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Dimension groups for interval maps

Fred Shultz

Abstract. With each piecewise monotonic map τ of the unit interval, a dimension triple is associated. The dimension triple, viewed as a $\mathbb{Z}[t, t^{-1}]$ module, is finitely generated, and generators are identified. Dimension groups are computed for Markov maps, unimodal maps, multimodal maps, and interval exchange maps. It is shown that the dimension group defined here is isomorphic to $K_0(A)$, where A is a C*-algebra (an "AI-algebra") defined in dynamical terms.

Contents

1. Introduction

Given a piecewise monotonic map τ of the unit interval into itself, our goal is to associate a dimension group $\mathrm{DG}(\tau)$, providing an invariant for the original map.

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In a dynamical context, dimension groups were introduced by Krieger [21]. Motivated by Elliott's classification of AF-algebras by their dimension groups [12], he gave a purely dynamical definition of the dimension group, via an equivalence relation defined by the action of an "ample" group of homeomorphisms on the space of closed and open subsets of a zero-dimensional metric space. For a shift of finite type, he associated an ample group, and in [22], showed that the dimension triple (consisting of an ordered dimension group with a canonical automorphism) completely determines an irreducible aperiodic shift of finite type up to shift equivalence.

Krieger's definition of a dimension group was extended to the context of a surjective local homeomorphism $\sigma : X \to X$, where X is a compact zero dimensional metric space, by Boyle, Fiebig, and Fiebig ([1]), who defined a dimension group called the "images group", and used this group as a tool in studying commuting local homeomorphisms.

We sketch the construction of the dimension group in [1]. If $\sigma : X \to X$, the transfer map $\mathcal{L} : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$ is defined by

$$\mathcal{L}f(x) = \sum_{\sigma y = x} f(y). \tag{1.1}$$

Then G_σ is defined to be the set of equivalence classes of functions in $C(X,\mathbb{Z})$, where $f \sim g$ if $\mathcal{L}^n f = \mathcal{L}^n g$ for some $n \geq 0$. Addition is given by $[f] + [g] = [f + g]$, and the positive cone consists of classes $[f]$ such that $\mathcal{L}^n f \geq 0$ for some $n \geq 0$. This is the same as defining clopen sets E, F in X to be equivalent if for some $n \geq 0$, σ^n is 1-1 on E and F, and $\sigma^n(E) = \sigma^n(F)$, and then building a group out of these equivalence classes by defining $[E] + [F] = [E \cup F]$ when E, F are disjoint. This latter definition is the one given in [1]; the authors then observe that it is equivalent to the definition above involving the transfer operator. If σ is surjective, this dimension group is the same as the stationary inductive limit of $\mathcal{L} : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$, cf. Lemma 3.8 and Equation (3.8) in the current paper.

A related approach was taken by Renault in [27]. If $X_1, X_2, \dots$ are compact metric spaces, and $T_n : C(X_n) \to C(X_{n+1})$ is a sequence of positive maps, Renault defined the associated dimension group to be the inductive limit of this sequence. If $X_n = X$ and $T_n = \mathcal{L}$ for all n, for a surjective local homeomorphism σ, the resulting dimension group is formally similar to that in [1]. However, the use of $C(X)$ instead of $C(X,\mathbb{Z})$ results in different dimension groups.

These definitions of dimension groups aren't directly applicable to interval maps, since such maps are rarely local homeomorphisms. We therefore associate with each piecewise monotonic map $\tau : [0,1] \to [0,1]$ a local homeomorphism. This is done by disconnecting the unit interval at a countable set of points, yielding a space X, and then lifting τ to a local homeomorphism $\sigma : X \to X$. The properties of σ are closely related to those of τ. A similar technique has long been used in studying interval maps, e.g., cf. [18, 14, 29, 32].

If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the associated local homeomorphism, the space X will be a compact subset of $\mathbb{R}$, but will not necessarily be zero-dimensional. However the map σ will have the property that each point has a clopen neighborhood on which σ is injective, and this allows us to define a dimension group in the same way as in [1]. We then define $\mathrm{DG}(\tau)$ to be the dimension group G_σ associated with σ.

The result is a triple $(\mathrm{DG}(\tau), \mathrm{DG}(\tau)^+, \mathcal{L}_*)$, where $\mathrm{DG}(\tau)$ is a dimension group with positive cone $\mathrm{DG}(\tau)^+$, and $\mathcal{L}_*$ is a positive endomorphism of $\mathrm{DG}(\tau)$, which will be an order automorphism if, for example, τ is surjective. In this case, $\mathrm{DG}(\tau)$ can be viewed as a $\mathbb{Z}[t, t^{-1}]$ module.

We now summarize this paper. Sections 2–3 lead up to the definition of the dimension group for a piecewise monotonic map (Definition 3.13). Sections 4–6 develop basic properties of the dimension group of a piecewise monotonic map, for example, characterizing when they are simple, and describing a canonical set of generators for the dimension module (Theorem 6.2), which is quite useful in computing dimension groups. Sections 7–11 compute the dimension group for various families of interval maps, some of which are sketched in Figure 1. Section 12 gives a dynamical description of a C*-algebra A_τ such that $K_0(A_\tau)$ is isomorphic to $\mathrm{DG}(\tau)$.

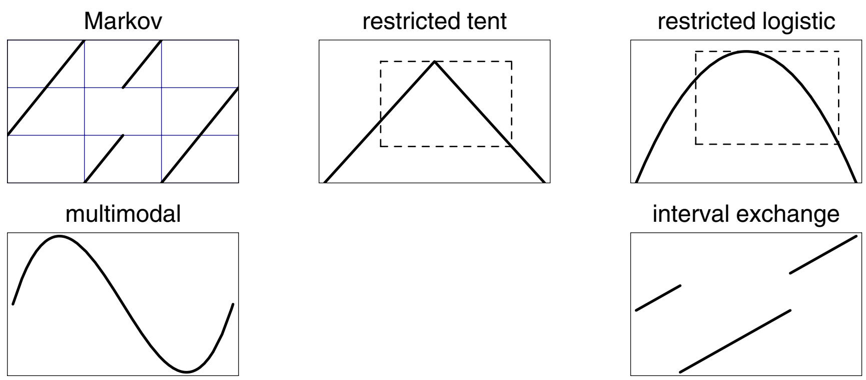

FIGURE 1. Examples of piecewise monotonic maps

This paper initiates a program to make use of dimension groups (and more generally, C*-algebras and their K-theory) to find invariants for interval maps. In [30], we investigate dimension groups for transitive piecewise monotonic maps. For such maps, we describe the order on the dimension group in concrete terms, and show that for some families of maps, the dimension triple is a complete invariant for conjugacy.

In the paper [8], two C*-algebras F_τ and O_τ are associated with piecewise monotonic maps, and the properties of these algebras are related to the dynamics. These algebras are analogs of the algebras F_A and O_A associated with shifts of finite type by Cuntz and Krieger [6], and F_τ is isomorphic to the algebra A_τ defined in Section 12 of this paper.

2. Piecewise monotonic maps and local homeomorphisms

Let $I = [0, 1]$. A map $\tau : I \to I$ is *piecewise monotonic* if there are points $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\tau|_{(a_{i-1}, a_i)}$ is continuous and strictly monotonic for $1 \le i \le n$. We denote by $\tau_i : [a_{i-1}, a_i] \to I$ the unique continuous extension of $\tau|_{(a_{i-1}, a_i)}$; note that τ_i will be a homeomorphism onto its range. We will refer to the ordered set $C = \{a_0, a_1, \ldots, a_n\}$ as the *partition associated with*

τ, or as the endpoints of the intervals of monotonicity. We will say this partition is *maximal* if the intervals (a_{i-1}, a_i) are the largest open intervals on which τ is continuous and strictly monotonic. For each piecewise monotonic function there is a unique maximal partition. We do not assume a partition associated with τ is maximal unless that is specifically stated. (We will be primarily interested in maximal partitions, but there will be a few cases where nonmaximal partitions are useful.)

Our goal in this section is to modify τ slightly so that it becomes a local homeomorphism σ on a larger space X, in such a way that properties of (I, τ) and (X, σ) are closely related. In general, points in the partition C cause trouble, since τ is usually neither locally injective nor an open map at such points, and may be discontinuous. We will disconnect I at points in C (and at points in their forward and backward orbit).

We have to be a little careful about the values of τ at points c in C where τ is discontinuous: what are relevant for our purposes are the left and right limits $\lim_{x\to c^\pm} \tau(x)$, not the actual values $\tau(c)$. If $\tau : I \to I$ is piecewise monotonic, we define a (possibly multivalued) function $\widehat{\tau}$ on I by setting $\widehat{\tau}(x)$ to be the set of left and right limits of τ at x. At points where τ is continuous, $\widehat{\tau}(x) = \{\tau(x)\}$, and we identify $\widehat{\tau}(x)$ with $\tau(x)$. If $A \subset I$, then $\widehat{\tau}(A) = \cup_{x\in A}\widehat{\tau}(x)$, and $\widehat{\tau}^{-1}(A) = \{x \in I \mid \widehat{\tau}(x) \cap A \neq \emptyset\}$. Observe that $\widehat{\tau}$ is a closed map, i.e., if A is a closed subset of I, then $\widehat{\tau}(A)$ is closed, since $\widehat{\tau}(A) = \cup_i \tau_i(A)$.

The *generalized orbit* of C is the smallest subset I_1 of I containing C and closed under $\widehat{\tau}$ and $\widehat{\tau}^{-1}$. This is the same as the smallest subset of I closed under τ_i and τ_i^{-1} for $1 \le i \le n$. We define $I_0 = I \setminus I_1$. Next we describe the space resulting from disconnecting I at points in I_1. (The term "disconnecting" we have borrowed from Spielberg [31].)

Definition 2.1. Let $I = [0, 1]$, and let I_0, I_1 be as above. The *disconnection of I at points in I_1* is the totally ordered set X which consists of a copy of I with the usual ordering, but with each point $x \in I_1 \setminus \{0, 1\}$ replaced by two points $x^- < x^+$. We equip X with the order topology, and define the *collapse map* $\pi : X \to I$ by $\pi(x^\pm) = x$ for $x \in I_1$, and $\pi(x) = x$ for $x \in I_0$. We write $X_1 = \pi^{-1}(I_1)$, and $X_0 = \pi^{-1}(I_0) = X \setminus X_1$.

If $x \in I_1$, then x^- is the smallest point in $\pi^{-1}(x)$, and x^+ is the largest. For $x \in I_0$, it will be convenient to define $x^- = x^+ = \widehat{x}$, where $\widehat{x}$ is the unique preimage under π of x. For any pair $a, b \in X$, we write $[a, b]_X$ for the order interval $\{x \in X \mid a \le x \le b\}$. More generally, if J is any interval in $\mathbb{R}$, then we write J_X instead of $J \cap X$.

Proposition 2.2. *Let X be the disconnection of I at points in I_1, and $\pi : X \to I$ the collapse map. Then X is homeomorphic to a compact subset of $\mathbb{R}$, and:*

1. *π is continuous and order preserving.*
2. *I_0 is dense in I, and X_0 is dense in X.*
3. *$\pi|_{X_0}$ is a homeomorphism from X_0 onto I_0.*
4. *X has no isolated points.*
5. *If $a, b \in I_1$, then $[a^+, b^-]_X$ is clopen in X, and every clopen subset of X is a finite disjoint union of such order intervals.*

Proof. Fix a listing of the elements of I_1, say $I_1 = \{x_1, x_2, \dots\}$. Define $\widehat{I}$ to be I with each point x_k in $I_1 \setminus \{0,1\}$ replaced by a pair of points $x_k^- < x_k^+$ and a gap of length 2^{-k} inserted between these points. Then $\widehat{I}$ is a compact subset of $\mathbb{R}$, and the order topology on $\widehat{I}$ coincides with the topology inherited from $\mathbb{R}$. Clearly $\widehat{I}$ is order isomorphic to the set X in Definition 2.1. (Hereafter we will identify X with $\widehat{I}$.)

(1) The inverse image of each open interval under π is an open order interval in X, so π is continuous. It follows at once from the definitions of π, and of the order on X, that π is order preserving.

(2) The complement of any finite subset of I is dense in I. By the Baire category theorem, since I_1 is countable, then I_0 is dense in I, and this implies density of X_0 in X.

(3) To prove that π restricted to X_0 is a homeomorphism, let B be a closed subset of X. We will show $\pi(B \cap X_0) = \pi(B) \cap I_0$. If $x \in \pi(B) \cap I_0$, then x has a preimage in B, and since $x \in I_0$, by definition of X_0, that preimage is also in X_0. Thus $\pi(B) \cap I_0 \subset \pi(B \cap X_0)$, and the opposite containment is clear. Hence $\pi(B \cap X_0)$ is closed in I_0, so $\pi|_{X_0}$ is a continuous, bijective, closed map from X_0 onto X_0, and thus is a homeomorphism.

(4) Since I_0 is infinite and dense in I, no point of I_0 is isolated in I_0, so by (3) no point of X_0 is isolated in X_0. By density of X_0 in X, no point of X is isolated.

(5) Observe that if $a \in I_1 \setminus \{1\}$, then a^+ is not a limit from the left, and if $b \in I_1 \setminus \{0\}$, then b^- is not a limit from the right. Thus for $a, b \in I_1$ with $a < b$, the set $[a^+, b^-]_X$ is clopen. Now let E be any clopen subset of X, and let $x \in E$. Since E is clopen, for each point x in E we can find an open order interval containing x and contained in E. The union of these intervals forms an open cover of the compact set E, so there is a finite subcover. The union of overlapping open order intervals is again an open order interval, so the intervals in this subcover can be expressed as a finite disjoint union of open order intervals. By construction, not only do these cover E, but they are contained in E, so E is equal to their disjoint union. Since each of these intervals is open, and each is the complement (in E) of the union of the others, each is also closed. Thus E is a disjoint union of a finite collection of clopen order intervals.

We will be done if we show that for each clopen order interval $[c, d]_X$, there exist $a, b \in X_1$ with $c = a^+$ and $d = b^-$. By density of I_0 in I, each point of I_0 is a limit from both the left and right of sequences from I_0, so each point of X_0 is a limit from both the left and right of sequences from X_0. Since $[c, d]_X$ is open, c cannot be a limit from the left, and d cannot be a limit from the right, so neither c nor d is in X_0. If $a \in I_1 \setminus \{0\}$ and $c = a^-$, then c would be a left limit, which is impossible. Thus $c = a^+$ for some $a \in I_1$, and similarly $d = b^-$. $\square$

We now turn to constructing a local homeomorphism $\sigma : X \to X$. Recall that if $a_0 < a_1 < \cdots < a_n$ is the partition associated with a piecewise monotonic map τ, we denote by τ_i the extension of $\tau|_{(a_{i-1}, a_i)}$ to a homeomorphism on $[a_{i-1}, a_i]$.

Theorem 2.3. *Let $\tau : I \to I$ be a piecewise monotonic map, $C = \{a_0, a_1, \dots, a_n\}$ the endpoints of the intervals of monotonicity, I_1 the generalized orbit of C, and X the disconnection of I at the points of I_1.*

(1) *The sets* $\widehat{J}_i = [a_{i-1}^+, a_i^-]_X$ *for* $i = 1, \dots, n$ *form a partition of* X *into clopen sets.*
(2) *There is a unique continuous map* $\sigma : X \to X$ *such that* $\pi \circ \sigma = \tau \circ \pi$ *on* X_0.
(3) *For* $1 \le i \le n$, σ *is a monotone homeomorphism from* $\widehat{J}_i$ *onto the clopen order interval with endpoints* $\sigma(a_{i-1}^+)$ *and* $\sigma(a_i^-)$.
(4) *For* $1 \le i \le n$, $\pi \circ \sigma = \tau_i \circ \pi$ *on* $\widehat{J}_i$.

Proof. For $1 \le i \le n$, let $J_i = [a_{i-1}, a_i]$, and define $\widehat{J}_i$ as in (1) above. By Proposition 2.2, the order intervals $\widehat{J}_1, \dots, \widehat{J}_n$ are clopen and form a partition of X.

Now we define σ on $\widehat{J}_1, \dots, \widehat{J}_n$. Fix i, and recall that τ_i is strictly monotonic. We will assume that τ_i is increasing; the decreasing case is similar. Then τ_i is a homeomorphism from J_i onto $[\tau_i(a_{i-1}), \tau_i(a_i)]$.

Since $\pi(\widehat{J}_i) = J_i$, each point in $\widehat{J}_i$ has the form x^+ or x^- for some $x \in J_i$. If $y \in \widehat{J}_i$ and $y = x^\pm$, then we define $\sigma(y) = \sigma(x^\pm) = \tau_i(x)^\pm$. This is well-defined, since if $y = x^+ = x^-$, then $x \in I_0$, so $\tau_i(x) \in I_0$, and thus $\tau_i(x)^+ = \tau_i(x)^-$. It follows that σ is an order isomorphism from $\widehat{J}_i$ onto $[\tau_i(a_{i-1})^+, \tau_i(a_i)^-]_X$. Since the topology on compact subsets of $\mathbb{R}$ coincides with the order topology, σ is an increasing homeomorphism from $\widehat{J}_i$ onto $\sigma(\widehat{J}_i)$, so (3) holds. Since $\tau_i(I_1) \subset I_1$, then by Proposition 2.2, $\sigma(\widehat{J}_i) = [\tau_i(a_{i-1})^+, \tau_i(a_i)^-]_X$ is clopen. Since each order interval $\widehat{J}_i$ is clopen, then σ is continuous on all of X. From the definition of σ, $\pi \circ \sigma = \tau \circ \pi$ on X_0, and so by continuity of σ and density of X_0, (2) follows. Now (4) is immediate. $\square$

From the proof above, if τ is increasing on the interval $J_i = [a_{i-1}, a_i]$, and if $c, d \in J_i \cap I_1$ with $c < d$, then

$$\sigma([c^+, d^-]_X) = [\tau_i(c)^+, \tau_i(d)^-]_X, \tag{2.1}$$

while if τ decreases on J_i then

$$\sigma([c^+, d^-]_X) = [\tau_i(d)^+, \tau_i(c)^-]_X. \tag{2.2}$$

Corollary 2.4. $\pi|_{X_0}$ *is a conjugacy from* (X_0, σ) *to* (I_0, τ).

Proof. By Proposition 2.2, $\pi|_{X_0}$ is a homeomorphism onto I_0, and by Theorem 2.3, π intertwines (X_0, σ) and (I_0, τ). $\square$

The map σ in Theorem 2.3 is a local homeomorphism, i.e., it is an open map, and each point admits an open neighborhood on which σ is a homeomorphism.

Definition 2.5. We will call the map $\sigma : X \to X$ in Theorem 2.3 the *local homeomorphism associated with* τ.

Example 2.6. Let $\tau : [0,1] \to [0,1]$ be the full tent map given by $\tau(x) = 1 - |2x - 1|$. Here the set C of endpoints of intervals of monotonicity is $\{0, 1/2, 1\}$. This set is invariant under τ, so the generalized orbit I_1 is $\cup_{n \ge 0} \tau^{-n}(C)$, which is just the set of dyadic rationals. The order intervals $\widehat{J}_1, \widehat{J}_2$ defined in Theorem 2.3 satisfy $\sigma(\widehat{J}_1) = \sigma(\widehat{J}_2) = X$. For $x \in X$, let $S(x) = s_0 s_1 s_2 \dots$, where $s_n = i$ if $\sigma^n x \in \widehat{J}_i$. Then S is a conjugacy from (X, σ) onto the full (one-sided) 2-shift.

For a general piecewise monotonic map $\tau : I \to I$, the associated local homeomorphism $\sigma : X \to X$ will not be a shift of finite type, nor even a subshift. For example, for a logistic map $\tau(x) = kx(1-x)$ with an attractive fixed point, X will contain nontrivial connected components.

In the remainder of this section, we will see that properties of a piecewise monotonic map $\tau : I \to I$ and the associated local homeomorphism σ are closely related. We illustrate this relationship for transitivity.

Definition 2.7. If X is any topological space, and $f : X \to X$ is a continuous map, then f is *transitive* if for each pair U, V of nonempty open sets, there exists $n \geq 0$ such that $f^n(U) \cap V \neq \emptyset$. We say f is *strongly transitive* if for every nonempty open set U, there exists n such that $\cup_{k=0}^{n} f^k(U) = X$.

Definition 2.8. If $\tau : I \to I$ is piecewise monotonic, we view τ as undefined at the set C of endpoints of intervals of monotonicity, and say τ is transitive if for each pair U, V of nonempty open sets, there exists $n \geq 0$ such that $\tau^n(U) \cap V \neq \emptyset$. We say τ is *strongly transitive* if for every nonempty open set U, there exists n such that $\cup_{k=0}^{n} \widehat{\tau}^k(U) = I$. (Recall that $\widehat{\tau}$ denotes the (possibly multivalued) function whose value at each point x is given by the left and right-hand limits of τ at x.)

It is well-known that for X compact metric with no isolated points, and $f : X \to X$ continuous, transitivity is equivalent to the existence of a point with a dense orbit. The same equivalence holds if $\tau : I \to I$ is piecewise monotonic, and is viewed as undefined on the set C of endpoints of intervals of monotonicity. (Apply the standard Baire category argument to show that transitivity implies the existence of a dense orbit, cf. e.g., [33, Theorem 5.9].)

If $\tau : I \to I$ is piecewise monotonic and continuous, then any dense orbit must eventually stay out of C. It follows that the definitions of transitivity in Definitions 2.7 and 2.8 are consistent. In addition, in this case $\tau = \widehat{\tau}$, so the definitions of strong transitivity also are consistent.

For general compact metric spaces X, transitive maps need not be strongly transitive. For example, the (two-sided) shift on the space of all biinfinite sequences of two symbols is transitive in the sense of Definition 2.7, but is not strongly transitive, since the complement of a fixed point is open and invariant. However, the next proposition shows that every piecewise monotonic transitive map is strongly transitive. (For continuous piecewise monotonic maps, this follows from [26, Thm. 2.5].) In the proof below, we will repeatedly use the fact that the collapse map $\pi : X \to I$, when restricted to X_0, is a conjugacy from (X_0, σ) onto (I_0, τ), cf. Corollary 2.4.

Proposition 2.9. *For a piecewise monotonic map $\tau : I \to I$, with associated local homeomorphism $\sigma : X \to X$, the following are equivalent:*

(1) *τ is transitive.*
(2) *τ is strongly transitive.*
(3) *σ is transitive.*
(4) *σ is strongly transitive.*

Proof. Assume τ is transitive. View τ as undefined on the set of endpoints of intervals of monotonicity. Let V be an open interval. Choose an open interval W such that $\overline{W} \subset V$. By [4, Proposition 2.6], or [15, Corollary on p. 382], there exists an N such that $\cup_0^\infty \tau^k(W) = \cup_0^N \tau^k(W)$. The right side is a finite union of open intervals. Since τ is transitive, the left side is dense, and thus $I \setminus \cup_0^N \tau^k(W)$ is finite. On the other hand, for each k, $\tau^k(W) \subset \widehat{\tau}^k(\overline{W}) \subset \widehat{\tau}^k(V)$. Since $\widehat{\tau}$ is a closed map, it follows that $\cup_0^N \widehat{\tau}^k(\overline{W}) = I$, so $\cup_0^N \widehat{\tau}^k(V) = I$. Thus τ is strongly transitive.

Strong transitivity of τ is equivalent to strong transitivity of $\tau|_{I_0}$. This in turn is equivalent to strong transitivity of $\sigma|_{X_0}$, and then to strong transitivity of σ on X. Finally, if σ is transitive on X, it is transitive on X_0, so τ is transitive on I_0, which in turn implies transitivity of τ on I. □

3. Dimension groups

In this section we will associate a dimension group $\mathrm{DG}(\tau)$ with each piecewise monotonic map τ. Our method will be to define a dimension group G_σ, where σ is the local homeomorphism associated with τ, and then define $\mathrm{DG}(\tau) = G_\sigma$. We start by defining a class of functions that includes the local homeomorphisms associated with piecewise monotonic maps, but also includes any local homeomorphism of a zero-dimensional metric space.

Definition 3.1. Let X be a topological space. A map $\sigma : X \to X$ is a *piecewise homeomorphism* if σ is continuous and open, and X admits a finite partition into clopen sets $X_1, X_2, \ldots, X_n$ such that σ is a homeomorphism from X_i onto $\sigma(X_i)$ for $i = 1, \ldots, n$.

If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the associated local homeomorphism, then σ is a piecewise homeomorphism, cf. Theorem 2.3. Every local homeomorphism σ on a compact zero-dimensional metric space X is a piecewise homeomorphism. In fact, each open neighborhood of a point in a zero-dimensional space contains a clopen neighborhood of that point, so there is a cover of X by clopen subsets on which σ is injective. Since X is compact, we can take a finite subcover, and the resulting partition that arises from all intersections of these subsets and their complements is the desired partition. Note that a composition of piecewise homeomorphisms is again a piecewise homeomorphism.

Hofbauer [15] and Keller [19] have studied the dynamics of maps that are *piecewise invertible*, i.e., that meet the requirements of a piecewise homeomorphism except for the requirement that the map be open.

We begin by defining G_σ as an ordered abelian group; later we will show G_σ is in fact a dimension group. Recall that an abelian group G is *ordered* if there is a subset G^+ such that $G = G^+ - G^+$, $G^+ \cap (-G^+) = \{0\}$ and $G^+ + G^+ \subset G^+$. Then for $x, y \in G$ with $y - x \in G^+$, we write $x \le y$. We write nx for the sum of n copies of x. An element u of an ordered abelian group G is an *order unit* if for each x in G there is a positive integer n such that $-nu \le x \le nu$.

The motivating idea for G_σ is to build a group out of equivalence classes of clopen subsets of X. As discussed in the introduction, this idea originated with Krieger [21], and was extended by Boyle–Fiebig–Fiebig [1]. In [1] a dimension group is associated with any surjective local homeomorphism on a zero-dimensional compact metric space. Our context is a slight generalization of that in [1], namely, piecewise homeomorphisms on an arbitrary compact metric space.

In what follows, a key role will be played by the *transfer map*. We will use this map both to define the dimension group for a piecewise homeomorphism, and to provide a key tool in computing this group.

Definition 3.2. Let X, Y be compact metric spaces. If $\sigma : Y \to X$ is any map which is finitely-many-to-one, the *transfer map* $\mathcal{L}_\sigma$ *associated with* σ is defined by

$$(\mathcal{L}_\sigma f)(x) = \sum_{\sigma(y)=x} f(y) \tag{3.1}$$

for each $f : Y \to \mathbb{C}$. (We will usually have $Y = X$.) We will write $\mathcal{L}$ instead of $\mathcal{L}_\sigma$ when the map σ intended is clear from the context.

We note some simple properties of $\mathcal{L}$, which are immediate consequences of the definition. If $f \geq 0$, then $\mathcal{L}f \geq 0$. If $f \geq 0$ and $\mathcal{L}f = 0$, then $f = 0$. If σ is injective on E, then

$$\mathcal{L}\chi_E = \chi_{\sigma(E)}. \tag{3.2}$$

If $\sigma : X \to X$ is a piecewise homeomorphism, then every function in $C(X, \mathbb{Z})$ can be written as a sum $\sum n_i \chi_{E_i}$ where σ is injective on E_i, so $\mathcal{L}$ maps $C(X, \mathbb{Z})$ into $C(X, \mathbb{Z})$. Observe that for $\sigma : X \to X$,

$$\mathcal{L}_{\sigma^n} = (\mathcal{L}_\sigma)^n. \tag{3.3}$$

If X is a compact metric space, we will write 1_X (or simply 1) for the function constantly 1 on X.

Definition 3.3. Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. Write $\mathcal{L}$ for $\mathcal{L}_\sigma$. We define an equivalence relation on $C(X, \mathbb{Z})$ by $f \sim g$ if $\mathcal{L}^n f = \mathcal{L}^n g$ for some $n \geq 0$, and write G_σ for the set of equivalence classes, and $G_\sigma^+ = \{[f] \mid f \geq 0\}$. We define addition on G_σ by $[f] + [g] = [f + g]$, and we order G_σ by $[f] \leq [g]$ if $[g - f] \in G_\sigma^+$. We call $[1_X]$ the *distinguished order unit* of G_σ.

It is easily verified that G_σ is an ordered abelian group. We will show G_σ is a dimension group, after developing some properties of the ordering on G_σ. When X is a zero-dimensional space and $\sigma : X \to X$ is a local homeomorphism, then G_σ is the same as the "images group" defined by Boyle–Fiebig–Fiebig in [1]. Our definition is different from theirs, but equivalent, as observed in [1, Rmk. 9.4].

Lemma 3.4. *Let X be a compact metric space and $\sigma : X \to X$ a piecewise homeomorphism. Then for each clopen subset F of $\sigma(X)$, there exists a clopen subset E of X such that σ is injective on E and $\sigma(E) = F$.*

Proof. Since σ is a piecewise homeomorphism, there is a partition $G_1, G_2, \ldots, G_n$ of X into clopen sets on which σ is injective. If $F \subset \sigma(G_i)$ for some i, then the set $E = G_i \cap \sigma^{-1}(F)$ has the desired properties. Otherwise, let $F_1, F_2, \ldots, F_k$ be the partition of F generated by the sets $\{F \cap \sigma(G_i) \mid 1 \leq i \leq n\}$. For each k, choose a clopen set E_k such that σ is injective on E_k and $\sigma(E_k) = F_k$. Then σ is injective on $E = E_1 \cup E_2 \cup \cdots E_k$, and $\sigma(E) = F$. $\square$

If $f \in C(X, \mathbb{Z})$, we define the *support* of f to be the set of x in X where f is nonzero, and denote this set by $\operatorname{supp} f$. Note that by the definition of $\mathcal{L}$ we have

$$\operatorname{supp} \mathcal{L}f \subset \sigma(\operatorname{supp} f), \tag{3.4}$$

with equality if $f \geq 0$. In particular, since $\mathcal{L}_\sigma^n = \mathcal{L}_{\sigma^n}$, we have

$$\operatorname{supp} \mathcal{L}^n f \subset \sigma^n(X). \tag{3.5}$$

A result similar to the following can be found in [1, Lemma 2.8].

Lemma 3.5. *Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. If $f \in C(X, \mathbb{Z})$, and the support of f is contained in $\sigma^n(X)$, then there exists $f_0 \in C(X, Z)$ such that $\mathcal{L}^n f_0 = f$. If $f \geq 0$, then f_0 can be chosen so that $f_0 \geq 0$.*

Proof. Let $f \in C(X, \mathbb{Z})$, with $\operatorname{supp} f \subset \sigma^n(X)$. Write $f = \sum_{i=1}^p n_i \chi_{E_i}$, with $E_1, \ldots, E_p$ disjoint, and $n_1, \ldots, n_p \in \mathbb{Z}$. We may assume that for each index i, $n_i \neq 0$. Then each E_i is contained in the support of f, and thus is contained in $\sigma^n(X)$. Applying Lemma 3.4 to σ^n, for each index i we can find a clopen set F_i such that σ^n is injective on F_i and $\sigma^n(F_i) = E_i$. Let $f_0 = \sum n_i \chi_{F_i}$. Then $\mathcal{L}^n f_0 = \sum n_i \chi_{\sigma^n(F_i)} = \sum n_i \chi_{E_i} = f$. If $f \geq 0$, we can arrange $n_i > 0$ for all i, so $f_0 \geq 0$. □

Lemma 3.6. *Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. In G_σ, $[f] \leq [g]$ iff $\mathcal{L}^n f \leq \mathcal{L}^n g$ for some $n \geq 0$.*

Proof. It suffices to show $[f] \geq [0]$ iff $\mathcal{L}^n f \geq 0$ for some $n \geq 0$. If $[f] \geq [0]$, then by definition there is $h \geq 0$ such that $f \sim h$, and therefore $\mathcal{L}^n f = \mathcal{L}^n h \geq 0$ for some $n \geq 0$. Conversely, fix n and suppose $\mathcal{L}^n f \geq 0$. By (3.5) and Lemma 3.5, there exists $h \geq 0$ such that $\mathcal{L}^n f = \mathcal{L}^n h$. Then $f \sim h$, so $[f] \geq [0]$. □

Definition 3.7. $\mathcal{L}_* : G_\sigma \to G_\sigma$ is defined by $\mathcal{L}_*[f] = [\mathcal{L} f]$.

This is an injective homomorphism. If σ is surjective, by Lemma 3.5 $\mathcal{L}$ is surjective, so $\mathcal{L}_*$ is surjective, and thus is an automorphism of G_σ. Note, however, that $\mathcal{L}_*$ is usually not unital.

If σ is injective on a clopen set E, then by (3.2)

$$\mathcal{L}_*[\chi_E] = [\chi_{\sigma(E)}]. \tag{3.6}$$

Notation. Let $G_1 \xrightarrow{T_1} G_2 \xrightarrow{T_2} G_3 \cdots$ be a sequence of ordered abelian groups with positive connecting homomorphisms. We review the construction of the inductive limit of this sequence. Let G_∞ be the set of sequences $(g_1, g_2, \ldots)$ such that $g_i \in G_i$ for each i, and such that there exists N such that $T_n g_n = g_{n+1}$ for all $n \geq N$. Make G_∞ into a group with coordinate wise addition. Consider two sequences (g_i), (h_i) to be equivalent if they eventually agree, i.e., if there exists N such that $g_i = h_i$ for $i \geq N$. The quotient group G is then an abelian group. Denote equivalence classes by square brackets. Define G^+ to be the set of equivalence classes of sequences that are eventually positive, and order G by $g \leq h$ if $h - g \in G^+$. Then G is the inductive limit in the category of ordered abelian groups. If G_1, G_2, … are identical, and T_1, T_2, … are the same map, we call G the *stationary* inductive limit with connecting maps $T_1 : G_1 \to G_1$.

Lemma 3.8. *Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. Then G_σ is isomorphic as an ordered group to the inductive limit*

$$C(X, \mathbb{Z}) \xrightarrow{\mathcal{L}} \mathcal{L}(C(X, \mathbb{Z}) \xrightarrow{\mathcal{L}} \mathcal{L}^2 C(X, \mathbb{Z}) \xrightarrow{\mathcal{L}} \cdots. \tag{3.7}$$

and this isomorphism carries the map $\mathcal{L}_*$ to the shift map

$$[(g_1, g_2, \dots)] \mapsto [(g_2, g_3, \dots)].$$

Proof. Let G be the inductive limit of this sequence. Every element of G has the form $[(0_0, 0_1, \dots, 0_{n-1}, \mathcal{L}^n g, \mathcal{L}^{n+1} g, \dots)] = [(g, \mathcal{L}g, \dots)]$, where 0_i denotes a zero in the i-th position. The map $\pi : G \to G_\sigma$ given by $\pi([(g, \mathcal{L}g, \dots)]) = [g]$ is an order isomorphism of G onto G_σ, and carries $\mathcal{L}_*$ to the shift map. ☐

If $\sigma : X \to X$ is surjective, then $\mathcal{L} : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ is also surjective (Lemma 3.5). Hence for surjective σ, the dimension group G_σ is isomorphic to the inductive limit of the sequence

$$C(X,\mathbb{Z}) \xrightarrow{\mathcal{L}} C(X,\mathbb{Z}) \xrightarrow{\mathcal{L}} C(X,\mathbb{Z}) \xrightarrow{\mathcal{L}} \cdots . \tag{3.8}$$

Lemma 3.9. *Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. For each $n \geq 0$, $\mathcal{L}^n C(X, \mathbb{Z})$ (with the order inherited from $C(X, \mathbb{Z})$) is isomorphic as an ordered group to $C(\sigma^n(X), \mathbb{Z})$.*

Proof. By (3.5), $\operatorname{supp} \mathcal{L}^n f \subset \sigma^n(X)$, so the map $f \mapsto f|_{\sigma^n(X)}$ is an order isomorphism from $\mathcal{L}^n C(X, \mathbb{Z})$ into $C(\sigma^n(X), \mathbb{Z})$. To prove this map is surjective, let $g \in C(\sigma^n(X), \mathbb{Z})$. Extend g to be zero on $X \setminus \sigma^n(X)$, so that g is in $C(X, \mathbb{Z})$ with support in $\sigma^n(X)$. There exists $f \in C(X, \mathbb{Z})$ with $L^n f = g$ by Lemma 3.5. Then $\mathcal{L}^n f|_{\sigma^n X} = g$, so $f \mapsto f|_{\sigma^n X}$ is surjective. ☐

Definition 3.10. An ordered abelian group G has the *Riesz decomposition property* if whenever $x_1, x_2, y_1, y_2 \in G$ with $x_i \leq y_j$ for $i, j = 1, 2$, then there exists $z \in G$ with $x_i \leq z \leq y_j$ for $i, j = 1, 2$. G is *unperforated* if $ng \geq 0$ for some $n > 0$ implies $g \geq 0$. A *dimension group* is a countable ordered abelian group G which is unperforated and which has the Riesz decomposition property.

A more common definition is that a dimension group is the inductive limit of a sequence of ordered groups Z^{n_i} (where Z^{n_i} is given the usual coordinate-wise order.) By a result of Effros, Handelman, and Shen [11], this is equivalent to the definition we have given. Two standard references on dimension groups are the books of Goodearl [13] and of Effros [10].

If G is a dimension group with a distinguished order unit u, then we refer to the pair (G, u) as a *unital dimension group*. A homomorphism between unital dimension groups (G_1, u_1), (G_2, u_2) is *unital* if it takes u_1 to u_2.

Below $C(X, \mathbb{Z})$ is viewed as an ordered abelian group with the usual addition and ordering of functions.

Lemma 3.11. *If X is a compact metric space, then $C(X, \mathbb{Z})$ is a dimension group, with order unit 1_X.*

Proof. We show G_σ is countable. Since X is a compact metric space, then the space $C(X)$ of continuous complex valued functions on X is a separable Banach space. Any two characteristic functions of clopen sets are a distance 2 apart in $C(X)$, so there are at most countably many clopen sets in X. Since every element of $C(X, \mathbb{Z})$ is a finite integral combination of characteristic subsets of clopen sets, then $C(X, \mathbb{Z})$ is also countable.

Suppose that f_i, g_j are in $C(X, \mathbb{Z})$ with $f_i \leq g_j$ for $i = 1, 2$, $j = 1, 2$. If $h = \max(f_1, f_2)$, then $f_i \leq h \leq g_j$ for $i = 1, 2$, $j = 1, 2$, so $C(X, \mathbb{Z})$ satisfies the Riesz decomposition property. It is evident that $C(X, \mathbb{Z})$ is unperforated. ☐

Corollary 3.12. *If X is a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism, then G_σ is a dimension group, with distinguished order unit $[1_X]$.*

Proof. Since $C(X,\mathbb{Z})$ is countable (cf. Lemma 3.11), and since G_σ is a quotient of $C(X,\mathbb{Z})$, then G_σ is countable. With the order inherited from $C(X,\mathbb{Z})$, $\mathcal{L}^n C(X,\mathbb{Z})$ is a dimension group, since it is isomorphic to the dimension group $C(\sigma^n(X),\mathbb{Z})$, cf. Lemma 3.9. Each of the properties defining a dimension group (being unperforated, and the Riesz decomposition property) are preserved by inductive limits, so by Lemma 3.8, G_σ is a dimension group. For any $f \in C(X,\mathbb{Z})$ there exists k such that $-k1_X \le f \le k1_X$, so $[1_X]$ is an order unit. □

Definition 3.13. If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the associated local homeomorphism, then $\mathrm{DG}(\tau)$ is defined to be the dimension group G_σ. Here we assume the partition C associated with τ is the maximal one; if not, then we instead write $\mathrm{DG}(\tau, C)$ for this dimension group.

4. Reduction to surjective maps

In this section, we will see that we can often reduce the computation of the dimension group $\mathrm{DG}(\tau)$ to the case where τ is surjective.

Definition 4.1. If X is a metric space and $\sigma : X \to X$ is continuous, we say σ is *eventually surjective* if $\sigma^{n+1}(X) = \sigma^n(X)$ for some $n \ge 0$. In this case we refer to $\sigma^n(X)$ as the *eventual range* of σ. If $\tau : I \to I$ is piecewise monotonic but not continuous, we say τ is eventually surjective if $\tau|_{I_0}$ is eventually surjective, and τ is surjective if $\widehat{\tau}$ is surjective (i.e., if $\widehat{\tau}(I) = I$).

If τ is piecewise monotonic and continuous, it is straightforward to check that the two definitions of eventual surjectivity in Definition 4.1 that are applicable are consistent.

Lemma 4.2. *If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the associated local homeomorphism, then σ is essentially surjective iff τ is essentially surjective, and σ is surjective iff τ is surjective.*

Proof. Since I_0 is dense in I, and $\widehat{\tau}$ is a closed map, surjectivity of τ is equivalent to surjectivity of $\tau|I_0$. By density of X_0 in X and continuity of σ, surjectivity of σ is equivalent to surjectivity of $\sigma|_{X_0}$. By the conjugacy of (I_0,τ) and (X_0,σ), surjectivity of τ is equivalent to surjectivity of σ.

Similarly, $\tau^{n+1}(I_0) = \tau^n(I_0)$ is equivalent to $\sigma^{n+1}(X_0) = \sigma^n(X_0)$, which in turn is equivalent to $\sigma^{n+1}X = \sigma^n X$, so eventual surjectivity of σ is equivalent to eventual surjectivity of τ. □

An alternative definition for eventual surjectivity of discontinuous τ might be that $\widehat{\tau}^{n+1}(I) = \widehat{\tau}^n(I)$ for some $n \ge 0$. This implies $\tau^{n+1}(I_0) = \tau^n(I_0)$, so τ is eventually surjective in the sense of Definition 4.1. The converse is also true, but is a little tedious to prove when τ is discontinuous, and won't be needed in the sequel, so we have chosen to define eventual surjectivity of τ as in Definition 4.1 instead.

Proposition 4.3. *If X is a compact metric space and $\sigma : X \to X$ is an eventually surjective piecewise homeomorphism with eventual range Y, then G_σ is isomorphic to the stationary inductive limit*

$$C(Y,\mathbb{Z}) \xrightarrow{\mathcal{L}} C(Y,\mathbb{Z}) \xrightarrow{\mathcal{L}} C(Y,\mathbb{Z}) \xrightarrow{\mathcal{L}} \cdots . \tag{4.1}$$

Proof. This follows from Lemmas 3.8 and 3.9. □

Example 4.4. If $\tau : [0,1] \to [0,1]$ is defined by $\tau(x) = kx(1-x)$, and $2 \le k < 4$, then $\tau([0,1]) = \tau([0,1/2]) = [0,k/4] \supset [0,1/2]$, so $\tau^2([0,1]) = \tau([0,1])$. Thus τ is eventually surjective, but is not surjective. By Lemma 4.2, the associated piecewise homeomorphism $\sigma : X \to X$ is also eventually surjective but not surjective. On the other hand, if $0 \le k < 2$, then the sets $\tau^n([0,1])$ are nested with intersection $\{0\}$, so τ is not eventually surjective.

The following is [1, Theorem 4.3], adapted to our current context.

Proposition 4.5. *Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. Then $\mathcal{L}_*$ is a homomorphism and an order isomorphism of G_σ onto its image, and is surjective iff σ is eventually surjective.*

Proof. We observed after Definition 3.7 that $\mathcal{L}_*$ is injective. Since $\mathcal{L}$ is a positive operator, it follows that $\mathcal{L}_*$ is positive. If $\mathcal{L}_*[f] \ge 0$, then $[\mathcal{L}f] \ge 0$, which by definition implies $\mathcal{L}^{n+1}f \ge 0$ for some $n \ge 0$, and thus $[f] \ge 0$. Hence $\mathcal{L}_*$ is an order isomorphism onto its image.

Next we show $\mathcal{L}_*$ is surjective if σ is eventually surjective. Suppose $\sigma^{n+1}(X) = \sigma^n(X)$, and let $f \in C[X,\mathbb{Z}]$. Then the support of $\mathcal{L}^n f$ is contained in $\sigma^n(X) = \sigma^{n+1}(X)$, so there exists $g \in C(X,\mathbb{Z})$ such that $\mathcal{L}^{n+1}g = \mathcal{L}^n f$. It follows that $\mathcal{L}_*[g] = [f]$.

Finally, suppose $\mathcal{L}_*$ is surjective. Choose $f \in C(X,\mathbb{Z})$ such that $\mathcal{L}_*[f] = [1]$. Then for some $n \ge 0$, $\mathcal{L}^{n+1}f = L^n 1$. Then

$$\sigma^n X = \operatorname{supp} \mathcal{L}^n 1 = \operatorname{supp} \mathcal{L}^{n+1} f \subset \sigma^{n+1} X,$$

so $\sigma^n X = \sigma^{n+1} X$. Thus σ is eventually surjective. □

Definition 4.6. Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. Then $(G_\sigma, G_\sigma^+, (\mathcal{L}_\sigma)_*)$ is the *dimension triple* associated with σ.

Suppose X_1, X_2 are compact metric spaces, $\sigma_i : X_i \to X_i$ for $i = 1,2$ are piecewise homeomorphisms, and $\psi : X_1 \to X_2$ is a piecewise homeomorphism that intertwines σ_1 and σ_2, i.e., $\psi \circ \sigma_1 = \sigma_2 \circ \psi$. Then $\mathcal{L}_\psi \mathcal{L}_{\sigma_1} = \mathcal{L}_{\psi\circ\sigma_1} = \mathcal{L}_{\sigma_2\circ\psi} = \mathcal{L}_{\sigma_2}\mathcal{L}_\psi$. It follows that $\mathcal{L}_\psi \mathcal{L}_{\sigma_1}^n = \mathcal{L}_{\sigma_2}^n \mathcal{L}_\psi$. Thus the map $(\mathcal{L}_\psi)_* : G_{\sigma_1} \to G_{\sigma_2}$ defined by $(\mathcal{L}_\psi)_*[f] = [\mathcal{L}_\psi f]$ is well-defined, and is a positive homomorphism that intertwines $(\mathcal{L}_{\sigma_1})_*$ and $(\mathcal{L}_{\sigma_2})_*$.

Proposition 4.7. *Let X be a compact metric space, and $\sigma : X \to X$ an eventually surjective piecewise homeomorphism. Let Y be the eventual range of σ, and choose N so that $\sigma^N(X) = Y$. Then there is a group and order isomorphism Φ from G_σ onto $G_{\sigma|_Y}$, which carries the automorphism $(\mathcal{L}_\sigma)_*$ to the automorphism $(\mathcal{L}_{\sigma|_Y})_*$, and carries the distinguished order unit $[1_X]$ to the order unit $[(\mathcal{L}^N 1)|_Y]$.*

Proof. Let $\psi : Y \to X$ be the inclusion map; note that ψ intertwines $\sigma|_Y$ and σ. By the remarks preceding this proposition, $\Phi_0 = (\mathcal{L}_\psi)_*$ is a positive homomorphism from $G_{\sigma|_Y}$ into G_σ, intertwining $(\mathcal{L}_{\sigma|_Y})_*$ and $(\mathcal{L}_\sigma)_*$. We will show that Φ_0 is a order isomorphism of $G_{\sigma|_Y}$ onto G_σ.

Observe that $\mathcal{L}_\psi f$ is the function that agrees with f on Y, and is zero off Y; in particular, $\mathcal{L}_\psi$ is 1-1 on $C(Y,\mathbb{Z})$, and $\mathcal{L}_\psi f \ge 0$ iff $f \ge 0$.

To prove that Φ_0 is surjective, let $f \in C(X,\mathbb{Z})$. Note that $\operatorname{supp} \mathcal{L}_\sigma^N f \subset \sigma^N(X) = Y$. Since σ is surjective on Y, Lemma 3.5 (applied to $\sigma|_Y$) implies that there exists $g \in C(Y,\mathbb{Z})$ such that $\mathcal{L}_{\sigma|_Y}^N g = (\mathcal{L}_\sigma^N f)|_Y$. Applying $\mathcal{L}_\psi$ to both sides gives $\mathcal{L}_\sigma^N \mathcal{L}_\psi g = \mathcal{L}_\psi(\mathcal{L}_\sigma^N f|_Y) = \mathcal{L}_\sigma^N f$ (since $\operatorname{supp} \mathcal{L}_\sigma^N f \subset Y$), so $\Phi_0([g]) = [\mathcal{L}_\psi g] = [f]$. Thus Φ_0 is surjective. Furthermore, if $[f] = [\mathcal{L}_\psi g] \geq 0$, then there exists n such that $\mathcal{L}_\sigma^n \mathcal{L}_\psi g \geq 0$. Then $\mathcal{L}_\psi L_{\sigma|_Y}^n g \geq 0$, which implies $L_{\sigma|_Y}^n g \geq 0$. Thus $[g] \geq 0$, so Φ_0 is an order isomorphism.

Finally, if $f = 1_X$ in the last paragraph, and $g \in C(Y,\mathbb{Z})$ is chosen so that $\mathcal{L}_{\sigma|_Y}^N g = (\mathcal{L}_\sigma^N 1_X)|_Y$, then $\Phi_0([g]) = [1_X]$. Thus $\Phi_0^{-1}[1_X] = (\mathcal{L}_{\sigma|_Y})_*^{-N}[(\mathcal{L}_\sigma^N 1_X)|_Y]$. Since $\sigma|_Y$ is surjective, by Proposition 4.5, $(\mathcal{L}_{\sigma|_Y}^N)_*$ is an order automorphism of $C(Y,\mathbb{Z})$. Now, $\Phi = (\mathcal{L}_{(\sigma|_Y)})_*^N \Phi_0^{-1}$ satisfies the conclusion of the proposition. $\square$

Remark. Suppose that $\tau : I \to I$ is continuous and piecewise monotonic, and let C be the set of endpoints of the associated maximal partition. If τ is eventually surjective, with eventual range J, then $\tau|_J$ is surjective, and we can reduce calculation of $\mathrm{DG}(\tau)$ to finding $\mathrm{DG}(\tau|_J, C')$ for a suitable partition C' as follows:

Choose N so that $J = \tau^N(X) = \tau^{N+1}(X)$. Since τ is continuous, then J is a closed interval. Let $C' = (C \cap J) \cup \tau^N(C)$. (If desired, we may omit from C' any points in $\tau^N(C)$ that are in the generalized orbit of $C \cap J$ under $\tau|_J$.) Associate with $\tau|_J$ the partition whose endpoints are the points in C'. Then the generalized orbit of C' under $\tau|_J$ will be $I_1 \cap J$. If $\sigma : X \to X$ is the local homeomorphism associated with τ, then σ will be eventually surjective; let Y be its eventual range. Then the local homeomorphism associated with $\tau|_J$ (with the partition C') will be $\sigma|_Y$. Thus by Proposition 4.7, $\mathrm{DG}(\tau) \cong \mathrm{DG}(\tau|_J, C')$.

Example 4.8. Let τ be the map on $[0,2]$ which has the values $\tau(0) = 0$, $\tau(1/2) = 1$, $\tau(1) = 0$, $\tau(3/2) = k < 1$, $\tau(2) = 0$, with τ linear between these points. The eventual range of τ is $J = \tau([0,2]) = [0,1]$. We take as a partition of $\tau|_J$ the nonmaximal partition $C' = \{0, 1/2, k, 1\}$. Then $\mathrm{DG}(\tau) \cong \mathrm{DG}(\tau|_J, C')$.

Example 4.9. Let τ be the logistic map in Example 4.4, with $2 \leq k < 4$. The eventual range of τ is $J = \tau([0,1]) = [0, \tau 1]$. We can take as partition for $\tau|_J$ the maximal partition $C' = \{0, 1/2, \tau 1\}$. Thus $\mathrm{DG}(\tau) \cong \mathrm{DG}(\tau|J)$.

5. Simplicity of the dimension group

An *order ideal* in a dimension group G is a subgroup J such that $a, b \in J$ and $a \leq c \leq b$ implies $c \in J$. An ideal J of a dimension group is *positively generated* if $J = J^+ - J^+$ where $J^+ = J \cap G^+$. A dimension group G is *simple* if it has no positively generated order ideals other than $\{0\}$ and G. We will use the following equivalent condition: G is simple iff every nonzero positive element of G is an order unit. We now examine conditions that guarantee that G_σ is simple.

Definition 5.1. Let X be a compact metric space and $\sigma : X \to X$ a continuous map. Then σ is (*topologically*) *exact* if for every open set V there exists $n \geq 0$ such that $\sigma^n(V) = X$. If $\tau : I \to I$ is piecewise monotonic (but not necessarily continuous everywhere), then τ is topologically exact if for every open set V there exists $n \geq 0$ such that $\widehat{\tau}^n(V) = I$. (Recall that if τ is continuous, then $\widehat{\tau} = \tau$, so these two definitions of exactness are consistent.)

If X is a compact metric space, and $\sigma : X \to X$ is topologically exact, then evidently σ is topologically mixing. The converse is not necessarily true, but is true for a broad class of piecewise monotonic maps. For a more thorough discussion of the relationship, see [30, Proposition 4.9].

Two examples of topologically exact maps are β-transformations, and tent maps with slopes $\pm s$ with $\sqrt{2} < s \leq 2$, restricted to a suitable interval [17].

Lemma 5.2. *Let $\tau : I \to I$ be piecewise monotonic, with associated local homeomorphism $\sigma : X \to X$. Then τ is topologically exact iff σ is topologically exact.*

Proof. We first show that τ is exact iff $\tau|_{I_0}$ is exact. Assume τ is exact, and let V be an open subset of I_0. Choose an open subset W of I such that $W \cap I_0 = V$. By exactness of τ, there exists n such that $\widehat{\tau}^n(W) = I$. Since I_0 is forward and backward invariant under $\widehat{\tau}$, then $\tau^n(V) = \tau^n(W \cap I_0) = I_0$, which proves that $\tau|_{I_0}$ is exact.

Conversely, suppose $\tau|_{I_0}$ is exact, and let W be a nonempty open subset of I. Choose a nonempty open set $V \subset W$ such that $\overline{V} \subset W$. Then $V \cap I_0$ is open in I_0, and is nonempty by density of I_0. Choose n so that $\tau^n(V \cap I_0) = I_0$. Then $\widehat{\tau}^n(V) \supset \tau^n(V \cap I_0) = I_0$. Since $\widehat{\tau}$ is a closed map, then $\widehat{\tau}^n(\overline{V})$ is a closed subset of I, containing the dense set I_0, so equals I. Then $\widehat{\tau}^n(W) = I$, which proves that $\widehat{\tau}$ is exact. $\square$

Recall that a compact metric space is zero-dimensional if there is a base of clopen sets, and that this is equivalent to being totally disconnected, cf. [16].

Theorem 5.3. *Let X be a compact metric space, and $\sigma : X \to X$ a piecewise homeomorphism. Then G_σ is simple iff the following hold:*

(1) *σ is eventually surjective.*
(2) *For every compact open subset V of the eventual range Y of σ, there exists $n \geq 0$ such that $\sigma^n(V) = Y$.*

In particular, if σ is eventually surjective and is topologically exact when restricted to its eventual range, then G_σ is simple. The converse is true if X is totally disconnected.

Proof. Suppose that (1) and (2) hold. By Proposition 4.7, if Y is the eventual range of σ, then G_σ is isomorphic to $G_{\sigma|_Y}$, so we may assume without loss of generality that σ is surjective. Let $f \in C(X, \mathbb{Z})^+$, with $f \neq 0$, and let E be a clopen set such that $\chi_E \leq f$. (Since $f \geq 0$ takes on integer values, such an E exists.) Choose $n \geq 0$ such that $\sigma^n(E) = X$, and choose a positive integer k such that $\mathcal{L}^n 1 \leq k1$. (Since $\mathcal{L}^n 1$ is bounded, such an integer k exists.) Then

$$\mathcal{L}_\sigma^n f \geq \mathcal{L}_{\sigma^n}(\chi_E) \geq \chi_{\sigma^n(E)} = 1_X \geq (1/k)\mathcal{L}^n 1_X. \tag{5.1}$$

Thus $k[f] \geq [1_X]$. Since $[1_X]$ is an order unit, then so is $[f]$, and this proves that G_σ is simple.

Conversely, suppose G_σ is simple. We first show that σ is eventually surjective. We have $X \supset \sigma(X) \supset \sigma^2(X) \cdots$. Since G_σ is simple, then there is some $k \geq 0$ such that $[1_X] \leq k[\chi_{\sigma(X)}]$. Then for some $n \geq 0$, $\mathcal{L}^n 1_X \leq k\mathcal{L}^n \chi_{\sigma(X)}$. The support of $\mathcal{L}^n 1_X$ is $\sigma^n(X)$, and the support of $\mathcal{L}^n \chi_{\sigma(X)}$ is $\sigma^{n+1}(X)$, so $\sigma^n(X) \subset \sigma^{n+1}(X)$. Since the opposite inclusion also holds, we must have equality, so σ is eventually surjective.

Now let Y be the eventual range of σ. If $Y \neq X$, replace σ by $\sigma|_Y$, so that we may assume σ is surjective. Let E be any nonempty clopen subset of X. By simplicity of G_σ, there exists $k > 0$ such that $k[\chi_E] \geq [1_X]$, and thus there exists $n \geq 0$ such that $k\mathcal{L}^n\chi_E \geq \mathcal{L}^n 1_X$. Comparing supports, we conclude that $\sigma^n(E) = X$, so (2) holds.

Finally, suppose that X is totally disconnected, and that G_σ is simple. Then by (1), σ is eventually surjective. Let Y be the eventual range of σ. Let V be any nonempty open subset of Y. Then V contains a nonempty clopen subset W, and so by (2) there exists n such that $\sigma^n(W) = Y$. Therefore $\sigma^n(V) = Y$, proving that σ is topologically exact on Y. □

Corollary 5.4. *If $\tau : I \to I$ is piecewise monotonic and topologically exact, then* $\mathrm{DG}(\tau)$ *is a simple dimension group.*

Proof. Let $\sigma : X \to X$ be the associated local homeomorphism, and assume τ is topologically exact. By Lemma 5.2, σ is topologically exact, so by Theorem 5.3, $\mathrm{DG}(\tau)$ is simple. □

By Theorem 5.3, simplicity of $\mathrm{DG}(\tau)$ is most informative when X is totally disconnected. We now explore when this occurs. Recall that the generalized orbit of a subset A of I is the smallest set containing A and closed under $\widehat{\tau}$ and $\widehat{\tau}^{-1}$.

Lemma 5.5. *Let $\tau : I \to I$ be piecewise monotonic, let C be the set of points in the associated partition, and let $\sigma : X \to X$ be the associated local homeomorphism. Then X is totally disconnected iff the generalized orbit I_1 of C is dense in I.*

Proof. Suppose that I_1 is dense in I. Let $d \in X$, and let $(a,b)_X$ contain d. We will assume $\pi(a) < \pi(d) < \pi(b)$; the cases where $\pi(a) = \pi(d)$ or $\pi(d) = \pi(b)$ can be proved in a similar manner. By density of I_1, we can choose $x, y \in I_1$ with $\pi(a) < x < \pi(d) < y < \pi(b)$. Then $a < x^+ < d < y^- < b$, and $[x^+, y^-]_X$ is a clopen order interval containing d and contained in $(a,b)_X$. Thus the clopen sets form a base for the topology of X, so X is totally disconnected.

Conversely, suppose that I_1 is not dense in I. Let (a,b) be an open interval in I containing no point of I_1. Then every point in (a,b) is in $I_0 = I \setminus I_1$, so every point in the open set $\pi^{-1}(a,b)$ is in X_0. Since each clopen subset of X contains points of X_1 (cf. Proposition 2.2), then $\pi^{-1}(a,b)$ is a nonempty open subset of X containing no nonempty clopen subset. Thus X is not totally disconnected. □

Below "interval" will always denote an interval that is not a single point.

Definition 5.6. Let $\tau : I \to I$ be piecewise monotonic, and J an open interval. Then J is a *homterval* if τ^n is a homeomorphism on J for all n.

Note that if J is a homterval for τ, then for each $p \geq 0$, J will be a homterval for τ^p. For additional background on homtervals, see the books of Collet–Eckmann [5, §II.5] or de Melo–van Strien [25, Lemma II.3.1]. The following result characterizes the existence of homtervals for polynomials.

Let $\tau : I \to I$ be piecewise monotonic and continuous. An *attracting periodic orbit* is a periodic orbit such that the set of points whose orbits converge to that periodic orbit has nonempty interior.

Proposition 5.7. *If $\tau : I \to I$ is a polynomial of degree ≥ 2, then τ has a homterval iff τ has an attractive periodic orbit.*

Proof. If f has a homterval, by [25, Lemma II.3.1 and Thm. A, p. 267], there exists an attractive periodic orbit. Conversely, let x_0 be an attractive periodic point of f, say $f^p(x_0) = x_0$. Let $h = f^{2p}$; then h has x_0 an attractive fixed point. Note $h'(x_0) = ((f^p)'(x_0))^2 \geq 0$. If $h'(x_0) > 0$, since x_0 is attractive, there must be an open interval V with endpoint x_0 such that $0 < h' < 1$ on V. By the mean value theorem, $h(V) \subset V$, and h is 1-1 on V. Thus V is a homterval for h, and therefore also for f.

If $h'(x_0) = 0$, choose $\delta > 0$ such that $0 < |h'(x)| < 1$ for $0 < |x - x_0| < \delta$. Let $W = (x_0 - \delta, x_0 + \delta) \cap I$. By the mean value theorem, h maps $W \setminus \{x_0\}$ into itself. If V is a component of $W \setminus \{x_0\}$, then V is a homterval for h, and thus for f. $\square$

Proposition 5.8. *Let $\tau : I \to I$ be piecewise monotonic, and let $\sigma : X \to X$ be the associated local homeomorphism. The following are equivalent:*

(1) *I contains no homterval.*
(2) *I_1 is dense in I.*
(3) *X is totally disconnected.*

In particular, these hold if τ is transitive.

Proof. (1) $\Rightarrow$ (2) If I contains no homterval, then for any open interval J, there exists n such that $\tau^n(J)$ meets C. Thus I_1 is dense in I.

(2) $\Leftrightarrow$ (3) Lemma 5.5.

(2) $\Rightarrow$ (1) Suppose that there exists a homterval J; we will show I_1 is not dense in I. We first show that either:

(a) there exists $p \geq 1$ and a homterval whose images under τ^p are disjoint, or
(b) there exists $p \geq 1$ and an interval fixed pointwise by τ^p.

Suppose the images of J under τ are not disjoint, so that $\tau^n(J) \cap \tau^{n+p}(J) \neq \emptyset$ for some $n \geq 0$ and $p \geq 1$. Then for $k \geq 0$, $\tau^{n+kp}(J) \cap \tau^{n+(k+1)p}(J) \neq \emptyset$, so $V = \cup_{k=0}^{\infty} \tau^{n+kp}(J)$ is an interval invariant under τ^p. Since J is a τ^p-homterval, therefore τ^p is continuous and monotonic on each of the overlapping open intervals $\tau^{n+kp}(J)$ for $k = 0, 1, \ldots$, so τ^p is a homeomorphism from V into V. Replacing τ^p by τ^{2p} if necessary, we may assume that τ^p is increasing on V.

If some point x in V is not fixed by τ^p, then the τ^p-orbit of x is a monotonic sequence converging to some point of $\overline{V}$. Let K be the open interval with endpoints x and $\tau^p(x)$. Then K is an τ^p-homterval whose images under τ^p are disjoint. Thus we've shown either (a) or (b) must hold. We will show in both cases that I_1 is not dense in I.

Let $p \geq 1$ and $h = \widehat{\tau}^p$, and suppose that J is a homterval whose images under h are disjoint. We will prove that $h^N(J) \cap I_1$ is finite for some $N \geq 0$, showing that I_1 is not dense in I. Let $D = \cup_{k=0}^{p} \widehat{\tau}^k(C)$. Observe that $\cup_{k=1}^{\infty} h^k(D) = \cup_{n=1}^{\infty} \widehat{\tau}^n(C)$ is invariant under $\widehat{\tau}$, and that the generalized orbit of C under τ is the same as the generalized orbit of D under h, i.e.,

$$(5.2) \qquad \begin{aligned} I_1 &= \{x \mid \widehat{\tau}^m x \cap \widehat{\tau}^n C \neq \emptyset \text{ for some } m, n \geq 0\} \\ &= \{x \mid h^j x \cap h^k D \neq \emptyset \text{ for some } j, k \geq 0\}. \end{aligned}$$

In the rest of this proof, "orbit" means the orbit under the map h unless otherwise specified.

Since the orbit of J consists of disjoint intervals, by replacing J by $h^k(J)$ for suitable $k \geq 0$, we can arrange that the orbit of J is disjoint from the finite set

$C \cup h(C)$. (Since $\widehat{\tau}$ may be multivalued at points in C, it simplifies the discussion to avoid such points where possible.)

Let D_1 consist of those points in D whose orbits miss C and meet the orbit of J. If $d \in D_1$, let n_d be the least integer such that $\mathrm{orbit}(d) \cap \tau^{n_d}(J) \neq \emptyset$, and let $N = \max_{d \in D_1} n_d$, and $J_N = \tau^N(J)$. (If $D_1 = \emptyset$, define $J_N = J$.) The orbit of each point d in D_1 meets J_{n_d}, and therefore meets J_N. Furthermore, $\mathrm{orbit}(d)$ meets J_N in a unique point y_d, since having $\tau^k d \in J_N$ and $\tau^{k+q} d \in J_N$ would imply $\tau^q J_N \cap J_N \neq \emptyset$, contrary to the assumption that the orbit of J consists of disjoint intervals.

Let $P = J_N \cap I_1$; we will prove P is finite. If $x \in P$, by (5.2) the orbit of x meets the orbit of some $d_0 \in D$. Then there is a sequence $d_0, d_1, \ldots, d_k$ such that $d_{i+1} \in h(d_i)$ for $0 \leq i \leq k-1$, and $d_k \in \mathrm{orbit}(x)$. By construction of J, the orbit of J misses C, so the orbit of x misses C; thus $d_k \notin C$. Let j be the least index such that $d_j, d_{j+1}, \ldots, d_k$ miss C, and set $d = d_j$. Then $d \in D_1$, and the orbit of d meets the orbit of x. Let y_d be the unique point where the orbit of d meets J_N. Distinct points in J_N have disjoint orbits, so $y_d = x$. Thus for each $x \in P$, there exists a point $d \in D_1$ whose orbit meets J_N exactly in the single point x. Since D_1 is finite, then P is finite, which finishes the proof that I_1 is not dense in I.

Suppose instead that there exists an interval V fixed pointwise by $h = \tau^p$ for some $p \geq 1$. Replacing V by a component of $V \setminus (C \cup \widehat{\tau} C)$ if necessary, we can assume that V is disjoint from $C \cup \widehat{\tau} C$. Let D_2 be the set of points in D whose orbits land in V and miss C. Each orbit of a point in D_2 is finite; let A be the union of such orbits. Then no point in C has an orbit that meets $V \setminus A$, so any component of $V \setminus A$ is disjoint from I_1. Thus I_1 is not dense in I.

Finally, suppose τ is transitive. Then τ is strongly transitive by Proposition 2.9. If V is any nonempty open subset of I, and $c \in C$, by strong transitivity there exists $n \geq 0$ such that $c \in \widehat{\tau}^n(V)$. Then V meets $\widehat{\tau}^{-n}(c)$, so $V \cap I_1 \neq \emptyset$. Thus I_1 is dense in I. □

Example 5.9. Let $1 < s \leq 2$ and let τ be the tent map with slopes $\pm s$. The length of each interval not containing the critical point $1/2$ is expanded by τ by a factor s, so for every open interval J, there exists n such that $1/2$ is in $\tau^n(J)$. Thus I_1 is dense in I, so if $\sigma : X \to X$ is the local homeomorphism associated with τ, then X is totally disconnected.

We have the following partial converse for Corollary 5.4.

Corollary 5.10. *If $\tau : I \to I$ is piecewise monotonic, is surjective, has no homtervals, and $\mathrm{DG}(\tau)$ is simple, then τ is topologically exact.*

Proof. Let $\sigma : X \to X$ be the associated local homeomorphism. Then X is totally disconnected (Proposition 5.8), and by hypothesis, the dimension group $G_\sigma = \mathrm{DG}(\tau)$ is simple. Since τ is surjective, then σ is surjective, cf. Lemma 4.2. By Theorem 5.3, σ is topologically exact on its eventual range, namely, X, and then by Lemma 5.2, τ is topologically exact. □

Example 5.11. Let $\tau : I \to I$ be a polynomial of degree ≥ 2, and let $\sigma : X \to X$ be the associated local homeomorphism. Then X will be totally disconnected iff τ has no attractive periodic orbit (Propositions 5.7 and 5.8).

6. Module structure of the dimension group

In this section we will generally work with eventually surjective piecewise monotonic maps $\tau : I \to I$, cf. Definition 4.1. For such maps, if $\sigma : X \to X$ is the associated local homeomorphism, then by Proposition 4.5, $(\mathcal{L}_\sigma)_*$ is an order automorphism of $\mathrm{DG}(\tau) = G_\sigma$. We use this to give $\mathrm{DG}(\tau)$ the structure of a module. We will show that this module is finitely generated, and will identify generators.

Definition 6.1. Let $\tau : I \to I$ be piecewise monotonic and eventually surjective. Then we view $\mathrm{DG}(\tau)$ as a $\mathbb{Z}[t, t^{-1}]$-module by defining

$$\left(\sum_{i=-n}^{m} z_i t^i\right)[f] = \left(\sum_{i=-n}^{m} z_i (\mathcal{L}_*)^i\right)[f] \tag{6.1}$$

for $z_{-n}, \ldots, z_m \in \mathbb{Z}$ and $[f] \in \mathrm{DG}(\tau)$. Thus $\mathrm{DG}(\tau)$ is both a dimension group and a $\mathbb{Z}[t, t^{-1}]$ module, and we call it the *dimension module* for τ.

Let $0 = a_0 < a_1 < \cdots < a_n = 1$ be the partition associated with τ. Recall that τ_i denotes the unique extension of $\tau|_{(a_{i-1}, a_i)}$ to a homeomorphism from $[a_{i-1}, a_i]$ onto its image.

If $a, b \in I_1$ with $a < b$, then we write $I(a, b)$ for the order interval $[a^+, b^-]_X$. If $a > b$, then we define $I(a, b) = I(b, a)$, and we set $I(a, a) = 0$. Recall that each such set $I(a, b)$ is clopen, and that every clopen subset of X is a finite disjoint union of such sets, cf. Proposition 2.2. We generally will not distinguish between an order interval E in X, its characteristic function $\chi_E \in C(X, \mathbb{Z})$, and the equivalence class $[\chi_E] \in \mathrm{DG}(\tau)$; it should be clear from the context which is intended. In particular, if $a, b \in I_1$, we consider $I(a, b)$ as a member of $\mathrm{DG}(\tau)$.

We also observe that if a, b are in $[a_{i-1}, a_i] \cap I_1$, then by Equations (2.1) and (2.2),

$$\sigma(I(a, b)) = I(\tau_i(a), \tau_i(b)). \tag{6.2}$$

If D is a finite subset of $\mathbb{R}$, we will say distinct points $x, y \in D$ are *adjacent* in D if there is no element of D between them.

Theorem 6.2. *Let $\tau : I \to I$ be piecewise monotonic and eventually surjective, with associated partition $C = \{a_0, a_1, \ldots, a_n\}$. Let M be any member of the set $\widehat{\tau}C = \{\tau_1(a_0), \tau_1(a_1), \tau_2(a_1), \tau_2(a_2) \ldots, \tau_n(a_{n-1}), \tau_n(a_n)\}$. Let*

$$\mathcal{J}_1 = \{\, I(c, d) \mid c,\ d \text{ are adjacent points in } \{\, a_0, a_1, \ldots, a_n, M \,\}\,\}, \tag{6.3}$$

and let $\mathcal{J}_2$ be the set of intervals corresponding to jumps at partition points, i.e.,

$$\mathcal{J}_2 = \{\, I(\tau_i(a_i), \tau_{i+1}(a_i)) \mid 1 \le i \le n-1 \,\}. \tag{6.4}$$

Then $\mathrm{DG}(\tau)$ is generated as a module by $\mathcal{J}_1 \cup \mathcal{J}_2$.

Proof. Let H be the submodule of $\mathrm{DG}(\tau)$ generated by $\mathcal{J}_1 \cup \mathcal{J}_2$. By Proposition 2.2, $\mathrm{DG}(\tau)$ is generated by the intervals $I(c, d)$ for $c, d \in I_1$, so it suffices to show each such interval is in H. Define $c \sim d$ if $I(c, d) \in H$. Then we need to show $c \sim d$ for all $c, d \in I_1$. This relation is clearly reflexive and symmetric. To show it is transitive, suppose $c \sim d$ and $d \sim e$. If $c \le d \le e$, then $I(c, d)$ and $I(d, e)$ are in H, and are disjoint intervals, so the sum $I(c, e)$ of their corresponding characteristic functions is also in H, and thus $c \sim e$. If $c \le e \le d$, then $I(c, e) = I(c, d) - I(e, d) \in H$, so again $c \sim e$. The case $e \le c \le d$ is similar.

By the definition of $\mathcal{J}_1$, each pair of adjacent points in $\{ a_0, a_1, \ldots, a_n, M \}$ is equivalent. By transitivity, all points in $\{ a_0, a_1, \ldots, a_n, M \}$ are equivalent. Suppose $c, d \in [a_{i-1}, a_i] \cap I_1$. We will show

$$c \sim d \iff \tau_i(c) \sim \tau_i(d). \tag{6.5}$$

Recall from Theorem 2.3 that σ is a homeomorphism on $[a_{i-1}^+, a_i^-]_X \supset I(c,d)$, so by (3.6) we have $\mathcal{L}_*(I(c,d)) = \sigma(I(c,d))$. From (6.2), $I(\tau_i(c), \tau_i(d)) = \mathcal{L}_*(I(c,d))$. Thus $I(\tau_i(c), \tau_i(d)) \in H$ iff $I(c,d) \in H$, which gives (6.5).

In particular, $\tau_i(a_{i-1}) \sim \tau_i(a_i)$ for $1 \le i \le n$. By the definition of $\mathcal{J}_2$, we have $\tau_i(a_i) \sim \tau_{i+1}(a_i)$ for $1 \le i \le n-1$. Then by transitivity we conclude that $\tau_1(a_0), \tau_1(a_1), \ldots, \tau_n(a_{n-1}), \tau_n(a_n)$ are equivalent. By definition, M is one of these numbers, so we conclude that each of these is also equivalent to each of $a_0, a_1, \ldots, a_n$.

Let P be the set of all points in I_1 equivalent to a_0. So far we have shown that P contains $a_0, a_1, \ldots, a_n$, and $\tau_1(a_0), \tau_1(a_1), \ldots, \tau_n(a_{n-1}), \tau_n(a_n)$. Let $x \in P$. Then for some index i, x is in $[a_{i-1}, a_i]$. Since $x \in P$, then $x \sim a_0$, and since $a_0 \sim a_{i-1}$, then $x \sim a_{i-1}$. By (6.5), $\tau_i(x) \sim \tau_i(a_{i-1}) \sim a_0$, so $\tau_i(x) \in P$. Thus P is closed under application of each τ_i.

Now suppose that $x \in P$ and $y \in [a_{i-1}, a_i]$ with $\tau_i(y) = x$. Since $\tau_i(a_i) \in P$, then $\tau_i(y) = x \sim \tau_i(a_i)$. By (6.5), $y \sim a_i$, so $y \in P$. Therefore P contains $\{a_0, a_1, \ldots, a_n\}$ and is closed under application of each τ_i and τ_i^{-1}, so must coincide with I_1. □

Corollary 6.3. *If $\tau : I \to I$ is a continuous, surjective piecewise monotonic map with associated partition $\{a_0, a_1, \ldots, a_n\}$, then* $\mathrm{DG}(\tau)$ *is generated as a module by* $I(a_0, a_1)$, $I(a_1, a_2)$, $\ldots$, $I(a_{n-1}, a_n)$.

Proof. Since τ is continuous, then all intervals in $\mathcal{J}_2$ in Theorem 6.2 are empty, so can be omitted. Since τ is continuous and monotonic on each interval $[a_i, a_{i+1}]$, its maximum must occur at one of $a_0, \ldots, a_n$. On the other hand, since τ is surjective, its maximum value must be a_n. Thus $a_n = \tau(a_j)$ for some j, so M as defined in Theorem 6.2 is equal to a_n. Now the result follows from Theorem 6.2. □

Example 6.4. Let $1 \le s \le 2$, and let $\tau : [0,1] \to [0,1]$ be the map $\tau(x) = sx$ for $0 \le x \le 1/2$ and $\tau(x) = s - sx$ for $x > 1/2$. By Theorem 6.2, for the maximal partition $\{0, 1/2, 1\}$, taking $M = \tau(0) = 0$, $\mathrm{DG}(\tau)$ is generated as a $Z[t, t^{-1}]$ module by $I(0, 1/2)$ and $I(1/2, 1)$. Furthermore, we have $\mathcal{L}_*(I(0,1/2)) = I(0, s/2) = \mathcal{L}_* I(1/2, 1)$, so in fact this module is singly generated, with generator $I(0, s/2)$.

Example 6.5. Let τ be the map $\tau(x) = 2x$ for $0 \le x \le 1/3$ and $\tau(x) = (4/3 - x)$ for $1/3 < x \le 1$. Then $\mathrm{DG}(\tau)$ is generated by the intervals $I(0, 1/3)$, $I(1/3, 1)$, and $I(2/3, 1)$. The last interval is included because of the jump discontinuity at $1/3$; note that $\tau_1(1/3) = 2/3$ while $\tau_2(1/3) = 1$.

7. Cyclic dimension modules

We will see later that the dimension modules of surjective unimodal maps are cyclic (i.e., singly generated). In the current section, we find conditions that guarantee that cyclic dimension modules are free. These conditions involve requirements that certain orbits of critical points are infinite and disjoint.

Let $\tau : I \to I$ be piecewise monotonic, with associated local homeomorphism $\sigma : X \to X$. We will identify functions in $C(X,\mathbb{Z})$ with functions in $L^1(\mathbb{R})$ in the following manner: if $g \in C(X,\mathbb{Z})$, there is a unique $f \in L^1(\mathbb{R})$ such that $(f \circ \pi)(x) = g(x)$ for $x \in X_1$, with $f(x) = 0$ for $x \notin I$. We will always choose f to be continuous except at a finite set of points in I_1. (This is possible by Proposition 2.2(5). For example, for $a, b \in I_1$, the function in $L^1(\mathbb{R})$ corresponding to the characteristic function of the order interval $[a^+, b^-]_X$ is the characteristic function of $[a, b]$.) Any two such choices for f will agree except on a finite set.

Since we identify functions that agree except at a finite number of points, then by a "discontinuity" for a function in $C(X,\mathbb{Z})$, viewed as a function on $\mathbb{R}$, we will mean a point where left and right limits exist but are different. We denote the set of discontinuities of g by $\mathcal{D}(g)$. Note that $\mathcal{D}(1_X) = \{0, 1\}$.

Next we will develop some basic facts about $\mathcal{D}(g)$. For the definition of $I(a, b)$, see the beginning of the previous section.

Let $\tau : I \to I$ be piecewise monotonic, with associated partition C, and let $C_0 = C \setminus \{0, 1\}$. For $g \in C(X,\mathbb{Z})$, let $x_0 < x_1 < \cdots < x_q$ be the points in C_0, together with the points where g is discontinuous. Then for some $n_1, \ldots, n_q$ in $\mathbb{Z}$, we can write

$$g = \sum_{i=1}^{q} n_i I(x_{i-1}, x_i), \tag{7.1}$$

and

$$\mathcal{L}g = \sum_i n_i \sigma(I(x_{i-1}, x_i)) = \sum_i n_i I(\tau_i(x_{i-1}), \tau_i(x_i)). \tag{7.2}$$

It follows that

$$\mathcal{D}(\mathcal{L}g) \subset \widehat{\tau}(\mathcal{D}(g)) \cup \widehat{\tau} C_0, \tag{7.3}$$

so for all $n \geq 0$,

$$\mathcal{D}(\mathcal{L}^n g) \subset \widehat{\tau}^n \mathcal{D}(g) \cup \bigcup_{k=1}^{n} \widehat{\tau}^k C_0. \tag{7.4}$$

Note in Lemma 7.1, we interpret $\tau e \notin \widehat{\tau} C_0$, where $C_0 = C \setminus \{0, 1\}$, to mean in particular that $e \notin C_0$, so that τ is single-valued at e.

Lemma 7.1. *Let $\tau : I \to I$ be piecewise monotonic, with associated partition C, and let $C_0 = C \setminus \{0, 1\}$. Let $e \in I_1$ satisfy $\tau e \notin \widehat{\tau} C_0$. If $g \in C(X,\mathbb{Z})$ is discontinuous at e, and g is continuous at all points in $\tau^{-1}(\tau e)$ except e, then $\mathcal{L}g$ is discontinuous at τe.*

Proof. Since we are viewing functions in $C(X,\mathbb{Z})$ as members of $L^1(\mathbb{R})$, if $a, b \in I_1$, we will write $\tau([a, b])$ for characteristic function of the closed interval $\overline{\tau((a, b))}$. With the notation in (7.1), since $\mathcal{D}(g) \subset \{x_0, \ldots, x_q\}$, there is some index j such that $e = x_j$. First consider the case $0 < j < q$. Since g is discontinuous at e, then $n_j \neq n_{j+1}$. Since $e \notin C_0$, then τ is monotonic and continuous on (x_{j-1}, x_{j+1}). Therefore $\tau([x_{j-1}, x_j])$ and $\tau([x_j, x_{j+1}]$ are closed intervals with just the endpoint $\tau x_j = \tau e$ in common. Since $n_j \neq n_{j+1}$, then $n_j \tau([x_{j-1}, x_j]) + n_{j+1} \tau([x_j, x_{j+1}])$ is discontinuous at $\tau x_j = \tau e$. On the other hand, for all $i \neq j$, either $x_i \in C_0$ or else x_i is a point of discontinuity of g. If $x_i \in C_0$, then $\tau e \notin \widehat{\tau} C_0$ implies that

$\tau e \notin \widehat{\tau} x_i$. If $x_i \in \mathcal{D}(g)$, by hypothesis $x_i \notin \tau^{-1}\tau e$, so again $\tau e \notin \widehat{\tau} x_i$. Thus none of the endpoints of the intervals $\tau([x_{i-1}, x_i])$ for $i \notin \{j, j+1\}$ is equal to τe, so by (7.2), $\mathcal{L}g$ is discontinuous at τe.

Now consider the remaining case where $j \in \{0, q\}$. We treat the case $j = q$; the other case is similar. Since g is discontinuous at $e = x_j$, then $n_j \neq 0$, so $n_j \tau([x_{j-1}, x_j])$ is discontinuous at $\tau x_j = \tau e$. As in the previous paragraph, none of the endpoints of the intervals $\tau([x_{i-1}, x_i])$ for $i \neq j$ is equal to τe, so by (7.2), $\mathcal{L}g$ is discontinuous at τe. □

Lemma 7.2. *Let $\tau : I \to I$ be piecewise monotonic, with associated partition C, and let $C_0 = C \setminus \{0, 1\}$. Let $g \in C(X, \mathbb{Z})$, and $e \in \mathcal{D}(g) \setminus C_0$. Suppose the following hold for all $n \geq 0$:*

(1) *$\tau^n e \notin \widehat{\tau}^k C_0$ for $k \leq n$.*
(2) *$\tau^n e \notin \widehat{\tau}^n(\mathcal{D}(g) \setminus \{e\})$.*

Then for all $n \geq 0$, $\mathcal{L}^n g$ is discontinuous at $\tau^n e$.

Proof. By (7.4),

$$\mathcal{D}(\mathcal{L}^n g) \subset \widehat{\tau}^n\big(\mathcal{D}(g) \setminus \{e\}\big) \cup \{\tau^n e\} \cup \bigcup_{k=1}^{n} \widehat{\tau}^k C_0. \tag{7.5}$$

We will prove

$$\tau^n e \in \mathcal{D}(\mathcal{L}^n g) \tag{7.6}$$

for all n, by induction on n. Suppose (7.6) holds for a particular $n \geq 0$. We will apply Lemma 7.1 with $\tau^n e$ in place of e. By (1), $\tau(\tau^n e) \notin \widehat{\tau} C_0$. Suppose $x \in \tau^{-1}\tau(\tau^n e) \cap \mathcal{D}(\mathcal{L}^n g)$, so that $\tau x = \tau^{n+1} e$. We will show that x is not in the first or third set on the right side of (7.5), so must be in the second. By (2), $\tau x = \tau^{n+1} e \notin \widehat{\tau}^{n+1}(\mathcal{D}(g) \setminus \{e\})$, so $x \notin \widehat{\tau}^n(\mathcal{D}(g) \setminus \{e\})$. Similarly, if $x \in \widehat{\tau}^k C_0$ with $k \leq n$, then $\tau^{n+1} e = \tau x \in \widehat{\tau}^{k+1} C_0$, which contradicts (1). Thus (7.5) implies $x = \tau^n e$, so we've shown

$$\tau^{-1}\tau(\tau^n e) \cap \mathcal{D}(\mathcal{L}^n g) = \{\tau^n e\}. \tag{7.7}$$

By Lemma 7.1, $\tau^{n+1} e \in \mathcal{D}(\mathcal{L}^{n+1} g)$, so by induction, $\tau^n e \in \mathcal{D}(\mathcal{L}^n g)$ for all $n \geq 0$. □

Recall that a $\mathbb{Z}[t, t^{-1}]$ module G is *cyclic* if there is an element $e \in G$ such that $G = \mathbb{Z}[t, t^{-1}]e$.

Proposition 7.3. *Let $\tau : I \to I$ be piecewise monotonic, with associated partition C. Assume that $\mathrm{DG}(\tau)$ is cyclic, and that there exists $a \in \{0, 1\}$ with an infinite orbit, such that*

$$\widehat{\tau}(C \setminus \{a\}) \subset C. \tag{7.8}$$

Then $\mathrm{DG}(\tau) \cong \mathbb{Z}[t, t^{-1}]$ as abelian groups, with the action of $\mathcal{L}_*$ given by multiplication by t.

Proof. Let $b \in C$ be adjacent to a (i.e., there are no points in C between a and b). Since the orbit of a is infinite, $\tau^n a \neq \tau^k a$ for $k \neq n$, and $\tau^n a \notin C$ for $n \geq 1$.

By (7.8) and Lemma 7.2 (with $e = a$ and $g = I(a, b)$), for each $n \geq 0$, $\mathcal{L}^n I(a, b)$ is discontinuous at $\tau^n a$. On the other hand, by (7.4), for $k < n$,

$$\mathcal{D}(\mathcal{L}^k I(a, b)) \subset \{\tau^k a\} \cup \widehat{\tau}^k b \cup \bigcup_{j=1}^{k} \widehat{\tau}^j C_0, \tag{7.9}$$

where $C_0 = C \setminus \{0, 1\}$. It follows that $\mathcal{L}^k I(a, b)$ is continuous at $\tau^n a$ for $k < n$. Therefore, for any polynomial $p \in \mathbb{Z}[t]$ of degree n, $p(\mathcal{L})I(a, b)$ is discontinuous at $\tau^n a$, and in particular is not identically zero.

By hypothesis, $\mathrm{DG}(\tau)$ is cyclic, so there exists $g \in C(X, \mathbb{Z})$ such that $[g]$ is a generator for $\mathrm{DG}(\tau)$. Suppose that $q(\mathcal{L}_*)[g] = 0$ for some $q \in \mathbb{Z}[t, t^{-1}]$. Choose $q' \in \mathbb{Z}[t, t^{-1}]$ so that $q'(\mathcal{L}_*)[g] = I(a, b)$. Then $q(\mathcal{L}_*)q'(\mathcal{L}_*)I(a, b) = 0$. It follows that there is a nonzero polynomial $q'' \in Z[t]$ such that $q''(\mathcal{L})I(a, b) = 0$, contrary to the result established in the first paragraph. We conclude that $p \to p(\mathcal{L}_*)[g]$ is an isomorphism from $\mathbb{Z}[t, t^{-1}]$ onto $\mathrm{DG}(\tau)$. □

We will apply Proposition 7.2 to find dimension groups of unimodal maps later in this paper, cf. Theorem 9.1, and we will show for multimodal maps that a similar requirement of infinite disjoint orbits for critical points leads to the dimension module being free, cf. Proposition 10.6.

8. Markov maps

Definition 8.1. A piecewise monotonic map τ is *Markov* if there is a partition $0 = b_0 < b_1 < \cdots < b_n = 1$, with each b_i being in I_1, such that for each i, τ is monotonic on (b_i, b_{i+1}), $\overline{\tau(b_i, b_{i+1})}$ is a union of intervals of the form $[b_j, b_{j+1}]$, and for some $k \geq 0$, $\widehat{\tau}^k(C) \subset \{b_0, b_1, \ldots, b_n\}$. We call such a partition a *Markov partition for* τ, and will refer to (b_0, b_1), (b_1, b_2), $\ldots$, (b_{n-1}, b_n) as the *partition intervals for* τ.

For a Markov map, we have both the original partition associated with τ, and the Markov partition. Since we don't require that τ be continuous on each interval (b_{i-1}, b_i), a Markov partition for τ may not even qualify as a "partition associated with τ" in the sense defined at the start of this paper.

If C is the set of endpoints in the partition associated with τ, then τ will be Markov iff the forward orbit of C under $\widehat{\tau}$ is finite. In that case, the points of the forward orbit of C will be the endpoints of a Markov partition. When τ is continuous, τ will be Markov iff each point in C is eventually periodic.

If $\tau : I \to I$ is piecewise monotonic, and $\sigma : X \to X$ is the associated local homeomorphism, a *Markov partition for* σ is a partition of X into clopen order intervals $E_1, \ldots E_k$ such that σ is monotonic on each E_i, and maps E_i onto a union of some of $E_1, \ldots, E_k$. If $0 = b_0 < b_1 < \cdots < b_n = 1$ is a Markov partition for τ, then defining $E_i = I(b_{i-1}, b_i)$ gives a Markov partition for σ. The associated zero-one *incidence matrix* A is given by $A_{ij} = 1$ iff $\sigma(E_i) \supset E_j$, or equivalently, if $\tau(b_{i-1}, b_i) \supset (b_{j-1}, b_j)$..

Note that if $E_1, \ldots, E_n$ is a Markov partition for σ, then $\{\sigma^k(X)\}$ will be a nested decreasing sequence of sets, with each being a union of some of $E_1, \ldots, E_n$. Thus σ will be eventually surjective (Definition 4.1), and similarly so will τ (Lemma 4.2).

Definition 8.2. If A is an $n \times n$ zero-one matrix, G_A denotes the stationary inductive limit $\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$ in the category of ordered abelian groups. Here A acts by right multiplication, and G_A will be a dimension group. The action of A induces an automorphism of G_A denoted A_*, so we view G_A as a $\mathbb{Z}[t, t^{-1}]$ module, and refer to (G_A, G_A^+, A_*) as the dimension triple associated with A.

Notation. Let G be an ordered abelian group, $T : G \to G$ a positive homomorphism, and $G_1, G_2, \dots$ a sequence of subgroups of G such that $T(G_i) \subset G_{i+1}$. If $g \in G_n$, we will write (g, n) for the sequence $(0, 0, \dots, 0, g, Tg, T^2g, \dots)$, where $n-1$ zeros precede g. We write $[(g, n)]$ (or simply $[g, n]$) for the equivalence class of this sequence in the inductive limit. (See the notation introduced after Definition 3.7). Every element of the inductive limit is of the form $[g, n]$, and $[g_1, n_1] = [g_2, n_2]$ with $n_2 \geq n_1$ iff $T^{n_2-n_1+k}g_1 = T^k g_2$ for some $k \geq 0$, or equivalently, iff $T^{n_2+k}g_1 = T^{n_1+k}g_2$. In particular, we represent elements of G_A as pairs $[v, k]$, where $v \in \mathbb{Z}^n$ and $k \in \mathbb{N}$, and the automorphism A_* of G_A is given by $A_*([v, k]) = [vA, k]$. Similarly, there is a canonical automorphism associated with any stationary inductive limit. For an exposition of inductive limits for dimension groups, see [28].

Lemma 8.3. *Let $C_1, C_2, \dots$ be subgroups of $C(X, \mathbb{Z})$ such that $\mathcal{L}C_n \subset C_{n+1}$ for all n, and let G be the inductive limit of the sequence $C_n \xrightarrow{\mathcal{L}} C_{n+1}$. Suppose the following conditions hold:*

(1) *For each $f \in C(X, \mathbb{Z})$ there exists $n \geq 0$ such that $\mathcal{L}^n f \in C_n$.*
(2) *For every $k \geq 0$ and $f \in C_k$, there exists $g \in C(X, \mathbb{Z})$ such that $\mathcal{L}_*^k[g] = [f]$.*

Then the map $\Phi : G \to \mathrm{DG}(\tau)$ defined by $\Phi([f, n]) = \mathcal{L}_*^{-n}[f]$ is an order isomorphism from the dimension group G onto the dimension group $\mathrm{DG}(\tau)$. If $C_1 = C_2 = C_3 \cdots$, then this isomorphism carries the canonical automorphism of this stationary inductive limit to $\mathcal{L}_*$.

Proof. Recall that $\mathcal{L}_*$ is injective, but need not be surjective. However, by (2), $\mathcal{L}_*^{-n}$ makes sense on C_n. Suppose that $f_1 \in C_{n_1}$ and $f_2 \in C_{n_2}$, and that $[f_1, n_1] = [f_2, n_2]$. Then $\mathcal{L}^{n_2+k}f_1 = \mathcal{L}^{n_1+k}f_2 \in C_{n_1+n_2+k}$ for some $k \geq 0$, so taking equivalence classes and applying $\mathcal{L}_*^{-n_1-n_2-k}$ gives $\mathcal{L}_*^{-n_1}[f_1] = \mathcal{L}_*^{-n_2}[f_2]$. Thus Φ is well-defined.

If $f_1 \in C_{n_1}$, $f_2 \in C_{n_2}$ and $\Phi([f_1, n_1]) = \Phi([f_2, n_2])$, then reversing the previous argument gives $\mathcal{L}_*^{n_2}[f_1] = \mathcal{L}_*^{n_1}[f_2]$, so for some $n \geq 0$, $\mathcal{L}^{n+n_2}f_1 = \mathcal{L}^{n+n_1}f_2 \in C_{n+n_1+n_2}$. Thus $[f_1, n_1] = [f_2, n_2]$, proving that Φ is 1-1.

For any $f \in C(X, \mathbb{Z})$, choose n so that $\mathcal{L}^n f \in C_n$. Then $\Phi[\mathcal{L}^n f, n] = \mathcal{L}_*^{-n}[\mathcal{L}^n f] = [f]$, so Φ is surjective. The remaining assertions are readily verified. ☐

Proposition 8.4. *Let $\tau : I \to I$ be piecewise monotonic and Markov, with associated local homeomorphism $\sigma : X \to X$. Let $E_1, E_2, \dots, E_q$ be the associated Markov partition for σ, with incidence matrix A, and define $\psi : \mathbb{Z}^q \to C(X, \mathbb{Z})$ by $\psi(z_1, z_2, \dots, z_q) = \sum_i z_i E_i$. Then the map $\Phi : G_A \to \mathrm{DG}(\tau)$ defined by $\Phi([v, n]) = \mathcal{L}_*^{-n}[\psi(v)]$ is an isomorphism from the dimension triple (G_A, G_A^+, A_*) onto the dimension triple $(\mathrm{DG}(\tau), \mathrm{DG}(\tau)^+, \mathcal{L}_*)$.*

Proof. Let M be the range of ψ. Then ψ is an order isomorphism from $\mathbb{Z}^q$ onto M, and M is invariant under $\mathcal{L}$ (by the Markov property of the partition). Observe that

$$\mathcal{L}(\psi(v)) = \psi(vA) \text{ for all } v \in \mathbb{Z}^q, \tag{8.1}$$

so ψ is an isomorphism from (G_A, G_A^+, A_*) onto the inductive limit of the stationary sequence $\mathcal{L} : M \to M$.

We next show that for any $f \in C(X, \mathbb{Z})$ there exists $n \geq 0$ such that $\mathcal{L}^n f \in M$. It suffices to prove this for $f = I(a,b)$ with $a, b \in I_1$. Note that $\mathcal{L}I(a,b)$ is a sum of intervals $I(c,d)$ with c, d either being images of a, b or being points in $\widehat{\tau}C$. By definition of I_1, for each point $x \in I_1$ there exists n such that $\tau^n(x)$ is in the forward orbit of C. It follows that for k large enough, $\mathcal{L}^k f$ is a sum of intervals of the form $I(x,y)$ with endpoints x, y in the forward orbit of C, and thus, by the definition of a Markov partition (Definition 8.1), for k large enough these endpoints will be in $\{b_0, \ldots, b_q\}$. Thus $\mathcal{L}^n f \in M$ for n large enough. Since τ is Markov, τ and σ are eventually surjective, so $\mathcal{L}_*$ is surjective (Proposition 4.5). Now the conclusion follows from Lemma 8.3 with each of C_1, C_2, $\ldots$ equal to M. □

Every zero-one square matrix (without zero rows) is the incidence matrix of a Markov map. To illustrate, the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{8.2}$$

is the incidence matrix for the Markov map pictured in Figure 1 on page 479, with respect to the Markov partition $B = \{0, 1/3, 2/3, 1\}$. In this example, the maximal partition associated with τ is $C = \{0, 1/3, 1/2, 2/3, 1\}$.

Let A be an $n \times n$ zero-one matrix (with no zero rows), and let (X_A, σ_A) be the associated one-sided shift of finite type. (In other words, X_A consists of sequences $x_0 x_1 x_2 \ldots$ with entries in $\{1, 2, \ldots, n\}$ such that $A_{x_i x_{i+1}} = 1$ for all i, and σ is the left shift.) Then, for an appropriate metric, X_A is a zero-dimensional metric space, and σ_A is a local homeomorphism. The dimension group G_{σ_A} (cf. Definition 3.3) is isomorphic to G_A. (For A irreducible, this is [1, Theorem 4.5]; the proof in [1] works in general. Alternatively, the proof of Proposition 8.4 can be adapted to prove this.) This suggests that (X, σ) might be conjugate to the one-sided shift (X_A, σ_A), and we now show that this happens precisely if τ has no homtervals (Definition 5.6).

Given a Markov partition $E_1, \ldots, E_n$ for $\sigma : X \to X$, with incidence matrix A, the *itinerary map* $S : X \to X_A$ is given by $S(x) = s_0 s_1 s_2 \ldots$, where $\sigma^k(x) \in E_{s_k}$. We say *itineraries separate points of* X if the itinerary map is 1-1. The following proposition shows that this property is independent of the choice of Markov partition.

Proposition 8.5. *Assume $\tau : I \to I$ is piecewise monotonic and Markov, with incidence matrix A. These are equivalent:*

(1) *τ has no homtervals.*
(2) *Itineraries separate points of X.*
(3) *(X, σ) is conjugate to the one-sided shift of finite type (X_A, σ_A).*

In particular, these equivalent conditions hold if τ is transitive.

Proof. Let $B = \{b_0, b_1, \ldots, b_q\}$ be the given Markov partition for τ, and let C be the partition associated with τ.

(1) ⇒ (2) Assume that there are no homtervals. We will show that the itinerary map $S : X \to X_A$ is 1-1. Suppose (to reach a contradiction) that there exists points $x < y$ in X with the same itinerary.

We first consider the case where $x = a^-$ and $y = a^+$ for $a \in I_1$. By definition of I_1, the orbit of a eventually lands on some point in the forward orbit of C, and by the definition of a Markov partition, the forward orbit of C is eventually contained in B. Thus there exists n such that $\widehat{\tau}^n a \subset B$, so $\pi(\sigma^n(a^\pm)) \subset B$. Suppose that $\sigma^k(a^-)$ and $\sigma^k(a^+)$ belong to the same Markov partition interval for $0 \le k \le n-1$. Then σ is monotonic on the Markov partition interval containing $\sigma^{n-1}(a^-)$ and $\sigma^{n-1}(a^+)$, so $\sigma^n(a^-) \ne \sigma^n(a^+)$. If $\sigma^n(a^-) = b^\pm$ for $b \in B$, then $\sigma^n(a^+) = b^\mp$, so the itineraries of $a^\pm$ are different, contrary to our assumption.

Now assume $(x, y)_X$ is not empty, and that x, y have the same itinerary. By density of X_0 in X, we may assume x and y are in X_0. Then $x' = \pi(x)$ and $y' = \pi(y)$ are distinct points in I_0 with the same itinerary with respect to the given Markov partition of I. Let $J = (x', y')$, and for $n \ge 0$ define J_n to be the open interval whose endpoints are $\tau^n(x')$ and $\tau^n(y')$. Since x and y have the same itinerary, τ is monotonic on J_n for all n. Since there are no homtervals, there is some n such that J_n contains a point $c \in C$. Then $\widehat{\tau}^k(c) \subset J_{n+k}$ for all $k \ge 0$. However, the orbit of c eventually lands in B, so J_q contains a point of B for some q. But then $\tau^q(x')$ and $\tau^q(y')$ would have different itineraries, a contradiction. Thus S is 1-1.

(2) ⇒ (3) Assume itineraries separate points, so that S is 1-1. To show that S is surjective, let $s = s_0 s_1 \cdots \in X_A$. For each $n \ge 0$, let $I_{s_0 s_1 \ldots s_n}$ be the set of points in X with the initial itinerary $s_0 s_1 \ldots s_n$. Then the points with itinerary s are those that are in $\cap_{n=0}^\infty I_{s_0 s_1 \ldots s_n}$. The sets $I_{s_0 s_1 \ldots s_n}$ for $n = 0, 1, \ldots$ are nested and compact, so surjectivity of the map S will follow if we show each such set is nonempty.

Observe that $I_{s_0 s_1 \ldots s_n} = I_{s_0} \cap \sigma^{-1}(I_{s_1 \ldots s_n})$. We use induction on the length of the partial itinerary. Suppose that these partial itinerary sets are nonempty for partial itineraries of length m. If $s_0 s_1$ is an allowed transition with respect to the incidence matrix A, by definition of A, $\sigma(I_{s_0}) \supset I_{s_1}$, so $\sigma(I_{s_0}) \supset I_{s_1 \ldots s_m}$. Thus $\sigma(I_{s_0}) \cap \sigma^{-1}(I_{s_1 \ldots s_m})$ is not empty, which completes the proof that S is surjective, so that (X, σ) is conjugate to (X_A, σ_A).

(3) ⇒ (1) If (X, σ) is conjugate to a shift of finite type, then X is totally disconnected. By Proposition 5.8, τ has no homtervals.

Finally, if τ is transitive, then τ has no homtervals (Proposition 5.8). □

Example 8.6. Let $\tau_0(x) = kx(1 - x)$, with c the unique critical point, and choose $k \approx 3.68$ so that $p = \tau^3(c)$ is fixed. Let τ denote $\tau_0|_J$, where $J = [\tau^2(c), \tau(c)]$, rescaled so that the domain of the restricted map is $[0, 1]$. (See the restricted logistic map in Figure 1 on page 479.) The map τ is Markov, with Markov partition $\{0, c, p, 1\}$. The incidence matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since τ is a polynomial, it will have homtervals iff it has an attracting periodic orbit (Proposition 5.7). For a quadratic, the orbit of the critical point will be attracted to any attracting periodic orbit ([9, p. 158]). For our particular polynomial τ_0, since the orbit of the critical point c lands on a repelling fixed point p, we conclude that there are no homtervals. If $\sigma : X \to X$ is the associated local homeomorphism, by Proposition 8.5, $(X, \sigma) \cong (X_A, \sigma_A)$.

If $\tau : I \to I$ is piecewise monotonic and Markov, the *directed graph associated with* τ has vertices consisting of the Markov partition intervals, with an edge from E to F if $\tau(E) \supset F$. This is the same as the directed graph associated with the incidence matrix for τ. By "directed graph" we will always mean a graph in which there are no multiple edges between pairs of vertices. A *loop* in a directed graph is a path that begins and ends at the same vertex. If $\tau : I \to I$ is piecewise linear and Markov, we will see we can determine whether there are homtervals by looking at the directed graph associated with τ.

Definition 8.7. Let $v_1, \ldots, v_n$ be vertices of a directed graph. If $v_1 v_2 \ldots v_n v_1$ is a loop, then an *exit* for this loop is an edge from some v_i to a vertex other than $v_{i+1 \bmod n}$. A directed graph satisfies *Condition L* if every loop has an exit.

Condition L appears in the study of C*-algebras associated with (possibly infinite) directed graphs, cf. [23]. An equivalent condition for finite graphs ("Condition I") was assumed in [6]. If a (finite) directed graph is irreducible, Condition L is equivalent to the associated incidence matrix not being a permutation matrix.

In Proposition 8.8, by a piecewise linear Markov map we mean a piecewise monotonic map, which is Markov, and which on each interval of the Markov partition is piecewise linear. We allow the slopes of the pieces to vary from interval to interval, but not within an interval of the partition. We don't require the map to be continuous on each partition interval, since we want to allow examples like the Markov map in Figure 1 on page 479.

Proposition 8.8. *Let $\tau : I \to I$ be piecewise linear and Markov, with incidence matrix A. These are equivalent:*

(1) *τ has no homtervals.*
(2) *Itineraries separate points of X.*
(3) *(X, σ) is conjugate to the one-sided shift of finite type (X_A, σ_A).*
(4) *In the directed graph associated with τ, every loop has an exit.*

Proof. By Proposition 8.5, it suffices to prove that (4) and (1) are equivalent. Suppose $v_1 v_2 \ldots v_n v_1$ is a loop without an exit in the directed graph associated with τ. By definition, each v_i is an open interval in the associated Markov partition for τ. Since τ is monotonic and piecewise linear on each v_i, and $\overline{\tau(v_i)} = \overline{v_{i+1 \bmod 1}}$ for each i, then τ must be a homomorphism of v_i onto $v_{i+1 \bmod n}$ for each i. Thus v_i is a homterval.

Conversely, assume that every loop has an exit. By conjugating τ by a suitable continuous piecewise linear map if necessary, we may assume that each interval in the Markov partition has the same length. If E is an interval in the Markov partition, since the slope of τ within E is constant, it must be an integer, equal to the number of partition intervals covered by $\tau(E)$. Let V be a homterval contained in a partition interval E_0. We claim that there exists $n \geq 1$ such that the length $\tau^n(V)$ is at least double that of V. If the slope of $\tau|_{E_0}$ is ± 1, since all partition intervals have the same length, then $E_1 = \tau(E_0)$ will also be a partition member. If

$E_0, \ldots, E_j$ have been defined, and the slope of τ on E_j is ± 1, define $E_{j+1} = \tau(E_j)$. Note that if $E_i = E_j$ for $i < j$, then $E_i, \ldots, E_j$ would be a loop without an exit, which is not allowed, so the sets $E_1, E_2, \ldots, E_j$ are distinct. Choose n maximal such that E_0, $E_2 = \tau(E_1)$, …, $E_n = \tau(E_{n-1})$ are all partition members. Then $\tau|_{E_n}$ has slope ≥ 2, so the (Lebesgue) measure of $\tau^n(V)$ is at least double the length of V. Furthermore, since V is a homterval, τ is 1-1 on $\tau^k(V)$ for all k, so the measure of $\tau^k(V)$ never decreases as k increases.

Now let V be an arbitrary homterval. Then we can write V as a disjoint union of homtervals $V_1, \ldots, V_k$, each contained in a partition interval, together with a finite set of points. Then there exists n such that for all i, $\tau^n(V_i)$ has measure at least double the length of V_i. Then the measure of $\tau^n(V)$ is at least double the length of V. This process can be repeated indefinitely, which is impossible since $\tau^n(V) \subset [0, 1]$ for all n. We conclude that τ has no homtervals. □

This provides an easy way to tell if Markov piecewise linear maps are transitive or topologically exact. Recall that a zero-one matrix is *irreducible* if in the associated directed graph, there is a path from any vertex to any other vertex, and is *irreducible and aperiodic* if it is irreducible and the greatest common divisor of the lengths of paths between any two vertices is 1. This is equivalent to A being *primitive*, i.e., to some power of A having strictly positive entries. It is well-known that the one-sided shift σ_A is transitive iff A is irreducible, (e.g., [20, Theorem 1.4.1]), and is topologically mixing iff A is primitive. It is readily verified that σ_A will be topologically exact iff it is topologically mixing.

Corollary 8.9. *Assume $\tau : I \to I$ is piecewise linear and Markov, with incidence matrix A. Then τ is topologically exact iff A is primitive, and is transitive iff A is irreducible and is not a permutation matrix. If A is irreducible, and is not a permutation matrix, and $\sigma : X \to X$ is the local homeomorphism associated with τ, then (X, σ) is conjugate to the one-sided shift (X_A, σ_A).*

Proof. Assume τ is transitive. By Proposition 8.5, (X, σ) is conjugate to the one-sided shift (X_A, σ_A). Transitivity of τ implies transitivity of σ (Proposition 2.9). Thus σ_A is transitive, so A is irreducible. If A were a permutation matrix, then A^p would be the identity for some p, so τ^p restricted to each interval of the Markov partition would be a monotonic bijection of that interval onto itself. This contradicts transitivity of τ, so A cannot be a permutation matrix.

Conversely, if A is irreducible and not a permutation matrix, then σ_A is transitive, and every loop has an exit, which implies that τ has no homtervals (Proposition 8.8). Having no homtervals implies that (X, σ) is conjugate to (X_A, σ_A) (Proposition 8.5). Thus σ and τ are transitive.

If τ is topologically exact, then it is also transitive, so again (X, σ) is conjugate to (X_A, σ_A). Then σ and σ_A are topologically exact, so A is primitive. Conversely, if A is primitive, one easily verifies that in the associated graph every loop has an exit, so (X, σ) will be conjugate to (X_A, σ_A). Primitivity of A implies that σ_A is topologically exact, and thus that σ and τ are topologically exact. □

Example 8.10. Let τ be the Markov map in Figure 1 on page 479, and let $\sigma : X \to X$ be the associated local homeomorphism. Then τ is Markov, so the dimension triples $(\mathrm{DG}(\tau), \mathrm{DG}(\tau)^+, \mathcal{L}_*)$ and (G_A, G_A^+, A_*) are isomorphic (Proposition 8.4).

The associated incidence matrix A is primitive, so (X,σ) is conjugate to (X_A,σ_A) and τ is topologically exact (Corollary 8.9).

9. Unimodal maps

If $\tau : I \to I$ is continuous, with just two intervals of monotonicity, without loss of generality, we may assume τ increases and then decreases. (If not, conjugate by the map $\phi(x) = 1 - x$.) We will say a continuous map $\tau : I \to I$ is *unimodal* if there exists c in $(0,1)$ such that τ is increasing on $[0,c]$ and decreasing on $[c,1]$. We are mainly interested in the case when τ is eventually surjective. In that case, for the purpose of computing the dimension group, we may as well assume that τ is surjective, cf. Proposition 4.7.

Theorem 9.1. *Let $\tau : I \to I$ be unimodal and surjective. Then the dimension module $\mathrm{DG}(\tau)$ is cyclic, with generator $I(0,1)$. If the critical point c is eventually periodic, then τ is Markov. In that case, if the incidence matrix is A, then $(\mathrm{DG}(\tau), \mathrm{DG}(\tau)^+, \mathcal{L}_*) \cong (G_A, G_A^+, A_*)$. If c is not eventually periodic, then $\mathrm{DG}(\tau) \cong \mathbb{Z}[t,t^{-1}]$ (as abelian groups), with the action of $\mathcal{L}_*$ given by multiplication by t.*

Proof. Suppose $\tau(1) = 0$. By Corollary 6.3, the module $\mathrm{DG}(\tau)$ is generated by $I(0,c)$ and $I(c,1)$, where c is the unique critical point. Since $\mathcal{L}_* I(c,1) = I(0,1)$, then $I(0,c) = I(0,1) - (\mathcal{L}_*)^{-1} I(0,1)$, so $\mathrm{DG}(\tau)$ is generated by $I(0,1)$, and thus is cyclic. The same conclusion applies if $\tau(0) = 0$, by a similar argument.

If c is eventually periodic, then τ is Markov with respect to the partition given by the forward orbit of c, together with the endpoint 0. Then $(\mathrm{DG}(\tau), \mathrm{DG}(\tau)^+, \mathcal{L}_*) \cong (G_A, G_A^+, A_*)$ follows from Proposition 8.4. If c is not eventually periodic, by surjectivity the unique critical point maps to 1, one endpoint of I must map to zero, and the other endpoint will have an infinite orbit. Thus $\mathrm{DG}(\tau) \cong \mathbb{Z}[t,t^{-1}]$ by Proposition 7.3. □

Given $s > 1$, we define the *restricted tent map* T_s by

$$T_s(x) = \begin{cases} 1 + s(x-c) & \text{if } x \le c \\ 1 - s(x-c) & \text{if } x > c, \end{cases} \tag{9.1}$$

where $c = 1 - 1/s$. (This is the usual symmetric tent map on $[0,1]$ with slopes $\pm s$, restricted to the interval $[\tau^2(1/2), \tau(1/2)]$, which is the interval of most interest for the dynamics. Then the map has been rescaled so that its domain is [0,1]. See Figure 1 on page 479.) Note that $\tau(c) = 1$ and $\tau(1) = 0$.

For use in the next example, we recall one way to compute the dimension group G_A (Definition 8.2), due to Handelman; see [24] for an exposition. If A is $n \times n$, view A as acting on $\mathbb{Q}^n$, and let $V_A \subset \mathbb{Q}^n$ be the eventual range of A, i.e., $V_A = \cap_{k=1}^{\infty} \mathbb{Q}^n A^k$. Let G be the set of vectors $v \in V_A$ such that $vA^k \in \mathbb{Z}^n$ for some $k \ge 0$, and G^+ those vectors such that $vA^k \in (\mathbb{Z}^n)^+$ for some $k \ge 0$. Then $(G_A, G_A^+, A_*) \cong (G, G^+, A)$.

Example 9.2. Let $\tau = T_2$ be the full tent map. Then τ is Markov, with incidence matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, so $\mathrm{DG}(\tau) \cong G_A$. Since the incidence matrix is primitive, τ is topologically exact (Corollary 8.9), as also is easy to verify directly. We compute

G_A. The eventual range of A in $\mathbb{Q}^2$ consists of multiples of $(1,1)$, and multiplication by A on the eventual range multiplies this vector by 2. Thus G_A is isomorphic to the space of vectors $q(1,1)$ for q in $\mathbb{Q}$ such that $q(1,1)A^k = 2^k q(1,1) \in \mathbb{Z}^2$ for some $k \geq 0$. It follows that $\mathrm{DG}(\tau)$ is isomorphic to the group of dyadic rationals, with the automorphism $\mathcal{L}_*$ given by multiplication by 2.

Example 9.3. Let τ be the restricted tent map $T_{\sqrt{2}}$, let c be the critical point, and let p be the fixed point. Then τ is Markov, with Markov partition $\{0, c, p, 1\}$, and τ^2 is conjugate to the tent map on each of the invariant intervals $[0,p]$ and $[p,1]$. Thus τ is transitive (as can also be seen by applying Corollary 8.9). By Example 9.2, $\mathrm{DG}(\tau)$ will be the direct sum of two copies of the dyadic rationals, with the generators of the summands being $I(0,p)$ and $I(p,1)$. The associated automorphism $\mathcal{L}_*$ will exchange the two summands, taking $(a,b) \in \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ to $(b, 2a)$.

Example 9.4. Let $s = 3/2$. Then the critical point of the restricted tent map T_s is not eventually periodic. (Indeed, if c is the critical point and $n \geq 3$, then $\tau^n(c) = a_n/2^{n-2}$, where a_n is odd.) By Theorem 9.1, $\mathrm{DG}(T_s) \cong \mathbb{Z}[t, t^{-1}]$ as abelian groups, with the action of $\mathcal{L}_*$ given by multiplication by t.

Example 9.5. Let $\tau(x) = kx(1-x)$, with k as in Example 8.6, restricted to the interval described there. Then τ is Markov, with incidence matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

As shown in Example 8.6, the associated local homeomorphism (X, σ) is conjugate to (X_A, σ_A). The map $T_{\sqrt{2}}$ is Markov, with the same incidence matrix. Since A is irreducible and not a permutation matrix, $T_{\sqrt{2}}$ also is conjugate to (X_A, σ_A) (Corollary 8.9). Thus τ is conjugate to $T_{\sqrt{2}}$, so $\mathrm{DG}(\tau)$ is as described in Example 9.3. Alternatively, $\mathrm{DG}(\tau)$ can be computed by applying Proposition 8.4.

10. Multimodal maps

By a multimodal map we mean a continuous piecewise monotonic map. In this section we will show that the dimension modules for multimodal maps are free modules, under the assumption that the critical points have orbits that are infinite and disjoint.

Definition 10.1. Let $\tau : I \to I$. If a set $B \subset I$ has the property that for $b_1, b_2 \in B$, and $n, m \geq 0$,

$$\tau^m b_1 = \tau^n b_2 \implies b_1 = b_2 \text{ and } m = n, \tag{10.1}$$

we say B satisfies the infinite disjoint orbit condition (IDOC).

Note that if B satisfies the IDOC, then τ is 1-1 on $\mathrm{orbit}(B)$. The IDOC condition has been used in studies of interval exchange maps, cf., e.g., [18].

Recall from Section 7 that we view functions in $C(X, \mathbb{Z})$ as functions in $L^1(\mathbb{R})$, continuous except on a finite set, and that $\mathcal{D}(g) \subset I_1$ denotes the set of points where $g \in L^1(\mathbb{R})$ has an essential (jump) discontinuity.

Lemma 10.2. *Let $\tau : I \to I$ be piecewise monotonic and continuous, with associated partition C, and associated local homeomorphism $\sigma : X \to X$. Let $B = \tau C \setminus \{0,1\}$. Assume:*

(1) *The orbits of points in B are infinite and disjoint.*
(2) *$\tau(\{0,1\}) \subset B$.*

Then the only solution in $C(X,\mathbb{Z})$ of $\mathcal{L}g = g$ is $g = 0$.

Proof. We start by establishing some basic facts about B that will also be used in the proof of Lemma 10.3. By the definition of B,

$$\tau C \subset B \cup \{0,1\}, \tag{10.2}$$

and since $\tau(\{0,1\}) \subset B$,

$$\tau^2 C \subset \tau B \cup B. \tag{10.3}$$

Combining (10.2) and (10.3) gives

$$\mathrm{orbit}(C) \setminus C \subset \mathrm{orbit}(B). \tag{10.4}$$

By (10.3) and the IDOC,

$$\tau^2(\mathrm{orbit}(B)) \cap \tau^2 C \subset \left(\bigcup_2^\infty \tau^n B\right) \cap (\tau B \cup B) = \emptyset. \tag{10.5}$$

Hence

$$\mathrm{orbit}(B) \cap C = \tau(\mathrm{orbit} B) \cap \tau C = \emptyset, \tag{10.6}$$

since if (10.6) failed, the result of applying τ^2 or τ to each equality would contradict (10.5).

By the definition of I_1, for each $a \in I_1$ there exists $m \geq 0$ such that $\tau^m a \in \mathrm{orbit}(C)$. By (10.3), $\tau^2 C \subset \mathrm{orbit}(B)$, and thus $\tau^{m+2} a \in \mathrm{orbit}(B)$. Hence

$$a \in I_1 \implies \tau^n a \in \mathrm{orbit}(B) \text{ for some } n \geq 0. \tag{10.7}$$

Now assume $g \in C(X,\mathbb{Z})$ and $\mathcal{L}g = g$. Choose $n \geq 0$ such that

$$\tau^n \mathcal{D}(g) \subset \mathrm{orbit}(B). \tag{10.8}$$

By (7.4), (10.2), and (10.3), using $\mathcal{L}g = g$,

$$\mathcal{D}(g) = \mathcal{D}(\mathcal{L}^n g) \subset \tau^n \mathcal{D}(g) \cup \bigcup_{k=1}^n \tau^k C \subset \mathrm{orbit}(B) \cup \{0,1\}. \tag{10.9}$$

Suppose that $\mathcal{D}(g) \not\subset \{0,1\}$, and let $N \geq 0$ be the largest integer such that $\mathcal{D}(g)$ contains a point of $\tau^N B$. (By the IDOC for B, the sets $\{\tau^n B \mid n \geq 0\}$ are disjoint, and $\mathcal{D}(g)$ is finite, so such an integer N exists.) Choose $e \in \mathcal{D}(g) \cap \tau^N B$. We now will verify that the hypotheses of Lemma 7.1 hold here. Since $e \in \mathrm{orbit}(B)$, then $\tau e \notin \tau C$ by (10.6). Let $x \in \mathcal{D}(g) \cap \tau^{-1}\tau e$. Then $\tau x = \tau e \notin \tau C$, so $x \notin C$. Thus $x \in \mathcal{D}(g) \setminus C$ and (10.9) imply that x is in $\mathrm{orbit}(B)$. By the IDOC, τ is 1-1 on $\mathrm{orbit}(B)$, so $\tau x = \tau e$ implies $x = e$. Now Lemma 7.1 implies that $\tau e \in \mathcal{D}(\mathcal{L}g)$. Then $\mathcal{D}(g) = \mathcal{D}(\mathcal{L}g)$ contains $\tau e \in \tau^{N+1} B$, which contradicts the maximality of N.

We conclude that $\mathcal{D}(g) \subset \{0,1\}$, and therefore $g \in \mathbb{Z}I(0,1)$. If $\mathcal{L}I(0,1) = I(0,1)$, since $\mathcal{L}I(0,1)(t)$ is the cardinality of $\sigma^{-1}(t)$, then σ would be bijective. It would follow that τ was bijective, and thus would be a homeomorphism, so $\tau(\{0,1\}) \subset \{0,1\}$, contrary to (2). Thus $g = 0$. $\square$

Lemma 10.3. *Let $\tau : I \to I$ be piecewise monotonic and continuous, with associated partition C, and associated local homeomorphism $\sigma : X \to X$. Let $B = \tau C \setminus \{0,1\}$. Assume:*

(1) *The orbits of points in B are infinite and disjoint.*
(2) *$\tau(\{0,1\}) \subset B$.*

If $f \in C(X,\mathbb{Z})$ and $\mathcal{D}(f - \mathcal{L}f) \subset C$, then $\mathcal{D}(f) \subset C$ and $\mathcal{L}f \in \mathbb{Z}I(0,1)$.

Proof. Assume $\mathcal{D}(f - \mathcal{L}f) \subset C$. We first prove

$$\mathcal{D}(f) \subset \mathrm{orbit}(C). \tag{10.10}$$

Suppose that $e_0 \in \mathcal{D}(f) \setminus \mathrm{orbit}(C)$. Observe that e_0 can't be periodic, since by (10.7), the orbit of e_0 lands in the orbit of B, and by the IDOC, no point of B can be periodic. Therefore the sets $\{\tau^{-k}e_0 \mid k \geq 0\}$ are disjoint, so there exists a maximal integer N such that $\tau^{-N}e_0 \cap \mathcal{D}(f) \neq \emptyset$. Choose $e \in \tau^{-N}e_0 \cap \mathcal{D}(f)$, and observe that

$$e \in \mathcal{D}(f) \setminus \mathrm{orbit}(C) \text{ and } \tau^{-1}e \cap \mathcal{D}(f) = \emptyset. \tag{10.11}$$

Then $e \notin \tau\mathcal{D}(f)$, and since $e \notin \mathrm{orbit}(C)$, in particular $e \notin \tau C$. By (7.4),

$$\mathcal{D}(\mathcal{L}f) \subset \tau(\mathcal{D}(f)) \cup \tau C, \tag{10.12}$$

so $e \notin \mathcal{D}(\mathcal{L}f)$. Since f is discontinuous at e and $\mathcal{L}f$ is continuous at e, then e is in $\mathcal{D}(f - \mathcal{L}f)$, which is contained in C by hypothesis, contradicting $e \notin \mathrm{orbit}(C)$. This proves (10.10).

We now prove

$$\tau(\mathcal{D}(f) \setminus C) \subset \mathcal{D}(f) \setminus C. \tag{10.13}$$

Let e be an arbitrary member of $\mathcal{D}(f) \setminus C$. By (10.10) and (10.4),

$$e \in \mathcal{D}(f) \setminus C \subset \mathrm{orbit}(C) \setminus C \subset \mathrm{orbit}(B). \tag{10.14}$$

By the same argument as in the proof of Lemma 10.2, the hypotheses of Lemma 7.1 are satisfied here, so $\tau e \in \mathcal{D}(\mathcal{L}f)$. By (10.6), $\tau e \in \mathrm{orbit}(B)$ implies that $\tau e \notin C$. By hypothesis, $\mathcal{D}(f - \mathcal{L}f) \subset C$, so $\tau e \notin \mathcal{D}(f - \mathcal{L}f)$, i.e., $f - \mathcal{L}f$ is continuous at τe. Since $\tau e \in \mathcal{D}(\mathcal{L}f)$, then $\mathcal{L}f$ is discontinuous at τe, so $f = (f - \mathcal{L}f) + \mathcal{L}f$ is discontinuous at τe. Hence $\tau e \in \mathcal{D}(f) \setminus C$, which finishes the proof of (10.13).

Since $\mathcal{D}(f) \setminus C \subset \mathrm{orbit}(B)$, by the IDOC any point in $\mathcal{D}(f) \setminus C$ has an infinite orbit, and this orbit stays in $\mathcal{D}(f) \setminus C$ by (10.13). Since $\mathcal{D}(f)$ is finite, it follows that $\mathcal{D}(f) \setminus C$ is empty, so we've proven

$$\mathcal{D}(f) \subset C.$$

By hypothesis $\mathcal{D}(f - \mathcal{L}f) \subset C$, so $\mathcal{D}(\mathcal{L}f) \subset C$. On the other hand, by (10.12),

$$\mathcal{D}(\mathcal{L}f) \subset \tau\mathcal{D}(f) \cup \tau C \subset \tau C. \tag{10.15}$$

By (10.6), $B \cap C = \emptyset$, so by (10.2),

$$\mathcal{D}(\mathcal{L}f) \subset \tau C \cap C \subset (B \cup \{0,1\}) \cap C = \{0,1\}. \tag{10.16}$$

If follows that $\mathcal{L}f \subset \mathbb{Z}I(0,1)$, which completes the proof of the lemma. $\square$

Lemma 10.4. *Let $E_1, \dots, E_n$ be elements in a $\mathbb{Z}[t]$ module M. Suppose the following hold:*

(1) *$E_1, \dots, E_n$ are independent in M viewed as a $\mathbb{Z}$ module.*
(2) *$(1-t)M \cap \sum_i \mathbb{Z}E_i = \{0\}$.*
(3) *$\ker(1-t) = \{0\}$.*

Then $E_1, E_2, \dots, E_n$ are independent in M viewed as a $\mathbb{Z}[t]$ module.

Proof. We first prove the implication (for $p_1, p_2, \dots, p_n \in \mathbb{Z}[t]$)

$$\sum_i p_i(t)E_i = 0 \implies p_1(1) = p_2(1) = \cdots = p_n(1) = 0. \tag{10.17}$$

If

$$0 = \sum_i p_i(t)E_i = \sum_i (p_i(t) - p_i(1))E_i + \sum_i p_i(1)E_i, \tag{10.18}$$

since each $p_i(t) - p_i(1)$ is divisible by $(1-t)$, then

$$\sum_i p_i(1)E_i = -\sum_i (p_i(t) - p_i(1))E_i \in (1-t)M, \tag{10.19}$$

which by (1) and (2) gives the implication (10.17).

Now let $p_1, p_2, \dots, p_n$ be arbitrary polynomials in $\mathbb{Z}[t]$, not all 0, and assume that $\sum_i p_i(t)E_i = 0$. Let k be the largest power such that $(1-t)^k$ divides each $p_i(t) - p_i(1)$ for $1 \le i \le n$. Write $p_i(t) - p_i(1) = (1-t)^k q_i(t)$, where $q_1, \dots, q_n \in \mathbb{Z}[t]$. Note, for use below, that by the maximality of k, there is an index j such that $(1-t)$ does not divide $q_j(t)$, and thus $q_j(1) \ne 0$.

Now by (3),

$$0 = (1-t)^k \sum_i q_i(t)E_i \tag{10.20}$$

implies that $\sum_i q_i(t)E_i = 0$, so by (10.17), $q_1(1) = q_2(1) = \cdots = q_n(1) = 0$. This contradicts $q_j(1) \ne 0$. Thus we conclude that $p_1 = p_2 = \cdots = p_n = 0$, and hence that $E_1, \dots, E_n$ are independent in the $\mathbb{Z}[t]$ module M. □

Lemma 10.5. *Let $\tau : I \to I$ be continuous, surjective, and piecewise monotonic, with associated partition $C = \{a_0, a_1, \dots, a_q\}$, and associated local homeomorphism $\sigma : X \to X$. Assume that the orbits of $a_1, a_2, \dots, a_{q-1}$ are infinite and disjoint, and that $\tau(\{0,1\}) \cap \{0,1\} = \emptyset$. Then $I(a_0, a_1)$, $I(a_1, a_2)$, $\dots$, $I(a_{q-2}, a_{q-1})$ are independent in $\mathrm{DG}(\tau)$.*

Proof. Let $B = \tau C \setminus \{0,1\}$. By surjectivity,

$$\{0,1\} \subset \tau(\{a_1, \dots, a_{q-1}\}), \tag{10.21}$$

and it follows that B satisfies the hypotheses of Lemma 10.2 and 10.3. Let $E_i = I(a_{i-1}, a_i)$ for $1 \le i \le q-1$.

We now verify the hypotheses of Lemma 10.4 with $n = q-1$ and $M = C(X, \mathbb{Z})$, viewed as a $\mathbb{Z}[t]$ module via the action of $\mathcal{L} = \mathcal{L}_\sigma$.

Since $E_1, \dots, E_{q-1}$ are disjoint, if $\sum_i z_i E_i = 0$, where $z_1, \dots, z_{q-1} \in \mathbb{Z}$, then $z_1 = \cdots = z_{q-1} = 0$, so $E_1, \dots, E_{q-1}$ are independent in M viewed as a $\mathbb{Z}$ module.

Next we show $(\mathrm{id} - \mathcal{L})C(X, \mathbb{Z}) \cap \sum_i \mathbb{Z}E_i = \{0\}$. Suppose that $g \in \sum_{i=1}^{q-1} \mathbb{Z}E_i$, with $g \ne 0$, and that $g = (\mathrm{id} - \mathcal{L})f$ for some $f \in C(X, \mathbb{Z})$. Then $\mathcal{D}(f - \mathcal{L}f) =$

$\mathcal{D}(g) \subset C$, so by Lemma 10.3, $\mathcal{D}(f) \subset C$, and $\mathcal{L}f = zI(0,1)$ for some $z \in \mathbb{Z}$. Let $f = \sum_{i=1}^{q} z_i I(a_{i-1}, a_i)$, where $z_1, \dots, z_q \in \mathbb{Z}$. Then

$$zI(0,1) = \mathcal{L}f = \sum_{i=1}^{q} z_i I(\tau a_{i-1}, \tau a_i). \tag{10.22}$$

By (10.21) and the IDOC for $\{a_1, \dots, a_{q-1}\}$, the points $\{\tau a_j \mid 0 \le j \le q\}$ are all distinct. If x is the first index in (10.22) for which z_x is nonzero, and y is the last such index, then the right side of (10.22) will be discontinuous at τa_{x-1} and τa_y, so $\{\tau a_{x-1}, \tau a_y\} \subset \{0,1\}$, and $z_x = z_y = z \neq 0$. Since $g \in \sum_{i=1}^{q-1} \mathbb{Z}E_i$ is continuous at 1,

$$g = f - \mathcal{L}f = \sum_{i=x}^{y} z_i I(a_{i-1}, a_i) - zI(0,1) \tag{10.23}$$

implies that $a_y = 1$. Since $\{\tau a_{x-1}, \tau a_y\} \subset \{0,1\}$, this contradicts our assumption that $\tau(\{0,1\}) \cap \{0,1\} = \emptyset$, so we have shown that g cannot be in $(\mathrm{id} - \mathcal{L})C(X,\mathbb{Z})$.

Finally, by Lemma 10.2, $\ker(\mathrm{id} - \mathcal{L}) = \{0\}$, so we have verified all the assumptions of Lemma 10.4. We conclude that $E_1, \dots, E_{q-1}$ are independent in the $\mathbb{Z}[t]$ module $C(X,\mathbb{Z})$. It follows that their equivalence classes are also independent in the $\mathbb{Z}[t,t^{-1}]$ module $\mathrm{DG}(\tau)$. □

Proposition 10.6. *Let $\tau : I \to I$ be multimodal and surjective, with associated partition $C = \{a_0, a_1, a_2, \dots, a_q\}$. If the orbits of $a_1, \dots, a_{q-1}$ are disjoint and infinite, and $\tau(\{0,1\}) \cap \{0,1\} = \emptyset$, then $\mathrm{DG}(\tau) \cong (\mathbb{Z}[t,t^{-1}])^{q-1}$ as (unordered) $\mathbb{Z}[t,t^{-1}]$ modules, with basis $I(a_0,a_1)$, $I(a_1,a_2)$, $\dots$, $I(a_{q-2},a_{q-1})$.*

Proof. By Corollary 6.3, $I(a_0,a_1), I(a_1,a_2) \dots, I(a_{q-1},a_q)$ generate $\mathrm{DG}(\tau)$. We will show $I(a_{q-1},a_q)$ is in the submodule generated by the first $q-1$ of these elements. It will follow that $I(a_0,a_1), I(a_1,a_2) \dots, I(a_{q-2},a_{q-1})$ generate $\mathrm{DG}(\tau)$, and are independent by Lemma 10.5, and thus these elements form a basis.

Since τ is surjective, we can choose indices x, y such that $\tau(a_x) = 0$ and $\tau(a_y) = 1$. For simplicity of notation, we suppose that $x < y$. Since $\tau(\{0,1\}) \cap \{0,1\} = \emptyset$, then $0 < x < y < q$. Next observe that for $a, b \in I_1$, $I(0,a) \pm I(a,b) = I(0,b)$, with the $\pm$ sign depending on whether $a < b$ or $a > b$. Let $b_i = \tau(a_{x+i})$ for $0 \le i \le y-x$. In particular, $b_0 = \tau(a_x) = 0$ and $b_{y-x} = \tau(a_y) = 1$. Then for appropriate choices of the $\pm$ signs in the equation below,

$$\begin{aligned}
&\mathcal{L}\left(I(a_x,a_{x+1}) \pm I(a_{x+1},a_{x+2}) \pm \cdots \pm I(a_{y-1},a_y)\right)\\
&= I(b_0,b_1) \pm I(b_1,b_2) \pm \cdots \pm I(b_{y-x-1},b_{y-x})\\
&= I(b_0,b_{y-x}) = I(0,1).
\end{aligned}$$

It follows that $I(0,1)$ is in the $\mathbb{Z}[t,t^{-1}]$ submodule of $\mathrm{DG}(\tau)$ generated by $I(a_0,a_1)$, $I(a_1,a_2)$, $\dots$, $I(a_{q-2},a_{q-1})$. Since

$$I(a_{q-1},a_q) = I(0,1) - I(a_0,a_1) - I(a_1,a_2) - \cdots - I(a_{q-2},a_{q-1}), \tag{10.24}$$

it follows that $I(a_0,a_1), I(a_1,a_2) \dots, I(a_{q-2},a_{q-1})$ generate $\mathrm{DG}(\tau)$. This completes the proof that the prescribed set of $q-1$ elements form a basis for the module $\mathrm{DG}(\tau)$, and thus that $\mathrm{DG}(\tau) \cong (\mathbb{Z}[t,t^{-1}])^{q-1}$. □

11. Interval exchange maps

Definition 11.1. A piecewise monotonic map $\tau : [0,1) \to [0,1)$ is an *interval exchange map* if τ is linear with slope 1 on each interval of monotonicity, is bijective, and is right continuous at all points. (See Figure 1 on page 479 for an example.) We will usually identify τ with its extension to a map from $[0,1]$ into $[0,1]$, defined to be left continuous at 1.

If $\{a_0, a_1, \ldots, a_n\}$ is the associated partition for an interval exchange map, then for each i, the lengths of $[a_{i-1}, a_i)$ and $\tau([a_{i-1}, a_i))$ will be the same, and the images of the intervals $\{\tau[a_{i-1}, a_i) \mid 1 \le i \le n\}$ will form a partition of $[0,1)$, which is the reason for the name "interval exchange map".

Definition 11.2. A map $\tau : I \to I$ is *essentially injective* if there are no disjoint intervals J, J' with $\tau(J) = \tau(J')$.

This is equivalent to τ being injective on the complement of the set of endpoints of intervals of monotonicity.

Lemma 11.3. *A piecewise monotonic map $\tau : I \to I$ is essentially injective iff the associated local homeomorphism $\sigma : X \to X$ is injective.*

Proof. If J, J' are disjoint intervals in I such that $\tau(J) = \tau(J')$, then $\tau(J \cap I_0) = \tau(J' \cap I_0)$. Since σ restricted to X_0 is conjugate to τ restricted to I_0, then σ is not 1-1.

Conversely, suppose σ is not 1-1. Note that for $z \in X$, $(\mathcal{L}1)(z)$ is the number of preimages of z. Let $z \in X$ have at least two distinct preimages. Since $\mathcal{L}1$ is integer valued and continuous, then $(\mathcal{L}1)(x) \ge 2$ for all x in an open neighborhood V of z. Thus each member of V has at least two preimages. It follows that each member of an open set in X_0 has two preimages, so the same is true for τ on an open set in I_0. It follows that τ is not essentially injective. $\square$

Definition 11.4. A piecewise monotonic map $\tau : I \to I$ is *essentially bijective* if τ is surjective and essentially injective, cf. Definitions 4.1 and 11.2.

A piecewise monotonic map $\tau : I \to I$ will be essentially bijective iff the associated local homeomorphism $\sigma : X \to X$ is bijective, cf. Lemma 11.3 and the remarks after Definition 4.1. Note that an interval exchange map $\tau : I \to I$ is bijective on $[0,1)$, but might only be essentially bijective on $I = [0,1]$.

For our purposes, the fact that an interval exchange map is linear on each interval of monotonicity will not play a role, which motivates the following definition:

Definition 11.5. A piecewise monotonic map $\tau : [0,1) \to [0,1)$ is a *generalized interval exchange map* if τ is increasing on each interval of monotonicity, is bijective, and is right continuous at all points. We will usually identify τ with its extension to a map from $[0,1]$ into $[0,1]$, defined to be left continuous at 1. (That map will be essentially bijective.)

We are now going to find the dimension group for (generalized) interval exchange maps that have certain orbits infinite and disjoint. (See also Theorem 9.1 and Proposition 10.6 for continuous maps with the IDOC property.)

Lemma 11.6. *Let $\tau : I \to I$ be piecewise monotonic, with associated partition $C = \{a_0, a_1, \ldots, a_n\}$, and associated local homeomorphism $\sigma : X \to X$. Let $E_i =$*

$I(a_{i-1}, a_i)$ for $1 \le i \le n-1$. If $\mathcal{L} = \mathcal{L}_\sigma$ is injective, and $\operatorname{orbit}(\widehat{\tau}C) \cap C \subset \{0,1\}$, *then $E_1, \ldots, E_{n-1}$ are independent in* $\mathrm{DG}(\tau)$.

Proof. Suppose first, to reach a contradiction, that $E_1, \ldots, E_{n-1}$ are not independent in the $\mathbb{Z}[t]$ module $C(X,\mathbb{Z})$, and choose $p_1, \ldots, p_{n-1}$ in $\mathbb{Z}[t]$, not all zero, satisfying $\sum_i p_i(\mathcal{L})E_i = 0$, with the sum of the degrees of the nonzero polynomials among $p_1, \ldots, p_{n-1}$ as small as possible. For $1 \le i \le n-1$, define $q_i(t) \in \mathbb{Z}[t]$ by $tq_i(t) = p_i(t) - p_i(0)$, and define

$$g_1 = \sum_{i=1}^{n-1} p_i(0)E_i, \text{ and } g_2 = \sum_{i=1}^{n-1} q_i(\mathcal{L})E_i. \tag{11.1}$$

Then $g_1 + \mathcal{L}g_2 = \sum_i p_i(\mathcal{L})E_i = 0$. Note by (7.4),

$$\mathcal{D}(\mathcal{L}g_2) \subset \widehat{\tau}(\mathcal{D}(g_2)) \cup \widehat{\tau}C. \tag{11.2}$$

Since $\cup_i \mathcal{D}(E_i) \subset C$, then $\mathcal{D}(g_2) \subset \operatorname{orbit}(C)$, so $\mathcal{D}(\mathcal{L}g_2) \subset \operatorname{orbit}(\widehat{\tau}C)$. Since $\mathcal{D}(g_1) \subset C \setminus \{1\}$ and $g_1 = -\mathcal{L}g_2$, by the assumption that $\operatorname{orbit}(\widehat{\tau}C) \cap C \subset \{0,1\}$,

$$\mathcal{D}(g_1) = \mathcal{D}(\mathcal{L}g_2) \subset \operatorname{orbit}(\widehat{\tau}C) \cap (C \setminus \{1\}) \subset \{0\}. \tag{11.3}$$

Thus $\mathcal{D}(g_1) = \mathcal{D}(\mathcal{L}g_2) = \{0\}$, hence $g_1 = 0$, and $\mathcal{L}g_2 = \mathcal{L}\sum_i q_i(\mathcal{L})E_i = 0$. By assumption, $\mathcal{L}$ is injective, so $\sum_i q_i(\mathcal{L})E_i = 0$, which contradicts the minimality of the sum of the degrees of the nonzero polynomials among $p_1, \ldots, p_{n-1}$. We conclude that $p_1, \ldots, p_{n-1}$ are all the zero polynomial. Thus $E_1, \ldots, E_{n-1}$ are independent in the $\mathbb{Z}[t]$ module $C(X,\mathbb{Z})$. It follows that their equivalence classes are independent in the $\mathbb{Z}[t, t^{-1}]$ module $\mathrm{DG}(\tau)$. □

We will write $\tau(a^-)$ for $\lim_{t\to a^-} \tau(t)$, and define $\tau(a^+)$ similarly.

Proposition 11.7. *Assume $\tau : I \to I$ is piecewise monotonic and essentially bijective, with associated partition $C = \{a_0, a_1, \ldots, a_n\}$. If* $\operatorname{orbit}(\widehat{\tau}C) \cap C \subset \{0,1\}$, *then the (unordered) module* $\mathrm{DG}(\tau)$ *is isomorphic to $\mathbb{Z}[t, t^{-1}]^{n-1} \oplus \mathbb{Z}$, where the action of $\mathbb{Z}[t, t^{-1}]$ on the first summand is the usual one, and t acts trivially on the second summand.*

Proof. Since τ is essentially bijective, the associated local homeomorphism $\sigma : X \to X$ is bijective. Then for $g \in C(X,\mathbb{Z})$, $\mathcal{L}_\sigma g = g \circ \sigma^{-1}$, so $\mathcal{L}_\sigma$ is bijective.

Let $E_i = I(a_{i-1}, a_i)$ for $1 \le i \le n$. By Lemma 11.6, $E_1, \ldots, E_{n-1}$ are independent in $\mathrm{DG}(\tau)$. Let $\widehat{\tau}C = \{b_0, b_1, \ldots, b_n\}$, in increasing order. Then each interval $I(b_{j-1}, b_j)$ is equal to $\mathcal{L}E_k$ for some k. The left and right limits of τ at partition points are in $\widehat{\tau}C$, so each jump $I(\tau(a_i^-), \tau(a_i^+))$ at partition points is of the form $I(b_j, b_k)$ for some $j < k$, and therefore is a sum of intervals of the form $\mathcal{L}E$ for $E \in \{E_1, \ldots, E_n\}$. Furthermore, by surjectivity, $0 \in \widehat{\tau}C$, so by Theorem 6.2, $\{E_1, \ldots, E_n\}$ is a set of generators for $\mathrm{DG}(\tau)$. Since $E_1 + E_2 \cdots + E_n = I(0,1)$, then the intervals $I(0,1)$, E_1, E_2, $\ldots$, E_{n-1} also generate.

Since σ is bijective, $\mathcal{L}_* I(0,1) = I(0,1)$, so for every $p \in \mathbb{Z}[t, t^{-1}]$, $p(\mathcal{L}_*)I(0,1) = p(1)I(0,1)$. Thus $\mathbb{Z}[t, t^{-1}]I(0,1) = \mathbb{Z}I(0,1)$, so every element of $\mathrm{DG}(\tau)$ can be written in the form $\sum_{i=1}^{n-1} p_i(\mathcal{L}_*)E_i + zI(0,1)$, with each $p_i \in \mathbb{Z}[t, t^{-1}]$, and $z \in \mathbb{Z}$. By the independence of $\{E_i \mid 1 \le i \le n-1\}$, no nonzero element $\sum_{i=1}^{n-1} p_i(\mathcal{L}_*)E_i$ is fixed by $\mathcal{L}_*$, so this representation is unique. □

When we have referred to the orbit of a point with respect to a piecewise monotonic map with some discontinuities, we have previously used the orbit with respect to the multivalued map $\widehat{\tau}$. In the next proposition, we need to refer to the orbit with respect to τ itself, and we will call this the τ-orbit to avoid confusion with our earlier usage. Recall that if a set B has the property that its points have orbits that are infinite and disjoint, we refer to this property as the IDOC for B.

Corollary 11.8. *Let $\tau : I \to I$ be a generalized interval exchange map, with associated partition $C = \{a_0, a_1, \ldots, a_n\}$. If the τ-orbits of $a_1, a_2, \ldots, a_{n-1}$ are infinite and disjoint, then $\mathrm{DG}(\tau) = C(X, \mathbb{Z})$ is isomorphic as an (unordered) $\mathbb{Z}[t, t^{-1}]$ module to $\mathbb{Z}[t, t^{-1}]^{n-1} \oplus \mathbb{Z}$ (where the action of $\mathcal{L}_*$ on the first summand is coordinatewise multiplication by t, and on the second summand is trivial).*

Proof. Let $C_0 = C \setminus \{0, 1\} = \{a_1, a_2, \ldots, a_{n-1}\}$. We'll prove

$$x \in C \implies \mathrm{orbit}(\widehat{\tau}x) \cap C_0 = \emptyset. \tag{11.4}$$

First consider the case $x = 0$. If $\tau 0 = 0$, then (11.4) is clear. If $\tau 0 \neq 0$, by surjectivity, we can choose $a \in C_0$ with $\tau a = 0$. Then $\mathrm{orbit}(\widehat{\tau}0) = \mathrm{orbit}(\tau^2 a)$, and $\mathrm{orbit}(\tau^2 a)$ misses C_0 by the IDOC.

Now suppose $x \in C_0$. Then the orbit of τx misses C_0 by the IDOC. Furthermore, if $x \in C_0 \cup \{1\}$, there exists $y \in C_0 \cup \{0\}$ such that $\tau y = \tau(x^-)$. Then $\mathrm{orbit}(\tau(x^-)) = \mathrm{orbit}(\tau y)$ misses C_0 by the previous part of this proof. Thus we've shown $\mathrm{orbit}(\widehat{\tau}C) \cap C_0 = \emptyset$.

As observed in the proof of Proposition 11.7, if $\sigma : X \to X$ is the local homeomorphism associated with τ, then essential bijectivity of τ implies that $\mathcal{L} = \mathcal{L}_\sigma$ is bijective, and so $\mathrm{DG}(\tau) = C(X, \mathbb{Z})$. The rest of the corollary follows from Proposition 11.7. □

Note that in Corollary 11.8, the order on $\mathrm{DG}(\tau) = C(X, \mathbb{Z})$ is just the usual pointwise order on $C(X, \mathbb{Z})$.

12. C*-algebras

Let $\tau : I \to I$ be a piecewise monotonic map, with associated partition $C = \{a_0, a_1, \ldots, a_n\}$. We now give a dynamical construction of a C*-algebra A_τ such that $K_0(A_\tau) \cong \mathrm{DG}(\tau)$. The construction will also display $\mathrm{DG}(\tau)$ explicitly as an inductive limit of groups of the form $\mathbb{Z}^{n_k}$, with the connecting maps easily computable from the map τ.

Define $C_0 = \{0, 1\}$, and for $n \geq 1$, define

$$C_n = \left(\bigcup_{k=1}^{2n} \widehat{\tau}^k C \right) \bigcap \widehat{\tau}^n(I).$$

Observe that the sequence $\{C_n\}$ has the following properties:

$$\widehat{\tau}(C) \subset C_1. \tag{12.1}$$

$$\widehat{\tau}(C_n) \subset C_{n+1}. \tag{12.2}$$

$$C_n \subset \widehat{\tau}^n I. \tag{12.3}$$

$$C_k \cap \widehat{\tau}^n I \subset C_n \text{ for } k \leq n. \tag{12.4}$$

$$\text{If } a \in I_1, \text{ then there exists } n \geq 0 \text{ such that } \widehat{\tau}^n(a) \subset C_n. \tag{12.5}$$

Any sequence $\{C_n\}$ of finite subsets of I_1 satisfying these five properties would work equally well in what follows. (Note that (12.5) could fail if we replaced the upper limit $2n$ by n in the union defining C_n.) We say $x, y \in C_n$ are *adjacent in* C_n if there is no point in C_n strictly between x and y. For $n \geq 0$, let $\mathcal{P}_n$ be the set of closed intervals contained in $\widehat{\tau}^n(I)$ whose endpoints are adjacent points in C_n. For each $Y \in \mathcal{P}_n$, choose a point x in the interior of Y and define $k(Y)$ to be the cardinality of $\tau^{-n}(x)$. (By the definition of $\mathcal{P}_n$, $k(Y)$ does not depend on the choice of x.) For $n \geq 0$, define

$$A_n = \oplus_{Y \in \mathcal{P}_n} C(Y, M_{k(Y)}). \tag{12.6}$$

Recall that for each i, τ_i denotes the continuous extension of $\tau|_{(a_{i-1}, a_i)}$ to a homeomorphism on $[a_{i-1}, a_i]$. Fix $n \geq 0$. If $Z \in \mathcal{P}_{n+1}$, for each index i, $\tau_i(I)$ either contains Z or is disjoint from the interior of Z. In the former case, there is exactly one $Y \in \mathcal{P}_n$ such that such that $\tau_i(Y) \supset Z$, or equivalently, $\tau_i^{-1}(Z) \subset Y$.

For $n \geq 0$, let X^n be the disjoint union of $\{Y \in \mathcal{P}_n\}$. We view A_n as a subalgebra of $C(X^n, \oplus_{Y \in \mathcal{P}_n} M_{k(Y)})$. Define $\phi_n : A_n \to A_{n+1}$ by

$$\phi_n(f) = \oplus_{Z \in \mathcal{P}_{n+1}} \mathrm{diag}\ (f \circ \tau_{i_1}^{-1}, \ldots, f \circ \tau_{i_p}^{-1}), \tag{12.7}$$

where for each $Z \in \mathcal{P}_{n+1}$ the indices $i_1, \ldots, i_p$ are the indices i such that $\tau_i(I)$ contains Z.

Let A_τ be the inductive limit of the sequence $\phi_n : A_n \to A_{n+1}$. Recall that an *interval algebra* is one isomorphic to $C([0,1], F)$, where F is a finite-dimensional C*-algebra, and an AI-algebra is an inductive limit of a sequence of interval algebras. Thus each A_n is an interval algebra, and A_τ is an AI-algebra.

Example 12.1. We illustrate this construction with the full tent map, cf. Example 9.2. Here $C_n = \{0, 1\}$ for all n, $\tau_1^{-1}(x) = x/2$ and $\tau_2^{-1}(x) = 1 - x/2$. Hence for all n, $A_n = C([0,1], M_{2^n})$, and $\phi_n : A_n \to A_{n+1}$ is given by $\phi_n(f) = \mathrm{diag}(f \circ \tau_1^{-1}, f \circ \tau_2^{-1})$. (This example also appears in [3] and [7].)

Example 12.2. Let τ be the restricted tent map $\tau = T_{\sqrt{2}}$. In this case the critical point c is eventually fixed, with $\tau^4 c = \tau^3 c$. Let $c = c_0, c_1, c_2, c_3, \cdots$ be the orbit of c, and let p be the fixed point of τ. Note that $0 = c_2 < c_0 = c < c_3 = p < c_1 = 1$. Then we can represent the imbeddings by the diagram of intervals in Figure 2.

The disjoint union of the intervals on each level of this diagram are the sets $X^0, X^1, X^2, \ldots$ described previously, with the connecting lines indicating the maps of intervals by τ. From this diagram, the algebras $A_0, A_1, \ldots$ and the maps $\phi_n : A_n \to A_{n+1}$ can also be read off. For example,

$$A_3 = C([c_2, c_3], M_2) \oplus C([c_3, c_1], M_4). \tag{12.8}$$

Here the dimensions of the matrix algebras for a given interval are the number of paths from the bottom to that interval. Since $K_0(C([0,1], M_n)) = \mathbb{Z}$, $K_0(A_n)$ is found by replacing each interval by a copy of $\mathbb{Z}$. The connecting lines indicate the connecting maps for the inductive limit of dimension groups, giving $\mathrm{DG}(\tau)$. In this particular example, the map τ is Markov, so the width of the diagram stabilizes. For general τ the diagram can become steadily wider, as can be seen from the examples discussed in the previous section.

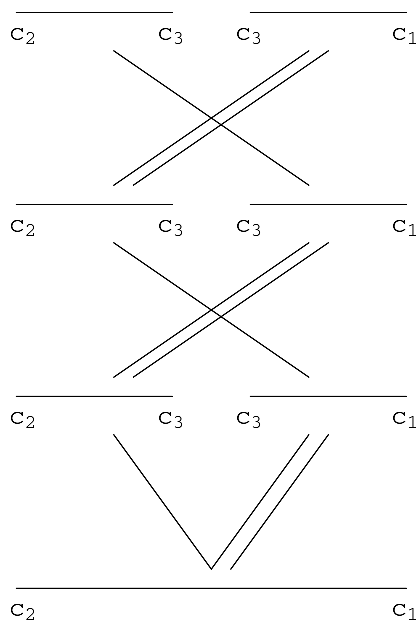

FIGURE 2. Inductive limit construction

The diagram in Figure 2, with intervals shrunk to points, also can be thought of as a Bratteli diagram, cf. [2]. The AF-algebra associated with that diagram would have the same K_0 group as A_τ. However, A_τ is not necessarily an AF-algebra.

Proposition 12.3. *If $\tau : I \to I$ is piecewise monotonic, $K_0(A_\tau)$ and $\mathrm{DG}(\tau)$ are isomorphic dimension groups.*

Proof. Recall that for each n, we view elements of A_n as maps from X^n into a suitable matrix algebra. We view elements of the matrix algebra $M_q(A_n)$ as functions from X^n into $\oplus_{Y\in\mathcal{P}_n} M_q(M_{k(Y)})$. For each projection p in a matrix algebra over A_n, define $\dim(p)$ to be the function $y \mapsto \dim\ p(y)$. Then $\dim(p) \in C(X^n, \mathbb{Z})$, and since X_n is a disjoint union of closed intervals, dim extends uniquely to an isomorphism of $K_0(A_n)$ onto $C(X^n, \mathbb{Z})$.

Let $\pi_n : \sigma^n(X) \to X^n$ be the collapse map (that collapses $x^\pm$ to the single point x except for the preimages of points in C_n.) For each $n \ge 0$, let $\widehat{C}(X^n, \mathbb{Z})$ denote the subgroup of $C(X, \mathbb{Z})$ generated by characteristic functions of intervals $I(a, b)$ with $a, b \in C_n$. Then $f \mapsto f \circ \pi_n$ is an order isomorphism from $C(X^n, \mathbb{Z})$ onto $\widehat{C}(X^n, \mathbb{Z})$. Observe that

$$\widehat{C}(X^n, \mathbb{Z}) = \{f \in C(X, \mathbb{Z}) \mid \mathcal{D}(f) \subset C_n\},$$

where $\mathcal{D}(f)$ is the set of discontinuities of f viewed as a function on $\mathbb{R}$, cf. Section 7. If $f \in \widehat{C}_n(X, \mathbb{Z})$, then $\operatorname{supp} f \subset \sigma^n(X)$, and $\operatorname{supp} \mathcal{L}f \subset \sigma^{n+1}(X)$. By (7.4) and (12.1)–(12.4),

$$\mathcal{D}(\mathcal{L}f) \subset \big(\widehat{\tau}(\mathcal{D}(f)) \cup \widehat{\tau}C\big) \cap \widehat{\tau}^{n+1} I \subset (\widehat{\tau}(C_n) \cup \widehat{\tau}C) \cap \widehat{\tau}^{n+1} I \subset C_{n+1}.$$

Thus $\mathcal{L}$ maps $\widehat{C}_n(X, \mathbb{Z})$ into $\widehat{C}_{n+1}(X, \mathbb{Z})$.

Consider the following commutative diagram:

$$
\begin{array}{ccccc}
K_0(A_n) & \xrightarrow{\dim} & C(X^n,\mathbb{Z}) & \longrightarrow & \widehat{C}_n(X,\mathbb{Z}) \\
\downarrow {\scriptstyle (\phi_n)_*} & & & & \downarrow {\scriptstyle \mathcal{L}} \\
K_0(A_{n+1}) & \xrightarrow{\dim} & C(X^{n+1},\mathbb{Z}) & \longrightarrow & \widehat{C}_{n+1}(X,\mathbb{Z}).
\end{array}
\tag{12.9}
$$

(The commutativity of this diagram can be verified by checking it on generators of $K_0(A_n)$, which are of the form $[1_Y e_{11}]$ for $Y \in \mathcal{P}_n$.) Since the horizontal maps are isomorphisms, by continuity of K_0 with respective to inductive limits, $K_0(A)$ is isomorphic to the inductive limit of the sequence $\mathcal{L} : \widehat{C}_n(X,\mathbb{Z}) \to \widehat{C}_{n+1}(X,\mathbb{Z})$. We will be done if we show this inductive limit is isomorphic to $\mathrm{DG}(\tau)$.

For that purpose, we apply Lemma 8.3. By (7.4), for $f \in C(X,\mathbb{Z})$,

$$
\begin{aligned}
\mathcal{D}(\mathcal{L}^n f) &\subset \widehat{\tau}^n \mathcal{D}(f) \cup \left(\left(\bigcup_{k=1}^{n} \widehat{\tau}^k C \right) \cap \widehat{\tau}^n I \right) \\
&\subset \widehat{\tau}^n \mathcal{D}(f) \cup C_n .
\end{aligned}
\tag{12.10}
$$

By (12.2) and (12.5), for each $f \in C(X,\mathbb{Z})$ there exists $n \geq 0$ such that $\widehat{\tau}^n \mathcal{D}(f) \subset C_n$, so (12.10) implies that $\mathcal{L}^n f \in \widehat{C}_n(X,\mathbb{Z})$. If $f \in \widehat{C}_k(X,\mathbb{Z})$, then $\operatorname{supp} f \subset \sigma^k(X)$, so $f \in \mathcal{L}^k C(X,\mathbb{Z})$ (Lemma 3.5). By Lemma 8.3, $\mathrm{DG}(\tau)$ is isomorphic to the inductive limit of the sequence $\mathcal{L} : C_n(X,\mathbb{Z}) \to C_{n+1}(X,\mathbb{Z})$, so $K_0(A)$ is isomorphic to $\mathrm{DG}(\tau)$. □

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WELLESLEY COLLEGE, WELLESLEY, MASSACHUSETTS 02481
fshultz@wellesley.edu

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