnegative. The kernel of Q_N is* 1*-dimensional with basis the constant function with value* 1*. The eigenvalues of Q_N with respect to the measure μ_N are all positive.*

Proof. Using Equation (4) and the notation for the weights on the edges of the model G_N in (3), we can expand the Dirichlet norm as

$$\begin{aligned}\|f\|_{\mathrm{Dir}}^2 &= \int_\Gamma \overline{f(x)}\ d\Delta(f) = \int_\Gamma \overline{f(x)} \left[\sum_{i=1}^N \{Q_N f\}(q_i)\ \delta_{q_i}(x)\right] \\ &= \sum_{q_i \in V_N} \overline{f(q_i)}\,\{Q_N f\}(q_i) \\ &= \sum_{q_i \in V_N} \left(|f(q_i)|^2 \sum_{q_k \in V_N} w_{ik} - \overline{f(q_i)} \sum_{q_j \in V_N} f(q_j) w_{ij} \right).\end{aligned}$$

Note that $w_{ii} = 0$ since G_N has no loop edges. Let us rearrange this last sum to be over the edges of G_N instead of over its vertices. Observe that summing over all pairs of vertices as above is equivalent to summing twice over all edges of the graph. We count $w_{ik}|f(q_i)|^2$ for each end of an edge $q_i q_k$. We also count $-w_{ij} f(q_i)\overline{f(q_j)}$ and $-w_{ij}\overline{f(q_i)}f(q_j)$ for each edge $q_i q_j$. Here we have implicitly taken advantage of the symmetry of Q_N. This yields

$$\begin{aligned}\|f\|_{\mathrm{Dir}}^2 &= \sum_{\text{edges } q_i q_k} w_{ik}\left(|f(q_i)|^2 + |f(q_k)|^2\right) \\ &\qquad - \sum_{\text{edges } q_i q_j} w_{ij}\left(f(q_i)\overline{f(q_j)} + \overline{f(q_i)} f(q_j)\right) \\ &= \sum_{\text{edges } q_i q_j} w_{ij}\,|f(q_i) - f(q_j)|^2 .\end{aligned} \tag{7}$$

Note that in these sums we count edge $q_i q_j = q_j q_i$ only once.

If $f \in$ Funct(V_N) is an ℓ^2-normalized eigenfunction for Q_N with respect to the measure μ_N with eigenvalue λ, then Equation (7) and the defining relation for an eigenfunction shows that $0 \le \|f\|_{\mathrm{Dir}}^2 = N\lambda$. Thus, λ is nonnegative. If $\lambda = 0$, then a more careful look at (7) shows that our eigenfunction f satisfies $f(q_i) = f(q_j)$ for every pair of adjacent vertices q_i, q_j. Since G_N is connected, we see that f is constant on the vertex set V_N. This proves the second assertion.

Suppose further that f has constant value M. Then $\int_\Gamma f d\mu_N = M$. If $f \in$ Funct$_{\mu_N}(V_N)$, we are forced to conclude that $M = 0$. That is, our function f is identically zero on V_N. This is horribly contrary to our definition of an eigenfunction, and so we conclude that $\lambda > 0$ in this case. □

A function $f \in \mathrm{Funct}(V_N)$ is called an *eigenfunction* for the operator φ_N if there exists an *eigenvalue* $\alpha \in \mathbb{C}$ such that $\varphi_N(f) = \alpha f$. Lemma 2 implies that any eigenfunction with nonzero eigenvalue must lie in $\mathrm{Funct}_{\mu_N}(V_N)$. Now we prove the theorem which relates the nonzero eigenvalues and eigenfunctions of the integral operator φ_N to the discrete Laplacian Q_N.

Theorem 3. *A nonzero function $f \in \mathrm{Funct}_{\mu_N}(V_N)$ is an eigenfunction of Q_N (with respect to the measure μ_N) with eigenvalue λ if and only if it is an eigenfunction of φ_N with eigenvalue $\frac{1}{N\lambda}$.*

Proof. Suppose f is an eigenfunction of φ_N in $\mathrm{Funct}_{\mu_N}(V_N)$ with eigenvalue α. Proposition 2 implies that for each $q \in V_N$

$$\alpha\{Q_N f\}(q) = \{Q_N\varphi_N(f)\}(q) = \frac{1}{N}f(q) - \left(\int_\Gamma f(x)dx_N\right)\mu_N(q).$$

For $g \in \mathrm{Funct}_{\mu_N}(V_N)$, we may multiply this last equality by $\overline{g(q)}$ and sum over all vertices $q \in V_N$ to see that

$$\alpha \sum_{q\in V_N} \{Q_N f\}(q)\overline{g(q)} = \frac{1}{N}\sum_{q\in V_N} f(q)\overline{g(q)}.$$

The last sum vanishes because g is orthogonal the the measure μ_N. If $\alpha = 0$, then $f \equiv 0$ by Lemma 1; but eigenfunctions are not identically zero. Thus f satisfies the defining equation (5) for an eigenfunction of Q_N with eigenvalue $\frac{1}{N\alpha}$.

Conversely, let $f \in \mathrm{Funct}_{\mu_N}(V_N)$ be an eigenfunction of Q_N with eigenvalue λ. Multiplying the result of Proposition 2 by $N\lambda$ and subtracting from the result in Proposition 3 shows

$$Q_N\{f - N\lambda\varphi_N(f)\} \equiv 0.$$

By Proposition 4, the kernel of Q_N is precisely the constant functions on Γ. Hence $f - N\lambda\varphi_N(f) \equiv M$ for some constant M. As f and $\varphi_N(f)$ are orthogonal to the measure μ_N, we know that integrating the equation $f - N\lambda\varphi_N(f) = M$ against μ_N produces $M = 0$. Since all eigenvalues of Q_N with respect to the measure μ_N are nonzero, we deduce that $\varphi_N(f) = \frac{1}{N\lambda}f$. □

Corollary 1. *All of the eigenvalues of φ_N acting on $\mathrm{Funct}_{\mu_N}(V_N)$ are positive.*

Proof. Apply the previous theorem and Proposition 4. □

Corollary 2. *There exists a basis of $\mathrm{Funct}_{\mu_N}(V_N)$ consisting of eigenfunctions of Q_N with respect to the measure μ_N.*

Proof. The space $\mathrm{Funct}_{\mu_N}(V_N)$ admits a basis of eigenfunctions of φ_N by Lemmas 2 and 1 and the finite-dimensional spectral theorem. Now apply the previous theorem. □

4. Reduction to a weaker form of the main theorem

Before embarking on the proof of the Main Theorem, we show the following reduction:

Lemma 3 (Reduction Lemma)**.** *Suppose that for each fixed $i \geq 1$ the following assertions are true:*

(A) $\lim_{N\to\infty} N\lambda_{i,N} = \lambda_i(\Gamma)$.

(B′) *There exists $N_0 = N_0(i)$ so that for all $N > N_0$, the dimension of the eigenspace for Q_N corresponding to the eigenvalue $\lambda_{i,N}$ satisfies $d_{i,N} \leq d_i(\Gamma)$.*

(C′) *The family $\mathcal{H}(i)$ is normal. The subsequential limits of $\mathcal{H}(i)$ lie in $\mathrm{Zh}_\mu(\Gamma)$, have unit L^2-norm, and are eigenfunctions of Δ corresponding to the eigenvalue $\lambda_i(\Gamma)$.*

Then Theorem 1 is true.

The proof of the Reduction Lemma will require a few preliminary results, which will take up the majority of this section.

Lemma 4. *Suppose $f \in \mathrm{Zh}(\Gamma)$ with the property that f is $\mathcal{C}^2$ on $\Gamma \setminus X$. Define f_N to be the unique function in $\mathrm{Funct}(V_N)$ which agrees with f on the vertex set V_N. Then*

$$\lim_{N\to\infty} \|f_N\|_{\mathrm{Dir}} = \|f\|_{\mathrm{Dir}}.$$

Proof. We begin by recalling Equation (7):

$$\|f_N\|^2_{\mathrm{Dir}} = \sum_{\text{edges } q_i q_j} w_{ij} \left|f_N(q_i) - f_N(q_j)\right|^2. \tag{8}$$

As Γ can be decomposed into a finite collection of segments $\{e\}$ using the preferred vertex set X, we can sum over the segments of Γ to get

$$\|f_N\|^2_{\mathrm{Dir}} = \sum_{\substack{\text{segments } e \\ \text{of } \Gamma}} \sum_{\substack{\text{edges } q_i q_j \\ \text{of } G_N \text{ with } q_i, q_j \in e}} w_{ij} \left|f_N(q_i) - f_N(q_j)\right|^2. \tag{9}$$

The integral is linear, and there are only finitely many segments of Γ over which we wish to integrate, so it suffices to show that the inner sum in (9) converges to $\int_e |f'(x)|^2 dx$ for any segment e of Γ (by Equation (2)). Hence, we may assume for the remainder of this proof that Γ consists of exactly one segment e; that is, Γ is isometric to a closed interval of length 1.

The single segment e of Γ admits an isometric parametrization $s_e : [0,1] \to e$. The vertex set V_N corresponds to a partition $0 = t_1 < t_2 < \cdots < t_N = 1$. We write $f_e = f \circ s_e$ for ease of notation. Now (8) takes the form

$$\begin{aligned}\|f_N\|^2_{\mathrm{Dir}} &= \sum_{i=1}^{N-1} w_{i(i+1)} \left|f_e(t_{i+1}) - f_e(t_i)\right|^2 \\ &= \sum_{i=1}^{N-1} (t_{i+1} - t_i) \left|\frac{f_e(t_{i+1}) - f_e(t_i)}{t_{i+1} - t_i}\right|^2.\end{aligned}$$

For each i there is $t_i^* \in (t_i, t_{i+1})$ so that $f_e(t_{i+1}) - f_e(t_i) = f_e'(t_i^*)(t_{i+1} - t_i)$ by the mean value theorem. Hence

$$\|f_N\|_{\mathrm{Dir}}^2 = \sum_{i=1}^{N-1} (t_{i+1} - t_i) \left| f_e'(t_i^*) \right|^2 .$$

But this last expression is just a Riemann approximation to $\int_0^1 |f_e'(x)|^2 dx$. By Proposition 5.2 in [BR], we find that f_e' is a continuous function on $[0,1]$ so that as N tends to infinity these approximations actually do limit to the desired integral. □

The following immediate corollary won't be of any use to us in our present task, but it is nice to include for completeness.

Corollary 3. *If $f, g \in \mathrm{Zh}(\Gamma)$, then we have the limit*

$$\lim_{N \to \infty} \langle f_N, g_N \rangle_{\mathrm{Dir}} \longrightarrow \langle f, g \rangle_{\mathrm{Dir}} ,$$

where f_N and g_N are affine approximations of f and g as in the statement of Lemma 4.

Proof. This is an easy consequence of Lemma 4 and the polarization identity

$$\langle f_N, g_N \rangle_{\mathrm{Dir}} = \frac{1}{4} \sum_{n=0}^{3} i^n \|f_N + i^n g_N\|_{\mathrm{Dir}}^2,$$

which may be checked easily by expanding the norms on the right-hand side. Here i denotes a fixed complex root of -1. □

Here is a technical lemma that will be needed in the Approximation Lemma (Lemma 6) and again in later sections.

Lemma 5. *Suppose for each N that k_N is a function in $\mathrm{Funct}(V_N)$ such that the sequence $\{k_N\}$ converges uniformly to a (continuous) function $k : \Gamma \to \mathbb{C}$. Then $\|k_N\|_{\ell^2} \to \|k\|_{L^2}$ as $N \to \infty$. If $\{k_N'\}$ is another such sequence of functions converging uniformly to some $k' : \Gamma \to \mathbb{C}$, then $\langle k_N, k_N' \rangle_{\ell^2} \to \langle k, k' \rangle_{L^2}$ as $N \to \infty$.*

Proof. Suppose N is so large that k_N is uniformly within ε of k. On one hand, we have

$$\begin{aligned}
\|k_N\|_{\ell^2}^2 &= \int_\Gamma |k_N(x)|^2 dx_N \\
&\leq \int_\Gamma \left(|k(x)| + \varepsilon \right)^2 dx_N \\
&= \|k\|_{\ell^2}^2 + 2\varepsilon \|k\|_{\ell^1} + \varepsilon^2 \\
&\longrightarrow \|k\|_{L^2}^2 + 2\varepsilon \|k\|_{L^1} + \varepsilon^2.
\end{aligned}$$

The convergence in the final step follows from weak convergence of dx_N to dx. Letting $\varepsilon \to 0$ shows that $\limsup_{N \to \infty} \|k_N\|_{\ell^2} \leq \|k\|_{L^2}$.

Similarly,

$$\begin{aligned}\int_\Gamma |k_N(x)|^2 dx_N &\geq \int_\Gamma \left(|k(x)| - \varepsilon\right)^2 dx_N \\ &= \|k\|_{\ell^2}^2 - 2\varepsilon\|k\|_{\ell^1} + \varepsilon^2 \\ &\longrightarrow \|k\|_{L^2}^2 - 2\varepsilon\|k\|_{L^1} + \varepsilon^2.\end{aligned}$$

Now we see that $\|k\|_{L^2} \leq \liminf_{N\to\infty} \|k_N\|_{\ell^2}$.

The final statement follows by the polarization identity as in the proof of Corollary 3. □

Lemma 6 (Approximation Lemma)**.** *Fix $m \geq 1$ and suppose $f \in \mathrm{Zh}_\mu(\Gamma)$ is an L^2-normalized eigenfunction of Δ with corresponding eigenvalue $\lambda_m(\Gamma)$. For each N, let f_N be the unique function in* $\mathrm{Funct}(V_N)$ *which agrees with f at the vertices in V_N. Assume also that assertions* (A), (B′), *and* (C′) *of the Reduction Lemma* (*Lemma* 3) *are satisfied for $j \leq m-1$. Given any subsequence of $\{G_N\}$, there exists a further subsequence such that for each $\varepsilon > 0$ there is a positive integer N_1 with the property that for every $N > N_1$ with G_N in our sub-subsequence, we can find a function $\widetilde{f}_N \in \mathrm{Funct}_{\mu_N}(V_N)$ for which the following conditions hold simultaneously:*

(i) $\|\widetilde{f}_N\|_{\ell^2} = 1$.

(ii) $\widetilde{f}_N$ *is ℓ^2-orthogonal to every $h \in \mathcal{H}_N(j)$, $j = 1, \ldots, m-1$.*

(iii) $\left|\|\widetilde{f}_N\|_{\mathrm{Dir}}^2 - \|f_N\|_{\mathrm{Dir}}^2\right| < \varepsilon$.

Proof. As condition (B′) of the Reduction Lemma holds, and each $d_j(\Gamma)$ is finite, we may pass to a subsequence of models $\{G_N\}$ such that $d_{j,N}$ is independent of N for every $j \leq m-1$ and all N sufficiently large in the subsequence. This implies that for large N, the set $\mathcal{T}_N = \bigcup_{j=1}^{m-1} \mathcal{H}_N(j)$ is finite with cardinality independent of N. Denote this cardinality by T, and let us label the elements of $\mathcal{T}_N$ as $h_N^1, \ldots, h_N^T$. The set $\mathcal{T}_N$ is an ℓ^2-orthonormal system by definition of the sets $\mathcal{H}_N(j)$ and the fact that eigenfunctions corresponding to distinct eigenvalues are ℓ^2-orthogonal (a consequence of the definition of eigenfunction).

Write $B_N = \int_\Gamma f d\mu_N$; we will abuse notation in what follows and let B_N also denote the constant function on Γ with value B_N. Define

$$\widetilde{f}_N = \frac{f_N - \sum_{j=1}^T \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j - B_N}{\left\| f_N - \sum_{j=1}^T \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j - B_N \right\|_{\ell^2}}.$$

It will be shown in a moment that for N sufficiently large lying in a certain subsequence, the denominator of this expression is nonzero.

The definition of $\mathcal{H}(j)$ implies that each h_N^j is orthogonal to the measure μ_N, so a small calculation shows $\widetilde{f}_N$ lies in $\mathrm{Funct}_{\mu_N}(V_N)$. Evidently condition (i) holds. Also, we note that B_N is an eigenfunction of φ_N acting on $\mathrm{Funct}(V_N)$ with corresponding eigenvalue zero. Any two eigenfunctions of a self-adjoint operator that have distinct associated eigenvalues are orthogonal, and hence B_N must be ℓ^2-orthogonal to each h_N^j. Another quick calculation shows that h_N^j is orthogonal to $\widetilde{f}_N$ for all $j \leq m-1$, which is condition (ii).

It remains to check that condition (iii) holds for our choice of $\widetilde{f}_N$ upon passage to a further subsequence. First observe that the definition of an eigenfunction for Q_N

implies that distinct elements of $\mathcal{T}_N$ are orthogonal with respect to the Dirichlet inner product. Also, $\langle B_N, g\rangle_{\mathrm{Dir}} = 0$ for all $g \in \mathrm{Funct}(V_N)$ since constant functions are in the kernel of Q_N. Expanding the Dirichlet norm and simplifying using these two observations gives

$$\left\| f_N - \sum_{j=1}^{T} \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j - B_N \right\|_{\mathrm{Dir}}^2 = \|f_N\|_{\mathrm{Dir}}^2 - \sum_{j=1}^{T} N\zeta_{j,N} \left| \left\langle f_N, h_N^j \right\rangle_{\ell^2} \right|^2,$$

where $\zeta_{j,N}$ is the eigenvalue associated to the eigenfunction h_N^j. Of course, $\zeta_{j,N} = \lambda_{k,N}$ for some $k \leq m-1$.

We have T sequences of functions $\{h_N^1\}, \ldots, \{h_N^T\}$, each of which lies in a particular $\mathcal{H}(j)$. The normality of each $\mathcal{H}(j)$ allows us to find a further subsequence of models and a set of functions $\mathcal{T} = \left\{h^1, \ldots, h^T\right\}$ so that $h_N^j \to h^j$ uniformly on Γ along this subsequence. For the rest of the proof we will assume that N lies in the subsequence we have just specified.

As $\mathcal{T}_N$ forms an ℓ^2-orthonormal set for each N, we see that $\mathcal{T}$ forms an L^2-orthonormal set of eigenfunctions for Δ (assertion (C$'$)). The associated eigenvalues of these functions are all strictly smaller than $\lambda_m(\Gamma)$, and hence each h^j is L^2-orthogonal to our initial function f. Lemma 5 implies that the inner product $\left\langle f_N, h_N^j \right\rangle_{\ell^2}$ tends to $\langle f, h^j \rangle_{L^2} = 0$ as N tends to infinity.

By Lemma 4, weak convergence of $\{dx_N\}$ to dx, the previous paragraph, and the hypothesis that the scaled eigenvalues $N\lambda_{j,N}$ converge to $\lambda_j(\Gamma)$ for each $j \leq m-1$, we conclude that

$$\text{(10)} \quad \|f_N\|_{\mathrm{Dir}}^2 - \sum_{j=1}^{T} N\zeta_{j,N} \left| \left\langle f_N, h_N^j \right\rangle_{\ell^2} \right|^2 \longrightarrow \|f\|_{\mathrm{Dir}}^2 - \sum_{j=1}^{T} \zeta_j \left| \langle f, h^j \rangle_{L^2} \right|^2 = \|f\|_{\mathrm{Dir}}^2$$

as N tends to infinity.

We also note that the function $\sum_{j=1}^{T} \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j + B_N$ tends to zero uniformly. Here $B_N \to 0$ by weak convergence of $\{\mu_N\}$ to μ. The Minkowski inequality and Lemma 5 imply that

$$\begin{aligned} \left\| f_N - \sum_{j=1}^{T} \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j - B_N \right\|_{\ell^2} &\leq \|f_N\|_{\ell^2} + \left\| \sum_{j=1}^{T} \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j + B_N \right\|_{\ell^2} \\ &= \|f\|_{L^2} + o(1), \end{aligned}$$

where the error term depends only on N. The Minkowski inequality also gives an identical lower bound so that

$$\text{(11)} \qquad \left\| f_N - \sum_{j=1}^{T} \left\langle f_N, h_N^j \right\rangle_{\ell^2} h_N^j - B_N \right\|_{\ell^2} \to \|f\|_{L^2} = 1$$

as N goes to infinity. This shows that the denominator of $\widetilde{f}_N$ is in fact nonzero for N sufficiently large.

Finally, we use the convergence relations (10) and (11) along with Lemma 4 to conclude that as N tends to infinity

$$\left\|\widetilde{f}_N\right\|^2_{\mathrm{Dir}} - \|f_N\|^2_{\mathrm{Dir}} = \frac{\|f_N\|^2_{\mathrm{Dir}} - \sum_{j=1}^{T} N\zeta_{j,N}\left|\left\langle f_N, h_N^j\right\rangle_{\ell^2}\right|^2}{\left\|f_N - \sum_{j=1}^{T}\left\langle f_N, h_N^j\right\rangle_{\ell^2} h_N^j - B_N\right\|^2_{\ell^2}} - \|f_N\|^2_{\mathrm{Dir}} \longrightarrow 0.$$

This completes the proof of condition (iii). □

And now we require one final proposition which gives us a characterization of the mth largest eigenvalue of Q_N as a minimum of the Dirichlet norm on the ℓ^2 unit circle.

Lemma 7. *Fix $N \geq \#X$. For a given $m \geq 1$, let $\mathcal{T}_N = \bigcup_{j=1}^{m-1} \mathcal{H}_N(j)$. Denote by $\mathcal{T}_N^{\perp}$ the ℓ^2-orthogonal complement of the span of $\mathcal{T}_N$ inside the space* $\mathrm{Funct}_{\mu_N}(V_N)$*. Provided that $\mathcal{T}_N^{\perp} \neq \emptyset$, we have*

$$N\lambda_{m,N} = \min_{\substack{g\in\mathcal{T}_N^{\perp}\\ g\not\equiv 0}} \frac{\|g\|^2_{\mathrm{Dir}}}{\|g\|^2_{\ell^2}}.$$

Proof. As $\mathrm{Funct}_{\mu_N}(V_N)$ admits a basis of eigenfunctions of Q_N (Corollary 2), there exists an ℓ^2-orthonormal basis $\{f_1, \ldots, f_r\}$ for $\mathcal{T}_N^{\perp}$. Let us assume that the corresponding eigenvalues are $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_r$. Note that $\gamma_1 = \lambda_{m,N}$ since $\mathcal{T}_N^{\perp}$ consists of all of the eigenfunctions of Q_N with eigenvalue strictly larger than $\lambda_{m-1,N}$.

To prove the proposition it suffices to consider the minimum over all $g \in \mathcal{T}_N^{\perp}$ with unit ℓ^2-norm. Using our basis, write $g = \sum_{j=1}^{r} a_j f_j$, where the complex numbers a_j satisfy $\sum_j |a_j|^2 = 1$. Since $\langle f_j, f_j\rangle_{\mathrm{Dir}} = N\gamma_j$, we see that

$$\langle g, g\rangle_{\mathrm{Dir}} = \sum_{j=1}^{r} |a_j|^2 \langle f_j, f_j\rangle_{\mathrm{Dir}} = N\sum_{j=1}^{r} |a_j|^2\gamma_j \geq N\gamma_1 \sum_{j=1}^{r} |a_j|^2 = N\lambda_{m,N}.$$

Note that if we take $g \in \mathrm{Funct}_{\mu_N}(V_N)$ to be an ℓ^2-normalized eigenfunction of Q_N with eigenvalue $\lambda_{m,N}$, then equality is actually achieved in this last computation. □

Finally, we may now return to:

Proof of Reduction Lemma. We assume that conditions (A), (B′), and (C′) hold for every $i \geq 1$. Clearly we only need to check that assertions (B) and (C) in the Main Theorem are true. If assertion (B) fails for some value of i, then we may choose a minimum such i. There exists a positive integer D strictly smaller than $d_i(\Gamma)$ and an infinite subsequence of models $\{G_N\}$ with $d_{i,N} = D$ for all G_N in the subsequence. For each such N, select distinct functions $h_N^1, \ldots, h_N^D \in \mathcal{H}_N(i)$. Using the normality of $\mathcal{H}(i)$ as given by hypothesis (C′), we may pass to a further subsequence and assume that there are D limit functions $h^1, \ldots, h^D$, with $h_N^j \to h^j$ uniformly along this subsequence for each j. Again using assertion (C′) we find that each h^j is an eigenfunction of Δ with associated eigenvalue $\lambda_i(\Gamma)$. As $D < d_i(\Gamma)$, there is some eigenfunction of Δ with eigenvalue $\lambda_i(\Gamma)$ which is not in the span of $\{h^1, \ldots, h^D\}$. Choose such a function with unit L^2-norm and denote it by f.

Now choose a sub-subsequence for which the conclusion of the Approximation Lemma is satisfied for the function f in the case $m = i+1$. Given $\varepsilon > 0$ and N large, we pick $\widetilde{f}_N$ as in the Approximation Lemma. We apply Lemma 7 in the case $m = i+1$ to see that

$$N\lambda_{i+1,N} = \min_{\substack{g \in \mathcal{T}_N^\perp \\ g \neq 0}} \frac{\|g\|_{\mathrm{Dir}}^2}{\|g\|_{\ell^2}^2} \leq \|\widetilde{f}_N\|_{\mathrm{Dir}}^2 < \|f_N\|_{\mathrm{Dir}}^2 + \varepsilon.$$

We remark that $\mathcal{T}_N^\perp \neq \emptyset$ for N sufficiently large since its dimension is at least $N - 1 - \sum_{j=1}^i d_j(\Gamma)$ by assertion (B′). Letting N go to infinity through our subsequence and using assertion (A) and Lemma 4, we find that

$$\lambda_{i+1}(\Gamma) \leq \|f\|_{\mathrm{Dir}}^2 + \varepsilon.$$

We know that $\langle f, g\rangle_{\mathrm{Dir}} = \lambda_i(\Gamma)\,\langle f, g\rangle_{L^2}$ for all $g \in \mathrm{Zh}_\mu(\Gamma)$ since f is an eigenfunction of Δ. In particular, setting $g = f$ tells us that $\|f\|_{\mathrm{Dir}}^2 = \lambda_i(\Gamma)$. Taking ε sufficiently small provides a contradiction since $\lambda_i(\Gamma) < \lambda_{i+1}(\Gamma)$. Thus assertion (B) must be true for all i.

Now we prove assertion (C). As assertion (B) holds for each fixed i, we know that $\mathcal{H}_N(i)$ consists of $d = d_i(\Gamma)$ pairwise ℓ^2-orthogonal functions on Γ when N is large. Let $h_N^1, \ldots, h_N^d$ be the elements in $\mathcal{H}_N(i)$. By normality of $\mathcal{H}(i)$, we can pass to a subsequence such that each of these d sequences $\{h_N^j\}$ converges. Now use Lemma 5 to see that the limit functions must be pairwise L^2-orthogonal and have unit L^2-norm. The dimension of the eigenspace of Δ corresponding to $\lambda_i(\Gamma)$ is d, so the subsequential limits just constructed form a basis for this eigenspace. □

5. Normality of the family $\mathcal{H}(i)$

For the remainder of the paper we assume that assertions (A), (B′), and (C′) of the Reduction Lemma hold up to the case $i-1$, and we prove that they are true for the case i.

Theorem 4 (Eigenvalue Convergence: Part 1)**.** *We have*

$$\limsup_{N\to\infty} N\lambda_{i,N} \leq \lambda_i(\Gamma).$$

Proof. Suppose the result is false and pick a subsequence of models and a positive constant δ so that $N\lambda_{i,N} > \lambda_i(\Gamma) + \delta$ for all N sufficiently large in our subsequence. Let f be an L^2-normalized eigenfunction for Δ in $\mathrm{Zh}_\mu(\Gamma)$ with eigenvalue $\lambda_i(\Gamma)$. We now select a further subsequence as in the Approximation Lemma of §4. For any $\varepsilon > 0$ and N sufficiently large in our sub-subsequence, we can find $\widetilde{f}_N$ and apply the Approximation Lemma and Lemma 7 to get

$$\lambda_i(\Gamma) + \delta < N\lambda_{i,N} = \min_{\substack{g \in \mathcal{T}_N^\perp \\ g \neq 0}} \frac{\|g\|_{\mathrm{Dir}}^2}{\|g\|_{\ell^2}^2} \leq \frac{\|\widetilde{f}_N\|_{\mathrm{Dir}}^2}{\|\widetilde{f}_N\|_{\ell^2}^2} < \|f_N\|_{\mathrm{Dir}}^2 + \varepsilon.$$

We remark that $\mathcal{T}_N^\perp$ is nonempty for N sufficiently large by assertion (B’) (which we are assuming holds up to $i-1$). Letting N tend to infinity through our subsequence and applying Lemma 4, we arrive at the statement

$$\lambda_i(\Gamma) + \delta \leq \|f\|_{\mathrm{Dir}}^2 + \varepsilon, \qquad \text{for any } \varepsilon > 0.$$

As $\|f\|_{\mathrm{Dir}}^2 = \lambda_i(\Gamma)$, taking ε sufficiently small produces a contradiction. □

Corollary 4. *The family $\mathcal{H}(i)$ has uniformly bounded Dirichlet norms.*

Proof. By definition of an eigenfunction of Q_N with respect to the measure μ_N, we have immediately that $\|h\|^2_{\mathrm{Dir}} = N\lambda_{i,N}$ for any $h \in \mathcal{H}_N(i)$. The previous theorem shows that $\|h\|^2_{\mathrm{Dir}}$ cannot be arbitrarily large for h lying in $\mathcal{H}(i)$. □

Theorem 5. (a) *The family $\mathcal{H}(i)$ is equicontinuous on Γ.*

(b) *$\mathcal{H}(i)$ is uniformly bounded on Γ.*

(c) *$\mathcal{H}(i)$ is a normal family.*

Proof. (a) Suppose M is a positive real number such that $\|h\|_{\mathrm{Dir}} \leq M$ for all $h \in \mathcal{H}(i)$. Now suppose that $\varepsilon > 0$ is given and that we have any pair $x, y \in \Gamma$ with $\mathrm{dist}(x, y) < \varepsilon^2/M^2$, where $\mathrm{dist}(\cdot,\cdot)$ is the metric on Γ. Let $\gamma \subset \Gamma$ be a unit speed path from x to y. The fundamental theorem of calculus and the Cauchy–Schwarz inequality imply that

$$\begin{aligned}|h(x) - h(y)| &= \left|\int_\gamma h'(t)dt\right| \\ &\leq \left(\int_\gamma dt\right)^{1/2} \left(\int_\gamma |h'(t)|^2\, dt\right)^{1/2} \\ &\leq \mathrm{dist}(x, y)^{1/2}\|h\|_{\mathrm{Dir}} < \varepsilon.\end{aligned}$$

The integrals in the above estimate should be viewed as path integrals in which we have implicitly chosen compatible orientations for consecutive segments along the path γ.

(b) It suffices to show that the family $\mathcal{H}(i)$ is uniformly bounded on $\bigcup_N V_N$ as this constitutes a dense subset of Γ and all of our functions are continuous.

Recall that if $h \in \mathcal{H}_N(i)$, then $\int_\Gamma h d\mu_N = 0$. By the proof of part (a) it follows that for any vertex $v \in V_N$ and $h \in \mathcal{H}_N(i)$,

$$\begin{aligned}|h(v)| &= \left|\int_\Gamma (h(v) - h(y))\, d\mu_N(y)\right| \\ &\leq \int_\Gamma |h(v) - h(y)|\, d|\mu_N|(y) \\ &\leq \int_\Gamma \mathrm{dist}(v, y)^{1/2}\|h\|_{\mathrm{Dir}} d|\mu_N|(y) \\ &\leq \|h\|_{\mathrm{Dir}}|\mu_N|(\Gamma).\end{aligned}$$

The last inequality follows because the distance between any two points in Γ is at most 1. Now Corollary 4 and the assumption that the measures μ_N have uniformly bounded total variation show that there is a constant M, independent of N, so that $|h(v)| \leq M$ for any choice of $h \in \mathcal{H}_N$ and $v \in V_N$.

Take any $h \in \mathcal{H}(i)$ now. Choose N so that $h \in \mathcal{H}_N(i)$. Note that if $x \in \Gamma$, then x lies on some segment e of Γ with respect to the vertex set V_N. By our choice of N, we find h is affine on e. Taking $y, z \in V_N$ to be the endpoints of e, we know that there is $t \in [0, 1]$ so that $h(x) = (1-t)h(y)+th(z)$. The triangle inequality implies $|h(x)| \leq \max\{|h(y)|, |h(z)|\} \leq \max_{v_i \in V_N} |h(v_i)| \leq M$.

(c) Apply the Arzela–Ascoli theorem using parts (a) and (b) to satisfy the necessary hypotheses. □

6. Convergence and approximation results

This section contains the proofs of various technical lemmas regarding the integral operator φ_μ and its relation to the discrete integral operator φ_N.

Lemma 8 ([Rud], p. 168, Exercise 16)**.** *Let $\{k_N\}$ be an equicontinuous sequence of functions on the metrized graph Γ such that $\{k_N\}$ converges pointwise. Then $\{k_N\}$ converges uniformly on Γ.*

Proof. Fix $\varepsilon > 0$. By equicontinuity there exists $\delta > 0$ such that $|k_N(x) - k_N(y)| < \varepsilon/3$ for every N and $x, y \in \Gamma$ with $\mathrm{dist}(x, y) < \delta$. By compactness, we can cover Γ with a finite number of balls of radius $\delta/2$. Select $y_1, \ldots, y_m \in \Gamma$ so that at least one y_i lies in each ball.

For $j = 1, \ldots, m$, let A_j be a positive real number such that $M, N > A_j$ implies $|k_N(y_j) - k_M(y_j)| < \varepsilon/3$. This is possible because $\{k_N\}$ is pointwise convergent. Set $A = \max\{A_j : j = 1, \ldots, m\}$.

Suppose $M, N > A$ and that $x \in \Gamma$. Pick y_i so that $\mathrm{dist}(x, y_i) < \delta$. Then

$$|k_N(x) - k_M(x)| \leq |k_N(x) - k_N(y_i)| + |k_N(y_i) - k_M(y_i)| + |k_M(y_i) - k_M(x)| < \varepsilon.$$

Note that our choice of M, N did not depend on x. This shows $\{k_N\}$ is uniformly Cauchy. □

Lemma 9. *Suppose that $\{k_N\}$ is a sequence of continuous functions on Γ which converges uniformly to a function $k : \Gamma \to \mathbb{C}$. Then $\varphi_\mu(k_N) \to \varphi_\mu(k)$ uniformly with N.*

Proof. Our strategy will be to use Lemma 8. Fix $x \in \Gamma$. Then, since $g_\mu(x, y)$ is continuous on Γ^2, it must be uniformly bounded (compactness). Also, since $\{k_N\}$ converges uniformly to a continuous function k, we find that $k_N(y)$ is uniformly bounded for all N, y. Thus Lebesgue dominated convergence guarantees that $\varphi_\mu(k_N) \to \varphi_\mu(k)$ pointwise.

Suppose that M is a positive real number such that $|k_N(y)| \leq M$ for all N, y. Also, since g_μ is continuous on a compact space, it must be uniformly continuous. Thus, given $\varepsilon > 0$, we can select $\delta > 0$ such that $|g_\mu(x, y) - g_\mu(x', y)| < \varepsilon/M$ whenever $x, x', y \in \Gamma$ with $\mathrm{dist}(x, x') < \delta$. Hence for all $x, x', y \in \Gamma$ such that $\mathrm{dist}(x, x') < \delta$, we have

$$|\varphi_\mu(k_N)(x) - \varphi_\mu(k_N)(x')| \leq \int_\Gamma |g_\mu(x, y) - g_\mu(x', y)| \cdot |k_N(y)|\, dy \leq \varepsilon.$$

This shows the desired equicontinuity. □

Lemma 10. *As $N \to \infty$, we have $g_{\mu_N} \to g_\mu$ uniformly on Γ^2.*

Proof. We intend to apply Lemma 8.

First we show pointwise convergence. By definition of g_{μ_N} and g_μ, it suffices to show:

- $j_{\mu_N}(x, y) \to j_\mu(x, y)$ for each pair $x, y \in \Gamma$.
- $\int_\Gamma j_{\mu_N}(x, y) d\mu_N(x) \to \int_\Gamma j_\mu(x, y) d\mu(x)$ for all $y \in \Gamma$.

The former assertion follows by weak convergence of measures. As for the second, fix $\varepsilon > 0$ and a basepoint $y_0 \in \Gamma$. As $j_z(x, y)$ is a continuous function on Γ^3 (a compact space), it is uniformly continuous. So there exists $\delta > 0$ such that for any

$x, x', z \in \Gamma$ with $\text{dist}(x, x') < \delta$, we have $|j_z(x, y_0) - j_z(x', y_0)| < \varepsilon$. Now for such x, x', it follows that

$$|j_{\mu_N}(x, y_0) - j_{\mu_N}(x', y_0)| \leq \int_\Gamma |j_z(x, y_0) - j_z(x', y_0)|\, d|\mu_N|(z) \leq \varepsilon|\mu_N|(\Gamma).$$

As the total variations $|\mu_N|(\Gamma)$ are uniformly bounded independent of N, we see $\{j_{\mu_N}(x, y_0)\}$ is equicontinuous in the variable x. By Lemma 8 we find $j_{\mu_N}(x, y_0) \to j_\mu(x, y_0)$ uniformly in x. For ease of notation, set $I_\nu(\sigma) = \int_\Gamma j_\nu(x, y_0) d\sigma(x)$ for any pair of measures ν, σ. Now suppose that N is so large that $j_{\mu_N}(x, y_0)$ is uniformly within ε of $j_\mu(x, y_0)$, and that $I_\mu(\mu_N)$ is within ε of $I_\mu(\mu)$ (by weak convergence). Then

$$\begin{aligned} |I_{\mu_N}(\mu_N) - I_\mu(\mu)| &\leq |I_{\mu_N}(\mu_N) - I_\mu(\mu_N)| + |I_\mu(\mu_N) - I_\mu(\mu)| \\ &\leq \int_\Gamma |j_{\mu_N}(x, y_0) - j_\mu(x, y_0)|\ \ d|\mu_N|(x) \\ &\qquad + |I_\mu(\mu_N) - I_\mu(\mu)| \\ &\leq \varepsilon|\mu_N|(\Gamma) + \varepsilon. \end{aligned} \tag{12}$$

By Proposition 1, we see that $\int_\Gamma j_{\mu_N}(x, y) d\mu_N(x)$ and $\int_\Gamma j_\mu(x, y) d\mu$ are independent of y. As ε was arbitrary, we conclude from (12) that $\int_\Gamma j_{\mu_N}(x, y) d\mu_N(x) \to \int_\Gamma j_\mu(x, y) d\mu$ pointwise for all $y \in \Gamma$.

All of this allows us to conclude that $g_{\mu_N}(x, y) \to g_\mu(x, y)$ pointwise on Γ^2. It remains to prove $\{g_{\mu_N}(x, y)\}$ is equicontinuous jointly in x and y. By uniform continuity of j on Γ^3, we know there exists $\delta > 0$ such that for all x, x', y, y', z with $\text{dist}((x, y), (x', y')) < \delta$ (some metric on the product space Γ^2), it must be that $|j_z(x, y) - j_z(x', y')| < \varepsilon$. Now for any $x, x', y, y' \in \Gamma$ with $\text{dist}((x, y), (x', y')) < \delta$,

$$\begin{aligned} |g_{\mu_N}(x, y) - g_{\mu_N}(x', y')| &= |j_{\mu_N}(x, y) - j_{\mu_N}(x', y')| \\ &\leq \int_\Gamma |j_z(x, y) - j_z(x', y')|\, d|\mu_N|(z) \leq \varepsilon|\mu_N|(\Gamma). \end{aligned}$$

This proves the equicontinuity of $\{g_{\mu_N}(x, y)\}$. □

Lemma 11. *Given any convergent sequence $\{h_N\} \subset \mathcal{H}(i)$ with limit function h such that $h_N \in \mathcal{H}_N(i)$ for each N, we find that $\varphi_N(h_N) \to \varphi_\mu(h)$ uniformly on Γ.*

Proof. We use Lemma 8 again. First we show pointwise convergence. Fix $x \in \Gamma$ and $\varepsilon > 0$. As $g_{\mu_N} \to g_\mu$ uniformly and $h_N \to h$ uniformly, we can suppose that N is large enough to guarantee $g_{\mu_N} h_N$ is uniformly within ε of $g_\mu h$ for all $x, y \in \Gamma$. We can also suppose that N is so large that $\int_\Gamma g_\mu(x, y) h(y) dy_N$ is within ε of $\int_\Gamma g_\mu(x, y) h(y) dy$. Now we have

$$\begin{aligned} |\varphi_N\{h_N\}(x) - \varphi_\mu\{h\}(x)| &= \left| \int_\Gamma g_{\mu_N}(x, y) h_N(y) dy_N - \int_\Gamma g_\mu(x, y) h(y) dy \right| \\ &\leq \left| \int_\Gamma g_{\mu_N}(x, y) h_N(y) dy_N - \int_\Gamma g_\mu(x, y) h(y) dy_N \right| \\ &\qquad + \left| \int_\Gamma g_\mu(x, y) h(y) dy_N - \int_\Gamma g_\mu(x, y) h(y) dy \right| \\ &< \int_\Gamma |g_{\mu_N}(x, y) h_N(y) - g_\mu(x, y) h(y)|\, dy_N + \varepsilon \\ &< 2\varepsilon. \end{aligned}$$

Hence pointwise convergence holds.

Now we wish to show equicontinuity of the sequence of functions $\{\varphi_N(h_N)\}$. The sequence $\{g_{\mu_N}(x,y)\}$ is equicontinuous on Γ^2 by the proof of Lemma 10, and the sequence $\{h_N\}$ is uniformly bounded by some positive constant M. Thus we have

$$|\varphi_N\{h_N\}(x) - \varphi_N\{h_N\}(x')| \leq M \int_\Gamma |g_{\mu_N}(x,y) - g_{\mu_N}(x',y)| dy_N,$$

and the integrand can be made arbitrarily small by taking x close to x'. □

Lemma 12. *Suppose that $\{h_N\} \subset \mathcal{H}(i)$ is any convergent sequence with limit function h such that $h_N \in \mathcal{H}_N(i)$ for each N. For each $\varepsilon > 0$ there exists a real number N_2 such that for all $N > N_2$, we have simultaneously:*

- $\|\varphi_\mu(h)\|_{L^2} > \|\varphi_N(h_N)\|_{\ell^2} - \varepsilon$.
- $\|h\|_{L^2} < \|h_N\|_{\ell^2} + \varepsilon$.

Proof. Observe that $h_N \to h$ uniformly, and so $\varphi_N(h_N) \to \varphi_\mu(h)$ uniformly by Lemma 11. Now use Lemma 5 to assert that for all N large, $\|h_N\|_{\ell^2}$ is within ε of $\|h\|_{L^2}$ and $\|\varphi_N(h_N)\|_{\ell^2}$ is within ε of $\|\varphi_\mu(h)\|_{L^2}$. □

Lemma 13. *Suppose $h : \Gamma \to \mathbb{C}$ is a continuous function such that $\int_\Gamma h du = 0$ and h is L^2-orthogonal to all of the eigenfunctions of Δ with associated eigenvalues $\lambda_1(\Gamma), \ldots, \lambda_{i-1}(\Gamma)$. For each $\varepsilon > 0$, there exists $\widetilde{h} \in \mathrm{Zh}_\mu(\Gamma)$ that satisfies the following two conditions:*

(i) *$\widetilde{h}$ is L^2-orthogonal to all of the eigenfunctions of Δ with associated eigenvalues $\lambda_1(\Gamma), \ldots, \lambda_{i-1}(\Gamma)$.*

(ii) $\frac{\|\varphi_\mu(\widetilde{h})\|_{L^2}}{\|\widetilde{h}\|_{L^2}} > \frac{\|\varphi_\mu(h)\|_{L^2} - \varepsilon}{\|h\|_{L^2} + \varepsilon}$.

Proof. Using the vertex set X, we may decompose Γ into a finite number of segments $\{e_1, \ldots, e_r\}$, each isometric to a closed interval. As h is continuous, we can apply the Stone–Weierstrass theorem to each segment of Γ to get uniform polynomial approximations of $h|_{e_j}$ with as great an accuracy as we desire. Moreover, we may define our approximations so that their values agree with the values of h at the endpoints of each segment, and hence we may glue our approximations together to get a uniform approximation of h. Evidently such approximations must lie in $\mathrm{Zh}(\Gamma)$.

Let us perform this procedure to construct a sequence of functions $\{f_n\} \subset \mathrm{Zh}(\Gamma)$ that converges uniformly to h. Let $\xi_1, \ldots, \xi_p$ be eigenfunctions of the Laplacian Δ that form an L^2-orthonormal basis for the direct sum of the eigenspaces associated with the eigenvalues $\lambda_1(\Gamma), \ldots, \lambda_{i-1}(\Gamma)$. Now set

$$H_n(x) = f_n(x) - \sum_{j=1}^{p} \langle f_n, \xi_j \rangle_{L^2} \xi_j(x) - \int_\Gamma f_n d\mu.$$

As each ξ_j is orthogonal to constant functions (constants are eigenfunctions of φ_μ with eigenvalue 0), we see that $H_n \in \mathrm{Zh}_\mu(\Gamma)$ for all n and condition (i) of the lemma is satisfied for all H_n. Since f_n tends uniformly to h, we see that the inner products $\langle f_n, \xi_j \rangle_{L^2}$ tend to zero as n tends to infinity. By dominated convergence, the integral $\int_\Gamma f_n d\mu$ tends to zero as well. Thus H_n converges uniformly to h on Γ.

By Lemma 9, we know that $\varphi_\mu(H_n) \to \varphi_\mu(h)$ uniformly. Since the L^2-norm respects this convergence, for any $\varepsilon > 0$ we may take n sufficiently large and set $\widetilde{h} = H_n$ to obtain condition (ii). □

The eigenvalues of φ_μ constitute a sequence of positive numbers tending to zero. If we label these distinct values by $\alpha_1(\Gamma) > \alpha_2(\Gamma) > \alpha_3(\Gamma) > \cdots$, then $\alpha_i(\Gamma) = 1/\lambda_i(\Gamma)$ for all i (see Theorem 2). We have the following classical characterization of the ith eigenvalue of φ_μ:

Proposition 5. *Let $\mathcal{S}(i)$ denote the span of the eigenfunctions of Δ with respect to the measure μ that are associated to the eigenvalues $\lambda_1(\Gamma), \ldots, \lambda_{i-1}(\Gamma)$. If $\mathcal{S}(i)^\perp$ is the L^2-orthogonal complement of this space inside* $\mathrm{Zh}_\mu(\Gamma)$*, then we find that*

$$\alpha_i(\Gamma) = \sup_{\substack{F \in \mathcal{S}(i)^\perp \\ F \neq 0}} \frac{\|\varphi_\mu(F)\|_{L^2}}{\|F\|_{L^2}}. \tag{13}$$

Moreover, there exists an eigenfunction $F \in \mathcal{S}(i)$ which realizes this supremum.

Proof. In Section 8 of [BR], it is proved that φ_μ is a compact Hermitian operator on $L^2(\Gamma)$; hence, by the spectral theorem, the eigenfunctions of φ_μ form an orthonormal basis for $L^2(\Gamma)$. In Section 15 of [BR] it is shown that the eigenfunctions of φ_μ must lie in $\mathrm{Zh}_\mu(\Gamma)$. We may now identify $\mathcal{S}(i)^\perp$ with a subspace W of $L^2(\Gamma)$ that is orthogonal to all of the eigenfunctions of Δ associated to the eigenvalues $\lambda_1(\Gamma), \ldots, \lambda_{i-1}(\Gamma)$; moreover, we see that φ_μ maps this subspace into itself.

The space $\mathrm{Zh}(\Gamma)$ is dense in $L^2(\Gamma)$ because, for example, smooth functions are dense in L^2. Thus the supremum on the right side of Equation (13) is nothing more than the operator norm of $\varphi_\mu|_W$. By [Yng] Theorem 8.10, we find that either $\|\varphi_\mu|_W\|$ or $-\|\varphi_\mu|_W\|$ is an eigenvalue of φ_μ acting on W. The nonzero eigenvalues of φ_μ are all positive, so we may eliminate the latter possibility. We conclude that there exists an eigenfunction $F \in W$ with associated eigenvalue α for which $\|\varphi_\mu|_W\| = \|\varphi_\mu(F)\|_{L^2}/\|F\|_{L^2} = \alpha$. Evidently $\alpha = \alpha_i(\Gamma)$ since the operator norm is defined as a supremum and $\alpha_i(\Gamma)$ is the largest eigenvalue of φ_μ acting on $\mathcal{S}(i)^\perp$. □

7. Proofs of assertions (A), (B′), and (C′)

Let $\alpha_{1,N} > \alpha_{2,N} > \alpha_{3,N} > \cdots$ denote the nonzero eigenvalues of the operator φ_N described in §3. There we saw that for each i, the eigenvalues of φ_N are related to the eigenvalues of Q_N with respect to the measure μ_N via $N\lambda_{i,N} = 1/\alpha_{i,N}$.

Recall that we are assuming assertions (A), (B′), and (C′) of the Reduction Lemma hold up to the case $i - 1$.

Theorem 6 (Eigenvalue Convergence: Part 2)**.** *We have*

$$\lambda_i(\Gamma) \leq \liminf_{N\to\infty} N\lambda_{i,N}.$$

Proof. Suppose the theorem is false. As $N\lambda_{i,N} = 1/\alpha_{i,N}$ and $\lambda_i(\Gamma) = 1/\alpha_i(\Gamma)$, our supposition is equivalent to $\alpha_i(\Gamma) < \limsup_{N\to\infty} \alpha_{i,N}$. We may pick a subsequence of $\{G_N\}$ and a positive constant η so that $\alpha_i(\Gamma) + \eta < \alpha_{i,N}$ for each N in the subsequence. For each such N, we also pick $h_N \in \mathcal{H}_N(i)$. By normality of $\mathcal{H}(i)$, upon passage to a further subsequence we may assume that $\{h_N\}$ tends uniformly to some continuous function h as N goes to infinity through this subsequence.

Fix $\varepsilon > 0$ and select $\widetilde{h}$ as in Lemma 13. For N large, the conclusions of Lemma 12 are true. It now follows that for any N sufficiently huge in our subsequence, we have

$$\begin{aligned}
\alpha_i(\Gamma) &= \sup_{\substack{F\in\mathcal{S}(i)^\perp \\ F\neq 0}} \frac{\|\varphi_\mu(F)\|_{L^2}}{\|F\|_{L^2}} && \text{by Lemma 5,} \\
&\geq \frac{\|\varphi_\mu(\widetilde{h})\|_{L^2}}{\|\widetilde{h}\|_{L^2}}, && \text{since } \widetilde{h} \in \mathcal{S}(i)^\perp, \\
&> \frac{\|\varphi_\mu(h)\|_{L^2} - \varepsilon}{\|h\|_{L^2} + \varepsilon} && \text{by Lemma 13,} \\
&\geq \frac{\|\varphi_N(h_N)\|_{\ell^2} - 2\varepsilon}{\|h_N\|_{\ell^2} + 2\varepsilon} && \text{by Lemma 12,} \\
&= \frac{\alpha_{i,N}\|h_N\|_{\ell^2} - 2\varepsilon}{\|h_N\|_{\ell^2} + 2\varepsilon}, && \text{since } h_N \text{ is an eigenfunction of } \varphi_N, \\
&= \frac{\alpha_{i,N} - 2\varepsilon}{1 + 2\varepsilon} \\
&> \frac{\alpha_i(\Gamma) + \eta - 2\varepsilon}{1 + 2\varepsilon}.
\end{aligned}$$

This furnishes us with a contradiction when ε is sufficiently small. □

Clearly the amalgamation of Theorems 4 and 6 prove that assertion (A) of our Main Theorem holds for i. As for assertion (B′) and the remainder of assertion (C′) in the Reduction Lemma, we have the following two corollaries.

Corollary 5. *The subsequential limits of $\mathcal{H}(i)$ are eigenfunctions of Δ with associated eigenvalue $\lambda_i(\Gamma)$. The limits have unit L^2-norm and lie in $\mathrm{Zh}_\mu(\Gamma)$.*

Proof. Let $\{h_N\}$ be a convergent subsequence of $\mathcal{H}(i)$ with limit function h. We know that $\varphi_N(h_N)(x) = \frac{1}{N\lambda_{i,N}} h_N(x)$ for all $x \in \Gamma$. Now let N go to infinity and apply Lemma 11 and assertion (A) to see that $\varphi_\mu(h) = \frac{1}{\lambda_i(\Gamma)} h$. Hence h is an eigenfunction of φ_μ. By Theorem 12.1 and Proposition 15.1 in [BR] we see that h is an eigenfunction of Δ lying in $\mathrm{Zh}_\mu(\Gamma)$. Lemma 5 shows $\|h\|_{L^2} = 1$. □

Corollary 6. *For N sufficiently large, it is true that $d_{i,N} \leq d_i(\Gamma)$.*

Proof. Set $d = d_i(\Gamma)$. Suppose that that $d_{i,N} > d$ for some infinite subsequence of $\{G_N\}$. For each such N, select distinct functions $h_N^1, \ldots, h_N^{d+1} \in \mathcal{H}_N(i)$. Using the normality of $\mathcal{H}(i)$, we may pass to a further subsequence and assume that there are $d+1$ limit functions $h^1, \ldots, h^{d+1}$, with $h_N^j \to h^j$ uniformly along this subsequence for each j. By Corollary 5, we find that each h^j is an eigenfunction of Δ with associated eigenvalue $\lambda_i(\Gamma)$. But the functions $h_N^1, \ldots, h_N^{d+1}$ are ℓ^2-orthogonal for each N, and Lemma 5 implies that the limit functions must be L^2-orthogonal. As the eigenspace corresponding to the eigenvalue $\lambda_i(\Gamma)$ has dimension d, this is an absurdity. □

We have now exhibited the induction step in our proof of the Main Theorem; i.e., the proof is complete.

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