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Perturbation theory for random walk in asymmetric random environment

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Abstract. In this paper the author continues his investigation into the scaling limit of a partial difference equation on the d-dimensional integer lattice $\mathbf{Z}^d$, corresponding to a translation invariant random walk perturbed by a random vector field. In a previous paper he obtained a formula for the effective diffusion constant. It is shown here that for the nearest neighbor walk in dimension $d \geq 3$ this effective diffusion constant is finite to all orders of perturbation theory. The proof uses Tutte's decomposition theorem for 2-connected graphs into 3-blocks.

Contents

1. Introduction

In this paper we continue the study of the homogenization problem for random walk in asymmetric random environment which was begun in [4]. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $b : \Omega \to \mathbf{R}$ be a bounded measurable function with mean 0. We assume that $\mathbf{Z}^d$ acts on Ω by translation operators $\tau_x : \Omega \to \Omega,\ x \in \mathbf{Z}^d$, which are measure preserving and satisfy the properties $\tau_x\tau_y = \tau_{x+y},\ x, y \in \mathbf{Z}^d$, $\tau_0 =$ identity. For $i = 1, \ldots, d$, let $\mathbf{e}_i \in \mathbf{Z}^d$ be the element with entry 1 in the ith position and 0 in the other positions. Suppose $\gamma \in \mathbf{C}$ and $f : \mathbf{R}^d \to \mathbf{R}$ is a C^∞ function with compact support. We shall be interested in solutions to the equation

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on the ε scaled integer lattice $\mathbf{Z}_\varepsilon^d = \varepsilon\mathbf{Z}^d$ given by

$$(1.1)\qquad u_\varepsilon(x,\omega) - \sum_{i=1}^{d} \frac{1}{2d}\left[u_\varepsilon(x+\varepsilon\mathbf{e}_i,\ \omega) + u_\varepsilon(x-\varepsilon\mathbf{e}_i,\ \omega)\right]$$
$$- \gamma\, b(\tau_{x/\varepsilon}\ \omega)\left[u_\varepsilon(x+\varepsilon\mathbf{e}_1,\ \omega) - u_\varepsilon(x-\varepsilon\mathbf{e}_1,\ \omega)\right] + \varepsilon^2 u_\varepsilon(x,\omega)$$
$$= \varepsilon^2\ f(x),\quad x\in\mathbf{Z}_\varepsilon^d,\quad \omega\in\Omega.$$

In Theorem 1.3 of [4] it was shown that if $\sup_{\omega\in\Omega}|b(\omega)| \le 1$ and $\gamma < \varepsilon/\sqrt{2d}$ then (1.1) has a unique solution $u_\varepsilon(x,\omega)$ in $L^2(\mathbf{Z}_\varepsilon^d)$ which is also analytic in γ.

Let $Y_x,\ x\in\mathbf{Z}^d$, be i.i.d. Bernoulli variables, $Y_x = \pm 1$ with equal probability. We now take the function $b(\omega)$ in (1.1) to be defined by $b(\tau_x\ \omega) = Y_x,\ x\in\mathbf{Z}^d$. In that case Theorem 1.3 of [4] gives an identity for the expectation value of the Fourier transform of the solution $u_\varepsilon(x,\omega)$ of (1.1). For an absolutely summable function $g:\mathbf{Z}_\varepsilon^d\to\mathbf{C}$ we define its Fourier transform $\hat{g}(\xi),\ \xi\in[-\pi/\varepsilon,\pi/\varepsilon]^d$, by

$$(1.2)\qquad \hat{g}(\xi) = \sum_{x\in\mathbf{Z}_\varepsilon^d} \varepsilon^d g(x) e^{ix\cdot\xi}.$$

Note that if $\hat{f}_\varepsilon(\xi)$ is the Fourier transform of the function $f:\mathbf{R}^d\to\mathbf{R}$ restricted to $\mathbf{Z}_\varepsilon^d$ then $\lim_{\varepsilon\to 0}\hat{f}_\varepsilon(\xi) = \hat{f}(\xi),\ \xi\in\mathbf{R}^d$, where $\hat{f}(\xi)$ is the Fourier transform of f. For $\zeta\in\mathbf{R}^d$ let $e(\zeta)$ be the d-dimensional vector $e(\zeta) = (e_1(\zeta),\dots,e_d(\zeta))$ with $e_k(\zeta) = 1 - e^{i\mathbf{e}_k\cdot\zeta},\ k=1,\dots,d.$ Denote by $\hat{u}_\varepsilon(\xi)$ the Fourier transform of the expectation of the solution $u_\varepsilon(x,\omega),\ x\in\mathbf{Z}_\varepsilon^d$, to (1.1). Theorem 1.3 of [4] states that there is a $d\times d$ matrix $q_{\gamma,\varepsilon}(\zeta)\ \zeta\in\mathbf{R}^d$, which is periodic on $[-\pi,\pi]$, such that

$$(1.3)\qquad \hat{u}_\varepsilon(\xi)\left[1 + \frac{1}{2d\varepsilon^2}|e(\varepsilon\xi)|^2 - \varepsilon^{-2} e(\varepsilon\xi) q_{\gamma,\varepsilon}(\varepsilon\xi) e(-\varepsilon\xi)\right] = \hat{f}_\varepsilon(\xi),$$
$$\xi\in[-\pi/\varepsilon,\pi/\varepsilon]^d.$$

The matrix $q_{\gamma,\varepsilon}(\zeta)$ is analytic in γ for $\gamma<\varepsilon/\sqrt{2d}$ and hence can be written as a convergent power series in this region. By Theorem 1.4 of [4] we may write

$$(1.4)\qquad q_{\gamma,\varepsilon}(\zeta) = \sum_{m=2}^{\infty} \gamma^{2m} q_{m,\varepsilon}(\zeta).$$

In [4] it was also shown that in dimension $d=1$ the functions $q_{m,\varepsilon}$ converge as $\varepsilon\to 0$ in the sense that for any compact set $K\subset\mathbf{R}$ then $\varepsilon^m\ q_{m,\varepsilon}(\varepsilon\,\xi)$ converges uniformly for $\xi\in K$ to a function $q_m(\xi)$ as $\varepsilon\to 0$. This limit is related to the limits obtained by Sinai [16] and Kesten [10] for one dimensional random walk in random environment.

In this paper we investigate the convergence properties of the matrices $q_{m,\varepsilon}(\zeta)$ as $\varepsilon\to 0$ in dimension $d>1$. In particular we prove the following:

Theorem 1.1. *For $d\ge 3$ there is a constant C_d depending only on d such that*

$$|q_{m,\varepsilon}(\zeta)| \le C_d^m\ m!,\ \ 0<\varepsilon<1,\ \ \zeta\in\mathbf{R}^d.$$

Furthermore, the matrix $q_{m,\varepsilon}(\zeta)$ converges uniformly for $\zeta\in[-\pi,\pi]^d$ as $\varepsilon\to 0$ to a matrix $q_m(\zeta)$.

In [4] it was shown that the matrix $q_m(\zeta) = [q_{m,k,k'}(\zeta)]$, $1 \le k, k' \le d$, has only one nonzero entry $q_{m,1,1}(0)$ for $\zeta = 0$. Hence taking the formal limit of (1.3) as $\varepsilon \to 0$ we obtain the effective homogenized equation for (1.1) in dimension $d \ge 3$,

$$\hat{u}(\xi)\left[1 + \frac{|\xi|^2}{2d} - (\mathbf{e}_1 \cdot \xi)^2 \sum_{m=2}^{\infty} \gamma^{2m} q_{m,1,1}(0)\right] = \hat{f}(\xi),\ \xi \in \mathbf{R}^d. \tag{1.5}$$

Theorem 1.1 shows that at the level of formal perturbation theory random walk in asymmetric environment is diffusive at large time for dimension $d \ge 3$, in agreement with the predictions of Fisher [9] and Derrida–Luck [6] . One should note here that since Theorem 1.1 gives only a factorial bound on the coefficients in the series (1.5) it does not imply that a homogenized limit even exists. The existence of the homogenized limit has been rigorously proven for small values of γ in [3, 19]. Nonperturbative arguments play a crucial role in the proof and it is unlikely that a proof can be given based on perturbation theory alone.

In contrast to this situation perturbation theory based proofs of the existence of homogenized limits can be given for random walks which satisfy some type of symmetry condition. This is the case for random walks in a symmetric environment [1] and for reversible random walks [5, 11]. In these situations one needs some restrictions on the strength of the environment noise. Otherwise one also has to use nonperturbative arguments, as for example in [2, 12, 13] which prove homogenization for symmetric walks without the smallness restriction on the environment noise of [1].

A distinguishing feature of the random walk in asymmetric environment is that its large time behavior depends on dimension. For dimension $d = 1$ Sinai [16] has shown that the large time behavior is subdiffusive. There has also been more recent rigorous work [2, 14, 17, 18] on (1.1) under the assumption $\langle \mathbf{b}(\cdot)\rangle \ne 0$. This situation is very different to the situation studied in Theorem 1.1 since one expects now the drift to dominate diffusion. The methods used in [17, 18] are related to methods used to prove Anderson localisation for the random Schrödinger equation.

The proof of Theorem 1.1 follows a similar strategy to that used to show perturbative renormalization in Euclidean field theories [15]. First one shows by a simple multiscale decomposition that a large class of Feynman graphs are completely convergent. This is the content of Lemma 2.6 and Corollary 2.1. The basic argument here goes back to Weinberg [21]. The paper of Feldman et al. [7] proves a very general version of Weinberg's theorem. Next one bounds an arbitrary Feynman graph by subdividing the graph into pieces which are completely convergent and then using the cancelation properties of the propagator. This is analogous to the renormalization procedure in Euclidean field theory [8, 15].

The graph subdivision in this paper is implemented by applying Tutte's decomposition theorem for 2-connected graphs into 3-blocks which is described in Chapter IV of [20]. Tutte's theorem does not appear to have been previously used to prove finiteness of Feynman graphs. In this paper we shall use the terminology of [20]. In particular the graphs we consider are multigraphs with multiple edges but no loops.

2. Proof of Theorem 1.1

For $\eta > 0,\ x \in \mathbf{Z}^d$, let $G_\eta(x)$ be the Green's function which satisfies the equation,

$$G_\eta(x) - \frac{1}{2d}\sum_{i=1}^{d}[G_\eta(x+\mathbf{e}_i)+G_\eta(x-\mathbf{e}_i)] + \eta\, G_\eta(x) = \delta(x),\ x \in Z^d,$$

where δ is the Kronecker δ function, $\delta(x) = 0,\ x \neq 0,\ \delta(0) = 1$. If $\hat{G}_\eta(\zeta),\ \zeta \in [-\pi,\pi]^d$, denotes the Fourier transform of $G_\eta(x)$ as defined by (1.2) with $\varepsilon = 1$ then we have

$$\hat{G}_\eta(\zeta) = 1/[\eta + |e(\zeta)|^2/2d],\ \zeta \in [-\pi,\pi]^d. \tag{2.1}$$

Lemma 2.1. *Let $h : \mathbf{Z}^d \to \mathbf{C}$ be an exponentially decreasing function with Fourier transform $\hat{h}$. Suppose further there is an α satisfying $0 < \alpha \le 1$ such that*

$$|\nabla^k \hat{h}(\zeta)| \le 1/|\zeta|^{k+1}, \quad 0 \le k \le d-1, \quad \zeta \in [-\pi,\pi]^d, \tag{2.2}$$

$$|\nabla^{d-1}\hat{h}(\zeta+\delta) - \nabla^{d-1}\hat{h}(\zeta)| \le |\delta|^\alpha/|\zeta|^{d+\alpha},\ |\delta| < |\zeta|/2,\ \zeta \in [-\pi,\pi]^d.$$

Then there is a constant $C_{d,\alpha}$ depending only on d,α such that

$$|h(x)| \le C_{d,\alpha}\Big/\left[1+|x|^{d-1}\right], \quad x \in \mathbf{Z}^d. \tag{2.3}$$

Proof. Since we are assuming h is exponentially decreasing it follows that $\hat{h}$ is C^∞ and h is given by the formula

$$h(x) = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\hat{h}(\zeta)e^{-ix\cdot\zeta}d\zeta, \quad x \in \mathbf{Z}^d. \tag{2.4}$$

From (2.4) and (2.2) with $k = 0$ we see that (2.3) holds if $|x| \le 1$, whence we shall assume $|x| > 1$. We write

$$h(x) = \int\limits_{|\zeta|<1/|x|} + \int\limits_{|\zeta|>1/|x|}. \tag{2.5}$$

It follows again from (2.2) with $k = 0$ that

$$\left|\ \int\limits_{|\zeta|<1/|x|}\ \right| \le C'_d\Big/\left[1+|x|^{d-1}\right],$$

for some constant C'_d depending only on d. We estimate the second term on the RHS of (2.5) by integrating by parts $d-1$ times. We may assume wlog that $x = (x_1,\dots,x_d)$ satisfies $|x_1| \ge |x|/\sqrt{d}$. We have then that

$$\begin{aligned}\int\limits_{|\zeta|>1/|x|} &= \frac{1}{(2\pi)^d x_1^{d-1}}\int\limits_{|\zeta|>1/|x|}\hat{h}(\zeta)\left(i\frac{\partial}{\partial\zeta_1}\right)^{d-1}e^{-ix\cdot\zeta}d\zeta\\ &= \frac{1}{(2\pi)^d x_1^{d-1}}\Bigg[\int\limits_{|\zeta|>1/|x|}\left(-i\frac{\partial}{\partial\zeta_1}\right)^{d-1}\hat{h}(\zeta)e^{-ix\cdot\zeta}d\zeta\\ &\quad+\sum_{r=0}^{d-2}\int\limits_{|\zeta|=1/|x|} i\left(-i\frac{\partial}{\partial\zeta_1}\right)^{r}\hat{h}(\zeta)\left(i\frac{\partial}{\partial\zeta_1}\right)^{d-2-r}e^{-ix\cdot\zeta}n_1 d\zeta\Bigg],\end{aligned}$$

where $\mathbf{n} = (n_1, \dots, n_d)$ is the inward pointing unit normal on the sphere $|\zeta| = 1/|x|$. From (2.2) with $k = 0, \dots, d-2$ it follows that the surface integral on $\{|\zeta| = 1/|x|\}$ in the last expression is bounded by the RHS of (2.3). We are left then to estimate the volume integral on $\{|\zeta| > 1/|x|\}$. From (2.2) with $k = d-1$ we obtain a bound which is logarithmically larger than the RHS of (2.3). We use the second inequality of (2.2) to improve this. For $\delta \in \mathbf{R}^d$ let $S_\delta = \{\zeta \in \mathbf{R}^d : \zeta + \delta \in [-\pi, \pi]^d, \ |\zeta + \delta| > |x|^{-1}\}$. Evidently we have that

$$\int_{|\zeta|>1/|x|} \left(-i\frac{\partial}{\partial \zeta_1}\right)^{d-1} \hat{h}(\zeta) e^{-ix\cdot\zeta} d\zeta = \int_{S_\delta} \left(-i\frac{\partial}{\partial \zeta_1}\right)^{d-1} \hat{h}(\zeta+\delta) e^{-ix\cdot(\zeta+\delta)} d\zeta, \tag{2.6}$$

for any $\delta \in \mathbf{R}^d$. We take $\delta = (\delta_1, \cdots, \delta_d)$ with $\delta_1 = \pi/x_1, \delta_j = 0, j \neq 1$. Then the LHS of (2.6) is given by

$$\begin{aligned} &\frac{1}{2}\int_{S_0\cap S_\delta} \left(-i\frac{\partial}{\partial \zeta_1}\right)^{d-1} \left[\hat{h}(\zeta) - \hat{h}(\zeta+\delta)\right] e^{-ix\cdot\zeta} d\zeta \\ &+ \frac{1}{2}\int_{S_0\setminus S_\delta} \left(-i\frac{\partial}{\partial \zeta_1}\right)^{d-1} \hat{h}(\zeta) e^{-ix\cdot\zeta} d\zeta \\ &- \frac{1}{2}\int_{S_\delta\setminus S_0} \left(-i\frac{\partial}{\partial \zeta_1}\right)^{d-1} \hat{h}(\zeta+\delta) e^{-ix\cdot\zeta} d\zeta. \end{aligned} \tag{2.7}$$

From (2.2) with $k = d-1$ the last two integrals in (2.7) are bounded by a constant C_d depending only on d. From the second inequality of (2.2) the first integral in (2.7) is bounded by a constant $C_{d,\alpha}$ depending only on d, α. □

Lemma 2.2. *For $\eta > 0$ let $K_\eta(x)$ be the function*

$$K_\eta(x) = G_\eta(x - \mathbf{e}_1) - G_\eta(x + \mathbf{e}_1), \ x \in \mathbf{Z}^d.$$

Suppose $g_k : \mathbf{Z}^d \to \mathbf{C}, \ k = 1, \dots, n$ are exponentially decreasing functions which satisfy $g_k(x) = -g_k(-x), \ |g_k(x)| \le 1/[1 + |x|^{d-1}]^3, \ x \in \mathbf{Z}^d, \ 1 \le k \le n$. Let $h : \mathbf{Z}^d \to \mathbf{C}$ be the convolution,

$$h = K_\eta * g_1 * K_\eta * g_2 * \cdots * K_\eta * g_n * K_\eta.$$

Then h is exponentially decreasing and satisfies

$$h(x) = -h(-x), \ |h(x)| \le C_d^n/[1 + |x|^{d-1}], \ x \in \mathbf{Z}^d,$$

for some constant C_d depending only on d.

Proof. It is evident that h is exponentially decreasing and that $h(x) = -h(-x)$. To obtain the bound on $|h(x)|$ we use Lemma 2.1. We have that

$$\hat{h}(\zeta) = \hat{K}_\eta(\zeta)^{n+1} \prod_{k=1}^n \hat{g}_k(\zeta).$$

From (2.1) we have that

$$\hat{K}_\eta(\zeta) = 2i \sin\zeta_1/[\eta + |e(\zeta)|^2/2d].$$

Since g_k is an odd function we have

$$\hat{g}_k(\zeta) = i \sum_{x\in\mathbf{Z}^d} g_k(x) \sin(x\cdot\zeta).$$

Hence if $d > 3$ there is a constant C_d depending only on d such that

$$|\nabla^k \hat{h}(\zeta)| \le C_d^n/|\zeta|^{k+1}, \ 0 \le k \le d.$$

The bound on $|h(x)|$ follows now from Lemma 2.1 if $d > 3$. For $d = 3$ the first inequality of (2.2) holds. The second inequality of (2.2) also holds for $\alpha < 1$ by observing that

$$\left| \sum_{x \in \mathbf{Z}^3} |x|^2 g_k(x) \left[\sin(x \cdot \zeta + x \cdot \delta) - \sin x \cdot \zeta\right] \right| \le C_\alpha |\delta|^\alpha,$$

where C_α depends only on α. □

By Theorem 1.4 of [4] the $d \times d$ matrix $q_{m,\varepsilon}(\zeta) = [q_{m,\varepsilon,k',k}(\zeta)]$ defined by (1.4) has nonzero entries only for $k' = 1, \ 1 \le k \le d$. If $\zeta = 0$ then there is just one nonzero entry, $q_{m,\varepsilon,1,1}(0)$. The function $q_{m,\varepsilon,1,k}(\zeta)$ is a sum of terms $K_{\varepsilon,k}(G, e, \zeta)$ where G is a connected graph with $2m$ edges, at most m vertices, and e is a distinguished edge of G. Denoting by $V[G]$ the set of vertices of G and $E[G]$ the set of edges, then $K_{\varepsilon,k}(G, e, \zeta)$ is given by the formula,

$$K_{\varepsilon,k}(G,e,\zeta) = \sum_{\{y_v \in \mathbf{Z}^d : v \in V[G]\}} (y_{e_-} \cdot \mathbf{e}_k) \delta(y_{e_+}) \prod_{e' \in E[G], \ e' \ne e} K_{\varepsilon^2}\left(y_{e'_+} - y_{e'_-}\right) \int_0^1 \exp[-ity_{e_-} \cdot \zeta] \, dt \Big/ \int_0^1 \exp[-it\mathbf{e}_k \cdot \zeta] \, dt. \tag{2.8}$$

In (2.8) to each vertex $v \in V[G]$ there is attached a variable $y_v \in \mathbf{Z}^d$ which is summed over. The notations f_+ and f_- are used for the vertices of an edge $f \in E[G]$. The Kronecker delta function is defined as usual by $\delta(y) = 0, \ y \ne 0, \ \delta(0) = 1$. The formula (2.8) generalizes Lemma 5.4 of [4]. The graphs G which occur have the property that the degree of every vertex is divisible by 4. The simplest such graph that can occur in (2.8) is therefore the graph G on 2 vertices with 4 edges, yielding an expression for $\zeta = 0$,

$$K_{\varepsilon,1}(G,e,0) = -\sum_{y \in \mathbf{Z}^d} (y.\mathbf{e}_1) \ K_{\varepsilon^2}(y)^3 \ .$$

The simplest such graph on 3 vertices has 6 edges, yielding an expression for $\zeta = 0$,

$$K_{\varepsilon,1}(G,e,0) = -\sum_{y,y' \in \mathbf{Z}^d} (y.\mathbf{e}_1) \ K_{\varepsilon^2}(y) \ K_{\varepsilon^2}(y-y')^2 \ K_{\varepsilon^2}(y')^2 \ .$$

The following proposition is the main step in proving Theorem 1.1.

Proposition 2.1. *Let G be a connected graph such that the degree of every vertex is divisible by 4. Then there is a constant, C_d depending only on d, such that*

$$|K_{\varepsilon,k}(G,e,\zeta)| \le C_d^{|E[G]|}, \ 0 < \varepsilon \le 1, \ \zeta \in \mathbf{C}, \tag{2.9}$$

$$\lim_{\varepsilon \to 0} K_{\varepsilon,k}(G,e,\zeta) \ \text{ exists uniformly for } \zeta \in \mathbf{C}. \tag{2.10}$$

We shall prove Proposition 2.1 in a series of lemmas. Our first goal will be to show that we may assume wlog that G is 2-connected. To do this we need an elementary result from graph theory.

Lemma 2.3. *Let G be a connected graph and suppose that the degree of all but one vertex of G is divisible by 4. Then all vertices of G have even degree and G has an odd number of edges.*

Proof. For $v \in V[G]$ let $d(v)$ be the degree of the vertex v. Then there is the identity

$$\sum_{v \in V[G]} d(v) = 2|E[G]|. \tag{2.11}$$

The result easily follows. □

Lemma 2.4. *Let G be a connected graph such that the degree of every vertex is divisible by 4. Let B be a block of the decomposition of G into 2-connected components. If $K_\varepsilon(G, e) \neq 0$ then the degree of every vertex of B is divisible by 4.*

Proof. We shall use induction on the number of blocks in the block decomposition of G. We may assume wlog that G has at least 2 blocks in which case there is an endblock B_0 which does not contain the distinguished edge $e \in E[G]$. Let $v_0 \in V[B_0]$ be the cut vertex of B_0 in G and $K_\varepsilon(B_0)$ be given by

$$K_\varepsilon(B_0) = \sum_{\{y_v \in \mathbf{Z}^d : v \in V[B_0]\}} \delta(y_{v_0}) \prod_{e' \in E[B_0]} K_{\varepsilon^2}\Big(y_{e'_+} - y_{e'_-}\Big).$$

Then it is clear that $K_\varepsilon(G, e) = K_\varepsilon(G \backslash B_0, e) K_\varepsilon(B_0)$. Since $K_\varepsilon(G, e) \neq 0$ it follows that $K_\varepsilon(B_0) \neq 0$ and $K_\varepsilon(G \backslash B_0, e) \neq 0$. If B_0 has an odd number of edges then $K_\varepsilon(B_0) = 0$ since $K_\eta(x) = -K_\eta(-x)$, $x \in \mathbf{Z}^d$. Hence B_0 has an even number of edges and it also has the property that the degree of every vertex other than v_0 is divisible by 4. It follows then from Lemma 2.1 that the degree of $v_0 \in V[B_0]$ is divisible by 4 whence the degree of $v_0 \in V[G \backslash B_0]$ is also divisible by 4. □

Next we show that if G is 3-edge connected then the result of Proposition 2.1 holds.

Lemma 2.5. *Suppose the graph G is 3-edge connected, the degree of each vertex of G is even and at least 4. Let G' be the graph obtained from G by the contraction of 2 vertices. Then G' is also 3-edge connected and the degree of each vertex of G' is even and at least 4.*

Proof. Obvious. □

Lemma 2.6. *Suppose the graph G is 3-edge connected, the degree of each vertex of G is even and at least 4. Let $e \in E[G]$ and for each edge $e' \in E[G] \backslash \{e\}$ let $n_{e'}$ be an arbitrary nonnegative integer. Then there is a constant C_d depending only on d such that there is the inequality,*

$$\sum_{\{y_v \in \mathbf{Z}^d : v \in V[G]\}} |y_{e_-}| \delta(y_{e_+}) \prod_{e' \in E[G],\ e' \neq e} 2^{-n_{e'}\, d/2} \exp\left[-2^{-n_{e'}} |y_{e'_+} - y_{e'_-}|\right] \leq C_d^{|V[G]|}. \tag{2.12}$$

Proof. The set $\{n_{e'} : e' \in E[G] \backslash \{e\}\} = \{N_1, N_2, \ldots, N_k\}$ with $0 \leq N_1 < N_2 < \cdots < N_k$. Let $G_1 = G$ and consider the graph G'_1 with edges $e' \in E[G] \backslash \{e\}$ satisfying $n_{e'} = N_1$. Contract each of the connected components of G'_1 to a single vertex. Thus we obtain a graph G_2 from G_1 by the contraction process. By Lemma 2.5 the graph G_2 is 3-edge connected, the degree of each vertex of G_2 is even and at least 4. Next we consider the subgraph G_2' of G_2 corresponding to the edges $e' \in E[G]\backslash\{e\}$ satisfying $n_{e'} = N_2$. We contract each of the connected components of G_2' to a single vertex, whence we obtain a graph G_3 from G_2. Proceeding in this manner we construct graphs $G_1, \dots, G_k$ and graphs $G_1', \dots G_k'$ where G_j' is a subgraph of G_j, $1 \le j \le k$. Note that each edge of G_k' corresponds to an edge $e' \in E[G]\backslash\{e\}$ with $n_{e'} = N_k$. Since G_k is 3-edge connected and G_k' differs from G_k by at most one edge corresponding to e it follows that G_k' is connected.

Next we construct a spanning tree T for the graph G. First let T_k be a spanning tree for G_k'. Now for each component of G_{k-1}' we can form a spanning tree. The graph G_{k-1} is obtained from G_k by splitting certain vertices into the components of G_{k-1}'. Thus the spanning trees for each component of G_{k-1}' together with T_k yield a spanning tree T_{k-1} for G_{k-1}. Similarly we obtain spanning trees T_j for G_j, $1 \le j \le k$. We put $T = T_1$.

For a subgraph H of G and $j = 1, \dots, k$ let

$$E_H[N_j] = \{e' \in E[H]\backslash\{e\} : n_{e'} = N_j\}.$$

We define an integer k_0, $1 \le k_0 \le k$ as follows: let e_- and e_+ be the vertices of the distinguished edge $e \in E[G]$. If e_- and e_+ correspond to different vertices of G_k in the contraction process then $k_0 = k$. If they correspond to the same vertex of G_k there is a $k_0 < k$ such that they correspond to the same vertex of G_j for $j > k_0$ but to different vertices of G_{k_0}. This defines k_0 uniquely.

We obtain inequalities relating the number of edges in $E_T[N_j]$ to the number of edges in $E_G[N_j]$. It is easy to see in fact that

$$2\sum_{j=r}^{k} |E_T[N_j]| + 2 \le \sum_{j=r}^{k} |E_G[N_j]|, \ r > k_0 \ , \tag{2.13}$$
$$2\sum_{j=r}^{k} |E_T[N_j]| + 1 \le \sum_{j=r}^{k} |E_G[N_j]|, \ r \le k_0 \ .$$

Now there exists a path P in T from e_- to e_+ such that every edge of the path is in $E_G[N_j]$ for some $j \le k_0$. It follows then from (2.13) that the LHS of (2.12) is bounded by

$$\sum_{f \in E[P]} \sum_{\{y_v \in \mathbf{Z}^d,\ v \in V[T]\}} \delta(y_{e_+}) |y_{f_+} - y_{f_-}| 2^{-3n_f\, d/2}$$
$$\exp\left[-2^{-n_f}|y_{f_+} - y_{f_-}|\right] \prod_{f' \in E[T]\backslash\{f\}} 2^{-n_{f'} d} \exp\left[-2^{-n_{f'}}|y_{f'_+} - y_{f'_-}|\right].$$

It is easy to see that the last expression is bounded by $C_d^{|V[T]|}$ for some constant C_d. □

Corollary 2.1. *Suppose the graph G is 3-edge connected, the degree of each vertex of G is even and at least 4. Let $e \in E[G]$ and for each edge $e' \in E[G]\backslash\{e\}$ let $K_{e'} : \mathbf{Z}^d \to \mathbf{R}$ be a function which satisfies the inequality,*

$$|K_{e'}(x)| \le 1 \Big/ \left[1 + |x|^{d-1}\right], \ x \in \mathbf{Z}^d. \tag{2.14}$$

Then there is a constant C_d depending only on d such that

$$\sum_{\{y \in \mathbf{Z}^d : v \in V[G]\}} |y_{e_-}| \delta(y_{e_+}) \prod_{e' \in E[G]\backslash\{e\}} \left| K_{e'}\left(y_{e'_+} - y_{e'_-}\right)\right| \le C_d^{|E[G]|}.$$

Proof. From (2.14) one sees there is a constant C_d depending only on d such that

$$|K_{e'}(x)| \le C_d \sum_{n=0}^{\infty} 2^{-n(d-1)} \exp\left[-2^{-n}|x|\right], \ x \in \mathbf{Z}^d.$$

The result follows from Lemma 2.6 since $d-1 > d/2$ for $d > 2$. □

Next we prove a more general result than Corollary 2.1.

Lemma 2.7. *Let G be a graph as in Corollary 2.1 with associated functions $K_{e'}$, $e' \in E[G]\backslash\{e\}$ which satisfy (2.14). Then there is a constant C_d depending only on d such that*

(2.15)
$$\sum_{\{y_v \in \mathbf{Z}^d,\ v \in V[G]\}} \delta(y_{e_-} - x)\delta(y_+) \prod_{e' \in E[G]\backslash\{e\}} |K_{e'}(y_{e'_+} - y_{e'_-})| \le C_d^{|E[G]|} \Big/ \left[1 + |x|^{d-1}\right]^3, \ x \in \mathbf{Z}^d.$$

Proof. The result follows from a generalization of Lemma 2.6 in the same way as Corollary 2.1. Using the notation of Lemma 2.6 we need to show that for some constant C_d depending only on d, there is the inequality,

(2.16)
$$\sum_{\{y_v \in \mathbf{Z}^d,\ v \in V[G]\}} \delta(y_{e_-} - x)\delta(y_+) \prod_{e' \in E[G]\backslash\{e\}} 2^{-n_{e'}\ d/2} \exp\left[-2^{-n_{e'}}|y_{e'_+} - y_{e'_-}|\right] \le C_d^{|V[G]|} 2^{-3N_{k_0}\ d/2} \exp\left[-2^{-(N_{k_0}+1)}|x|\right], \ x \in \mathbf{Z}^d.$$

The inequality (2.16) is obtained by summing out all variables except those attached to vertices along the path P in T from e_- to e_+. Now for $e' \in E[P]$ we have $n_{e'} \le N_{k_0}$ and at least one e' has $n_{e'} = N_{k_0}$. Now (2.16) follows by first taking out the exponential factor $\exp[-2^{-(N_{k_0}+1)}|x|]$ using the fact that

$$\exp\left[-\sum_{e' \in E[P]} 2^{-(n_{e'}+1)}|y_{e'_+} - y_{e'_-}|\right] \le \exp[-2^{-(N_{k_0}+1)}|x|].$$

Then we remove an edge e' satisfying $n_{e'} = N_{k_0}$ and sum over the remaining variables. To finish the proof of (2.15) we simply observe that from (2.16) the LHS of (2.15) is bounded above by

$$C_d^{|E[G]|} \sum_{\{e_1,e_2,e_3 \in E[G]\backslash\{e\}\}} \left(\prod_{e' \in E[G]\backslash\{e\}} \sum_{n_{e'}=0}^{\infty} \right) \prod_{e' \in E[G]\backslash\{e,e_1,e_2,e_3\}} 2^{-n_{e'}(d-1\ -\ d/2)} \prod_{i=1}^{3} 2^{-n_{e_i}(d-1)} \exp\left[-2^{-(n_{e_i}+3)}|x|\right],$$

where C_d depends only on d and the sum over e_1, e_2, e_3 is over all distinct sets of edges $\{e_1, e_2, e_3\} \subset E[G]\backslash\{e\}$. □

We turn to the proof of Proposition 2.1 in the general case where by Lemma 2.4 we may assume that G is 2-connected. We also assume that G is not 3-connected since in that case Proposition 2.1 already follows from Corollary 2.1. Hence G has a nontrivial Tutte decomposition [20] into 3-blocks. The 3-blocks of G are either 3-connected graphs, n-linkages with $n \geq 3$ or cycles. The following lemma shows that if there are no 3-blocks which are cycles then G is 3-edge connected whence Proposition 2.1 follows from Corollary 2.1.

Lemma 2.8. *Let G be a graph, T a tree graph and W_t, $t \in V[T]$, be a system of subgraphs of G with the properties:*

(a) $\bigcup_{t \in V[T]} W_t = G$.
(b) *Every edge of G belongs to exactly one W_t.*
(c) *If $t, t' \in T$ are adjacent then $W_t \cap W_{t'}$ consists of precisely 2 vertices. If $t, t'' \in T$ are not adjacent then $W_t \cap W_{t''} \subset W_t \cap W_{t'}$ where t' is the vertex adjacent to t which lies on the path in T from t'' to t.*

For $t \in T$ let W_t^* be the graph obtained from W_t by adding an edge joining the pair of vertices in $W_t \cap W_{t'}$ for all $t' \in T$ adjacent to t. Suppose that the graphs W_t^* are 3-edge connected for all $t \in T$. Then G is 3-edge connected.

Proof. It is easy to see that G is 3-edge connected if T has just 2 vertices. More generally the result follows by induction on contraction of an edge of T. □

Proof of Proposition 2.1. If G has no 3-blocks which are cycles then the result follows from Lemma 2.6 and Corollary 2.1. Suppose then G has exactly one 3-block which is a cycle. Let T be the tree graph corresponding to the Tutte decomposition which has 3-blocks of G as its vertices. Let $t_0 \in V[T]$ be the vertex corresponding to the 3-block which is a cycle. Since the degree of every vertex of G is at least 4, the vertex t_0 cannot be an end vertex of T, whence $T \backslash \{t_0\}$ has at least 2 components. Suppose $T \backslash \{t_0\}$ has m components $T_1, \ldots, T_m$. These correspond to m virtual edges $f_1, \ldots, f_m$ of the cycle, which we assume are in order on the cycle with f_{j+1} following f_j, $1 \leq j \leq m-1$. Note that the f_j, $1 \leq j \leq m$, do not necessarily constitute all edges of the cycle since there can be edges which belong to G. Using the notation of Lemma 2.8 we put $H_j' = \cup_{t \in T_j} W_j$, $1 \leq j \leq m$. Let H_j be the graph consisting of the union of H_j' and the edge f_j, $1 \leq j \leq m$. By Lemma 2.8 H_j is 3-edge connected. Suppose now that the cycle has m edges whence $G = \cup_{j=1}^m H_j'$. Since each H_j is 3-edge connected it is clear that G is 3-edge connected whence the result follows from Corollary 2.1. Alternatively let us assume that the cycle has more than m edges. Then the graph with edges $f_1, \ldots, f_m$ has $k \leq m$ components $F_1, \ldots, F_k$ and each component F_i is a simple path with 2 vertices of degree 1 which we denote by F_i^+ and F_i^-, $1 \leq i \leq k$. We put $G_i' = \bigcup_{f_j \in F_i} H_j'$, $1 \leq i \leq k$, and let G_i be the graph which is the union of G_i' and the edge $F_i^- F_i^+$, $1 \leq i \leq k$. Since each H_j is 3-edge connected it follows that G_i is 3-edge connected, $1 \leq i \leq k$. One also easily sees that the degree of every vertex of G_i is divisible by 4.

We consider how to write $K_{\varepsilon,k}(G, e, \zeta)$ in terms of a convolution of functions defined by the graphs G_i, $1 \leq i \leq k$. For $1 < i \leq k$ we define functions $g_i(x)$, $x \in \mathbf{Z}^d$, by

$$g_i(x) = \sum_{\{y_v \in \mathbf{Z}^d,\ v \in V[G_i]\}} \delta\left(y_{F_i^-} - x\right) \delta\left(y_{F_i^+}\right) \prod_{e' \in E[G_i']} K_{\varepsilon^2}\left(y_{e'_+} - y_{e'_-}\right).$$

It is evident that $g_i(-x) = -g_i(x)$. By Lemma 2.7 one also has that

$$|g_i(x)| \leq C_d^{|E[G_i]|} \Big/ \left[1 + |x|^{d-1}\right]^3, \quad x \in \mathbf{Z}^d.$$

First we consider the case when e is an edge of the cycle. We may assume $e_+ = F_k^-$, $e_- = F_1^+$. Let h be the function,

$$h(x) = g_1 * K_{\varepsilon^2} * g_2 * \cdots * K_{\varepsilon^2} * g_k(x),\ x \in \mathbf{Z}^d.$$

Then there is the identity,

$$\int_0^1 \exp[-it\mathbf{e}_k \cdot \zeta]\ dt\ K_{\varepsilon,k}(G,e,\zeta) = \int_0^1 i\ \frac{\partial \hat{h}}{\partial \zeta_k}(t\zeta)\ dt.$$

By the argument of Lemma 2.2 there is a constant C_d depending only on d such that

$$|\nabla \hat{h}(\zeta)| \le C_d^{|E[G]|}, \quad \zeta \in \mathbf{C}.$$

Hence the inequality (2.9) of Proposition 2.1 holds. Next we consider the case when e is not an edge of the cycle so we may wlog assume that $e \in E[G_1]$. Now for $e' \in E[G_1]$, $e' \ne e$, $F_1^- F_1^+$ we have $K_{e'}(x) = K_{\varepsilon^2}(x)$, $x \in \mathbf{Z}^d$. We define now $K_f(x)$ for $f = F_1^-\ F_1^+$ by

$$K_f(x) = K_{\varepsilon^2} * g_2 * K_{\varepsilon^2} * g_3 * \cdots * K_{\varepsilon^2} * g_k * K_{\varepsilon^2}(x),\ x \in \mathbf{Z}^d. \tag{2.17}$$

By Lemma 2.2 we have that $K_f(x) = -K_f(-x)$, $x \in \mathbf{Z}^d$ and

$$|K_f(x)| \le C_d^{|E[G]| \setminus |E[G_1]|} \Big/ \left[1 + |x|^{d-1}\right],\ x \in \mathbf{Z}^d,$$

for some constant C_d depending only on d. Hence by Corollary 2.1 applied to the graph G_1 with distinguished edge e the inequality (2.9) of Proposition 2.1 follows.

Next we consider the situation where the Tutte decomposition of G has more than one 3-block which is a cycle. Letting T be the tree graph corresponding to the Tutte decomposition of G, we can easily see that there exists a vertex $t_0 \in T$ which has the properties:

(a) t_0 corresponds to a 3-block which is a cycle.
(b) Exactly one component of $T \backslash \{t_0\}$ contains all 3-blocks which are cycles.
(c) The edge e belongs to a 3-block which is contained in the unique component defined in (b).

Now we proceed exactly as before to eliminate t_0 and thus reducing the number of 3 blocks which are cycles, whence (2.9) follows by induction.

We finally prove that the limit (2.10) exists. Since for $d \ge 2$ $\lim_{\varepsilon \to 0} K_{\varepsilon^2}(x) = K_0(x)$, $x \in \mathbf{Z}^d$, (2.10) follows from Corollary 2.1 in the case G is 3-edge connected. Consider the situation when G has exactly one 3-block which is a cycle. We denote the kernel K_f of (2.17) by $K_{f,\varepsilon}$ to denote the dependence on ε. It is clear from the proof of Lemma 2.2 that $\lim_{\varepsilon \to 0} K_{f,\varepsilon}(x) = K_{f,0}(x)$. Hence (2.10) follows as before. A similar argument gives (2.10) when G has more than one 3-block which is a cycle. □

Proof of Theorem 1.1. From Lemma 5.3 and 5.4 of [4] it follows that there is a universal constant $C > 1$ such that $q_{m,\varepsilon}(\zeta)$ is a sum of less than $C^m\ m!$ terms each of which has the form (2.8) and such that the number of edges of the graph is $2m$. The result follows now from Proposition 2.1. □

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