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Error inequalities for a corrected interpolating polynomial

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Abstract. A corrected interpolating polynomial is derived. Error inequalities of Ostrowski type for the corrected interpolating polynomial are established. Some similar inequalities are also obtained.

Contents

1. Introduction

Many error inequalities in polynomial interpolation can be found in [2] and [12]. These error bounds for interpolating polynomials are usually expressed by means of the norms $\|\cdot\|_p$, $1 \le p \le \infty$. Some new error inequalities in polynomial interpolation can be found in [16]. In this paper we derive error inequalities for a corrected interpolating polynomial. Similar inequalities are obtained in numerical integration. For example see [3]–[11], [14] and [15]. In some of the mentioned papers we can find estimations for errors of quadrature formulas which are expressed by means of the differences $\Gamma_k - \gamma_k$, $S - \gamma_k$, $\Gamma_k - S$, where Γ_k, γ_k are real numbers such that $\gamma_k \le f^{(k)}(t) \le \Gamma_k$, $t \in [a,b]$ (k is a positive integer while $[a,b]$ is an interval of integration) and $S = \left[f^{(k-1)}(b) - f^{(k-1)}(a)\right]/(b-a)$. It is shown that the estimations expressed in such a way can be much better than estimations expressed by means of the norms $\left\|f^{(k)}\right\|_p$, $1 \le p \le \infty$. Furthermore, it is well-known that corrected quadrature formulas have better estimations of errors than corresponding original formulas.

As we know there is a close relationship between interpolation polynomials and quadrature rules. Thus, it is a natural try to establish similar error inequalities in

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polynomial interpolation. The usual procedure is to find an interpolating formula and then we write a corresponding quadrature formula. Here we reverse the procedure. We have some results for quadrature formulas and we derive corresponding results for interpolating polynomials.

We first establish general error inequalities, expressed by means of $\left\|f^{(k)}-P_m\right\|$, where P_m is any polynomial of degree m and then we obtain inequalities of the above mentioned types. For that purpose we derive a representation of remainder in corrected interpolating polynomial. This is done in Section 2. In the same section we give a relationship between the corrected interpolating polynomial and a corresponding quadrature rule. In Section 3 we obtain the error inequalities.

Finally, we emphasize that the usual error inequalities in polynomial interpolation (for the Lagrange interpolating polynomial $L_n(x)$) are given by means of the $(n+1)$th derivative while in this paper we can find these error inequalities expressed by means of the kth derivative for $k=1,2,\ldots,n$.

2. Corrected interpolating polynomial

Let $\pi=\{a=x_0<x_1<\cdots<x_n=b\}$ be a given subdivision of the interval $[a,b]$ and let $f:[a,b]\to R$ be a given function. The Lagrange interpolation polynomial is given by

$$L_n(x)=\sum_{i=0}^{n}p_{ni}(x)f(x_i), \tag{2.1}$$

where

$$p_{ni}(x)=\frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}, \tag{2.2}$$

for $i=0,1,\ldots,n$. We have the Cauchy relations ([12, pp. 160-161]),

$$\sum_{i=0}^{n}p_{ni}(x)=1 \tag{2.3}$$

and

$$\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^j=0,\ \ j=1,2,\ldots,n. \tag{2.4}$$

Let $\pi=\{x_0=a<x_1<\cdots<x_n=b\}$ be a given uniform subdivision of the interval $[a,b]$, i.e., $x_i=x_0+ih$, $h=(b-a)/n$, $i=0,1,2,\ldots,n$. Then the Lagrange interpolating polynomial is given by

$$\begin{aligned} L_n(x)&=L_n(x_0+th)\\ &=(-1)^n\frac{t(t-1)\cdots(t-n)}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\frac{f(x_i)}{t-i}, \end{aligned}$$

where $t\notin\{0,1,2,\ldots,n\}$, $0<t<n$.

As we know the divided difference of the first order of the function f is given by

$$f\left[x_0;x_1\right]=\frac{f(x_1)-f(x_0)}{x_1-x_0}.$$

The divided difference of order n is defined via the divided differences of order $n-1$ by the recurrence formula

$$f[x_0;x_1;\ldots;x_n] = \frac{f[x_1;x_2;\ldots;x_n] - f[x_0;x_1;\ldots;x_{n-1}]}{x_n - x_1}.$$

The following lemma is valid ([2, p. 86]):

Lemma 2.1. *The nth-order divided difference satisfies the relation*

$$f[x_0;x_1;\ldots;x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

The interpolating polynomial can be written in the Newton form as

$$\begin{aligned} L_n(x) &= f(x_0) + \sum_{i=0}^{n-1}(x-x_0)\cdots(x-x_i) f[x_0;\ldots;x_{i+1}] \\ &= f(x_0) + \sum_{i=0}^{n-1}\omega_i(x) f[x_0;\ldots;x_{i+1}], \end{aligned}$$

where

$$\omega_i(x) = (x-x_0)(x-x_1)\cdots(x-x_i), \tag{2.5}$$

for $i = 0,1,2,\ldots,n$.

We also recall some properties of Euler polynomials. The Euler polynomials are defined by the relation

$$\frac{2e^{xt}}{e^t+1} = \sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!}, \quad |t| < \pi,$$

such that

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x, \ldots \tag{2.6}$$

We have

$$E_k'(x) = kE_{k-1}(x) \text{ or } \int E_k(x)dx = \frac{E_{k+1}(x)}{k+1},\ k = 1,2,\ldots, \tag{2.7}$$

$$\int_0^1 E_n(t)E_m(t)dt = (-1)^n 4(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2} \tag{2.8}$$

and

$$\int_0^1 E_n(t)dt = \frac{E_{n+1}(1) - E_{n+1}(0)}{n+1},\ n = 1,2,\ldots. \tag{2.9}$$

We also have

$$E_k(0) = -E_k(1) = -2\frac{2^{k+1}-1}{k+1}B_{k+1}, \tag{2.10}$$

where B_k are Bernoulli numbers such that

$$E_1(0) = -\frac{1}{2}, \quad E_2(0) = 0, \quad E_3(0) = \frac{1}{4}, \quad E_4(0) = 0, \ldots$$

or generally $E_{2n}(0) = 0$. Further properties of Euler polynomials can be found in [1].

Lemma 2.2. *Let $P_m(t)$ be any polynomial of degree $\le m$ and let π be a given subdivision of the interval $[a,b]$. Then*

$$\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^k \int_{x_i}^{x} P_m(t) E_k\left(\frac{t-x_i}{x-x_i}\right) dt = 0,$$

for $0 \le k+m \le n-1$ and $x \in [a,b]$, where $E_k(t)$ are Euler polynomials.

Proof. Let x be a real number. Then we have

$$P_m(t) = \sum_{j=0}^{m} c_j (x-t)^j,$$

for some coefficients $c_j = c_j(x)$, $j = 0,1,\dots,m$. (This is a consequence of the Taylor formula.) Thus,

$$\int_{x_i}^{x} P_m(t) E_k\left(\frac{t-x_i}{x-x_i}\right) dt = \sum_{j=0}^{m} c_j \int_{x_i}^{x} (x-t)^j E_k\left(\frac{t-x_i}{x-x_i}\right) dt.$$

We now calculate

$$\begin{aligned}\int_{x_i}^{x} (x-t)^j E_k\left(\frac{t-x_i}{x-x_i}\right) dt &= \int_0^{x-x_i} (x-x_i-u)^j E_k\left(\frac{u}{x-x_i}\right) du \\ &= (x-x_i)^j \int_0^{x-x_i} \left(1-\frac{u}{x-x_i}\right)^j E_k\left(\frac{u}{x-x_i}\right) du \\ &= (x-x_i)^{j+1} \int_0^1 (1-v)^j E_k(v) dv.\end{aligned}$$

Integrating by parts and using the property (2.7), we obtain

$$\int_0^1 (1-v)^j E_k(v) dv = -\frac{1}{k+1} E_{k+1}(0) + \frac{j}{k+1} \int_0^1 (1-v)^{j-1} E_{k+1}(v) dv.$$

In a similar way we get

$$\int_0^1 (1-v)^{j-1} E_{k+1}(v) dv = -\frac{1}{k+2} E_{k+2}(0) + \frac{j-1}{k+2} \int_0^1 (1-v)^{j-2} E_{k+2}(v) dv.$$

Hence, we have

$$\begin{aligned}\int_0^1 (1-v)^j E_k(v) dv = &-\frac{1}{k+1} E_{k+1}(0) - \frac{j}{(k+1)(k+2)} E_{k+2}(0) \\ &+ \frac{j(j-1)}{(k+1)(k+2)} \int_0^1 (1-v)^{j-2} E_{k+2}(v) dv.\end{aligned}$$

Continuing in this way we get

$$\int_0^1 (1-v)^j E_k(v) dv = -\sum_{l=0}^{j-1} \frac{k!j!}{(k+l+1)!(j-l)!} E_{k+l+1}(0) - 2\frac{k!j!E_{k+j+1}(0)}{(k+j+1)!},$$

for $j \ge 1$, since $\int_0^1 E_{k+j}(v) dv = -2\frac{E_{k+j+1}(0)}{k+j+1}$. For $j=0$, $\int_0^1 (1-v)^0 E_k(v) dv = -\frac{2E_{k+1}(0)}{k+1}$. Thus,

$$\int_{x_i}^{x} (x-t)^j E_k\left(\frac{t-x_i}{x-x_i}\right) dt = (x-x_i)^{j+1} D_{kj},$$

where

$$D_{kj} = \begin{cases} -\frac{2E_{k+1}(0)}{k+1}, & j = 0 \\ -\sum\limits_{l=0}^{j-1} \frac{k!j!}{(k+l+1)!(j-l)!} E_{k+l+1}(0) - 2\frac{k!j!E_{k+j+1}(0)}{(k+j+1)!}, & 1 \le j \le m. \end{cases}$$

It follows that

$$\int_{x_i}^{x} P_m(t) E_k \left(\frac{t - x_i}{x - x_i} \right) dt = \sum_{j=0}^{m} c_j D_{kj} (x - x_i)^{j+1}.$$

Finally, we get

$$\begin{aligned} &\sum_{i=0}^{n} p_{ni}(x)(x - x_i)^k \int_{x_i}^{x} P_m(t) E_k \left(\frac{t - x_i}{x - x_i} \right) dt \\ &= \sum_{j=0}^{m} c_j D_{kj} \sum_{i=0}^{n} p_{ni}(x)(x - x_i)^{k+j+1} = 0, \end{aligned}$$

for $0 \le k + m \le n - 1$, since (2.4) holds. □

Theorem 2.1. *Under the assumptions of Lemma 2.2 suppose that $f \in C^{k+1}(a, b)$. Then*

$$f(x) = L_n(x) + \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j E_j(0)}{j!} g_j \left[x_0; x_1; \ldots; x_n\right] + R_{k,m}(x), \tag{2.11}$$

where

$$R_{k,m}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)(x - x_i)^k \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_m(t) \right] E_k \left(\frac{t - x_i}{x - x_i} \right) dt \tag{2.12}$$

and

$$g_j(t) = (x - t)^{j-1} f^{(j)}(t), \quad j = 1, 2, \ldots, k.$$

Proof. We have

$$\begin{aligned} R_{k,m}(x) = &\frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)(x - x_i)^k \int_{x_i}^{x} f^{(k+1)}(t) E_k \left(\frac{t - x_i}{x - x_i} \right) dt \\ &- \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)(x - x_i)^k \int_{x_i}^{x} P_m(t) E_k \left(\frac{t - x_i}{x - x_i} \right) dt. \end{aligned}$$

From the above relation and Lemma 2.2 it follows that

$$R_{k,m}(x) = R_k(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)(x - x_i)^k \int_{x_i}^{x} f^{(k+1)}(t) E_k \left(\frac{t - x_i}{x - x_i} \right) dt.$$

Integrating by parts, we obtain

$$\begin{aligned}&\frac{(-1)^k}{k!}(x-x_i)^k\int_{x_i}^{x} f^{(k+1)}(t)E_k\left(\frac{t-x_i}{x-x_i}\right)dt\\&=\frac{(-1)^k}{k!}(x-x_i)^k\left[E_k(1)f^{(k)}(x)-E_k(0)f^{(k)}(x_i)\right]\\&\quad+\frac{(-1)^{k-1}}{(k-1)!}(x-x_i)^{k-1}\int_{x_i}^{x} f^{(k)}(t)E_{k-1}\left(\frac{t-x_i}{x-x_i}\right)dt,\end{aligned}$$

since (2.7) holds. In a similar way we get

$$\begin{aligned}&\frac{(-1)^{k-1}}{(k-1)!}(x-x_i)^{k-1}\int_{x_i}^{x} f^{(k)}(t)E_{k-1}\left(\frac{t-x_i}{x-x_i}\right)dt\\&=\frac{(-1)^{k-1}}{(k-1)!}(x-x_i)^{k-1}\left[E_{k-1}(1)f^{(k-1)}(x)-E_{k-1}(0)f^{(k-1)}(x_i)\right]\\&\quad+\frac{(-1)^{k-2}}{(k-2)!}(x-x_i)^{k-2}\int_{x_i}^{x} f^{(k-1)}(t)E_{k-2}\left(\frac{t-x_i}{x-x_i}\right)dt.\end{aligned}$$

Continuing in this way we get

$$\begin{aligned}&\frac{(-1)^k}{k!}(x-x_i)^k\int_{x_i}^{x} f^{(k+1)}(t)E_k\left(\frac{t-x_i}{x-x_i}\right)dt\\&=\sum_{j=1}^{k}\frac{(-1)^j}{j!}(x-x_i)^j\left[E_j(1)f^{(j)}(x)-E_j(0)f^{(j)}(x_i)\right]+\int_{x_i}^{x} f'(t)dt\\&=\sum_{j=1}^{k}\frac{(-1)^j}{j!}(x-x_i)^j\left[E_j(1)f^{(j)}(x)-E_j(0)f^{(j)}(x_i)\right]+f(x)-f(x_i).\end{aligned}$$

Then we have

$$\begin{aligned}&R_k(x)\\&=\sum_{i=0}^{n}p_{ni}(x)\left[f(x)-f(x_i)\right]\\&\quad+\sum_{i=0}^{n}p_{ni}(x)\sum_{j=1}^{k}\frac{(-1)^j}{j!}(x-x_i)^j\left[E_j(1)f^{(j)}(x)-E_j(0)f^{(j)}(x_i)\right]\\&=f(x)-L_n(x)+\sum_{j=1}^{k}\frac{(-1)^j}{j!}\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^j\left[E_j(1)f^{(j)}(x)-E_j(0)f^{(j)}(x_i)\right]\\&=f(x)-L_n(x)-\sum_{j=1}^{k}\frac{(-1)^j}{j!}E_j(0)\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^jf^{(j)}(x_i),\end{aligned}$$

since (2.3) holds and

$$\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^jE_j(1)f^{(j)}(x)=0,$$

for $1 \le j \le n$. We have

$$\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^j f^{(j)}(x_i)$$
$$= \omega_n(x) \sum_{i=0}^{n} \frac{(x-x_i)^{j-1} f^{(j)}(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

and

$$g_j[x_0;x_1;\ldots;x_n] = \sum_{i=0}^{n} \frac{(x-x_i)^{j-1} f^{(j)}(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)},$$

by Lemma 2.1, so that

$$R_k(x) = f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j}{j!} E_j(0) g_j[x_0;x_1;\ldots;x_n].$$

This completes the proof. □

Remark 2.1. Since $E_{2j}(0) = 0$, $j = 1, 2, \ldots$, we can write

$$f(x) = L_n(x) - \omega_n(x) \sum_{j=0}^{\left[\frac{k-1}{2}\right]} \frac{E_{2j+1}(0)}{(2j+1)!} g_{2j+1}[x_0;x_1;\ldots;x_n] + R_{k,m}(x).$$

Example 2.1. If we choose $n = 1$ in (2.11) then we get

$$f(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$
$$+ \frac{1}{2}(x-x_0)(x-x_1)\frac{f'(x_1)-f'(x_0)}{x_1-x_0} + R.$$

Thus,

$$\int_{x_0}^{x_1} f(x)dx = \frac{f(x_0)+f(x_1)}{2}(x_1-x_0)$$
$$- \frac{(x_1-x_0)^2}{12}\left[f'(x_1)-f'(x_0)\right] + \int_{x_0}^{x_1} R dx,$$

since

$$\int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1} dx = \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0} dx = x_1 - x_0,$$

$$\frac{1}{2}\int_{x_0}^{x_1} \frac{(x-x_0)(x-x_1)}{x_1-x_0} dx = -\frac{(x_1-x_0)^2}{12}.$$

We see that the above quadrature formula is the well-known corrected trapezoidal rule.

Remark 2.2. If we generalize the above example then we find that the following conclusions are valid: If we integrate (2.11) from x_0 to x_n then we get a corrected Newton-Cotes formula. (Of course, we suppose that the partition π is uniform.) It is well-known that the corrected formulas have better estimations of errors than corresponding original quadrature formulas.

3. Error inequalities

We now introduce the notations

$$C_k(x) = \sum_{i=0}^{n} \frac{|x-x_i|^k}{|x_i-x_0|\cdots|x_i-x_{i-1}|\,|x_i-x_{i+1}|\cdots|x_i-x_n|}, \tag{3.1}$$

$$F_k(x) = \sum_{i=0}^{n} \frac{(S_{ki}-\gamma_{k+1})\,|x-x_i|^k}{|x_i-x_0|\cdots|x_i-x_{i-1}|\,|x_i-x_{i+1}|\cdots|x_i-x_n|}, \tag{3.2}$$

$$D_k(x) = \sum_{i=0}^{n} \frac{(\Gamma_{k+1}-S_{ki})\,|x-x_i|^k}{|x_i-x_0|\cdots|x_i-x_{i-1}|\,|x_i-x_{i+1}|\cdots|x_i-x_n|}, \tag{3.3}$$

where $S_{ki} = \left[f^{(k)}(x) - f^{(k)}(x_i)\right]/(x-x_i)$, $i = 0,1,\ldots,n$ and γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \le f^{(k+1)}(t) \le \Gamma_{k+1}$, $t \in [a,b]$, $k = 0,1,\ldots,n-1$. We also define the constant

$$\delta_k = \frac{\sqrt{|B_{2k+2}|}}{\sqrt{(2k+2)!}}\sqrt{2^{2k+2}-1}. \tag{3.4}$$

Let $g \in C(a,b)$. As we know among all algebraic polynomials of degree $\le m$ there exists the only polynomial $P_m^*(t)$ having the property that

$$\|g - P_m^*\|_\infty \le \|g - P_m\|_\infty\,,$$

where $P_m \in \Pi_m$ is an arbitrary polynomial of degree $\le m$. We define

$$G_m(g) = \|g - P_m^*\| = \inf_{P_m\in\Pi_m} \|g - P_m\|_\infty\,. \tag{3.5}$$

Theorem 3.1. *Under the assumptions of Theorem 2.1 we have*

$$\left| f(x) - L_n(x) - \omega_n(x)\sum_{j=1}^{k} \frac{(-1)^j E_j(0)}{j!} g_j\left[x_0;x_1;\ldots;x_n\right] \right| \le 2\delta_k G_m(f^{(k+1)}) C_k(x)\,|\omega_n(x)|\,,$$

where $C_k(\cdot)$, δ_k and $G_m(\cdot)$ are defined by (3.1), (3.4) and (3.5), respectively.

Proof. We have

$$\begin{aligned}\left(\int_{x_i}^{x} E_k\left(\frac{t-x_i}{x-x_i}\right)dt\right)^2 &\le |x-x_i| \left|\int_{x_i}^{x}\left(E_k\left(\frac{t-x_i}{x-x_i}\right)\right)^2 dt\right| \\ &= (x-x_i)^2 \int_0^1 (E_k(u))^2\,du \\ &= 4\frac{(k!)^2}{(2k+2)!}(2^{2k+2}-1)(x-x_i)^2\,|B_{2k+2}|\,,\end{aligned}$$

since (2.8) holds.

Let $P_m(t) = P_m^*(t)$, where $P_m^*(t)$ is defined by (3.5) for the function $g(t) = f^{(k+1)}(t)$. Then we have

$$\begin{aligned}
|R_{k,m}(x)| &= \left| \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^k \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_m^*(t) \right] E_k\left(\frac{t-x_i}{x-x_i} \right) dt \right| \\
&\leq \frac{1}{k!} \sum_{i=0}^{n} \left| p_{ni}(x)(x-x_i)^k \right| \left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_m^*(t) \right] E_k\left(\frac{t-x_i}{x-x_i} \right) dt \right| \\
&\leq \frac{\left\| f^{(k+1)} - P_m^* \right\|_\infty}{k!} \frac{2k!\sqrt{|B_{2k+2}|}}{\sqrt{(2k+2)!}} \sqrt{2^{2k+2}-1} C_k(x) \left| \omega_n(x) \right| \\
&= 2 \frac{G_m(f^{(k+1)})\sqrt{|B_{2k+2}|}}{\sqrt{(2k+2)!}} \sqrt{2^{2k+2}-1} C_k(x) \left| \omega_n(x) \right|
\end{aligned}$$

and

$$R_{k,m}(x) = f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j}{j!} E_j(0) g_j \left[x_0; x_1; \ldots; x_n \right].$$

This completes the proof. □

Remark 3.1. The above estimate has only theoretical importance, since it is difficult to find the polynomial P^*. In fact, we can find P^* only for some special cases of functions. However, we can use the estimate to obtain some practical estimations — see Theorem 3.2.

Theorem 3.2. *Let the assumptions of Theorem 2.1 hold. If γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \leq f^{(k+1)}(t) \leq \Gamma_{k+1}$, $t \in [a,b]$, $k = 0, 1, \ldots, n-1$, then*

$$\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j E_j(0)}{j!} g_j \left[x_0; x_1; \ldots; x_n \right] \right| \\ \leq \delta_k \left[\Gamma_{k+1} - \gamma_{k+1} \right] C_k(x) \left| \omega_n(x) \right|,$$

where ω_n, δ_k and $C_k(\cdot)$ are defined by (2.5), (3.4) and (3.1), respectively. We also have

$$\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j B_j}{j!} g_j \left[x_0; x_1; \ldots; x_n \right] \right| \leq 2\delta_k \left| \omega_n(x) \right| F_k(x)$$

and

$$\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j B_j}{j!} g_j \left[x_0; x_1; \ldots; x_n \right] \right| \leq 2\delta_k \left| \omega_n(x) \right| D_k(x),$$

where $F_k(\cdot)$ and $D_k(\cdot)$ are defined by (3.2) and (3.3), respectively.

Proof. We set $P_m(t) = (\Gamma_{k+1} + \gamma_{k+1})/2$ in (2.12). Then we have

$$\begin{aligned}
&\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j}{j!} E_j(0) g_j \left[x_0; x_1; \ldots; x_n\right] \right| \\
&= |R_{k,m}(x)| \\
&\leq \frac{1}{k!} \sum_{i=0}^{n} \left| p_{ni}(x)(x - x_i)^k \right| \left\| f^{(k+1)} - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right\|_\infty \left| \int_{x_i}^{x} E_k\left(\frac{t - x_i}{x - x_i}\right) dt \right|.
\end{aligned}$$

We also have

$$\left\| f^{(k+1)} - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right\|_\infty \leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{2}$$

and

$$\left| \int_{x_i}^{x} E_k\left(\frac{t - x_i}{x - x_i}\right) dt \right| \leq \frac{2k!}{\sqrt{(2k+2)!}} \sqrt{2^{2k+2} - 1}\, |x - x_i| \sqrt{|B_{2k+2}|}.$$

From the above three relations we get

$$\begin{aligned}
&\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j}{j!} E_j(0) g_j \left[x_0; x_1; \ldots; x_n\right] \right| \\
&\leq \frac{\Gamma_{k+1} - \gamma_{k+1}}{\sqrt{(2k+2)!}} \sqrt{2^{2k+2} - 1} \sqrt{|B_{2k+2}|} \sum_{i=0}^{n} |p_{ni}(x)|\, |x - x_i|^{k+1} \\
&= \frac{\Gamma_{k+1} - \gamma_{k+1}}{\sqrt{(2k+2)!}} \sqrt{2^{2k+2} - 1} \sqrt{|B_{2k+2}|} C_k(x) \left|\omega_n(x)\right|.
\end{aligned}$$

The first inequality is proved.

We now set $P_m(t) = \gamma_{k+1}$ in (2.12). Then we have

$$\begin{aligned}
&\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j}{j!} E_j(0) g_j \left[x_0; x_1; \ldots; x_n\right] \right| \\
&= |R_{k,m}(x)| \\
&\leq \frac{1}{k!} \sum_{i=0}^{n} \left| p_{ni}(x)(x - x_i)^k \right| \left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - \gamma_{k+1} \right] E_k\left(\frac{t - x_i}{x - x_i}\right) dt \right|.
\end{aligned}$$

We also have

$$\begin{aligned}
\left| \int_{x_i}^{x} \left[f^{(k+1)}(t) - \gamma_{k+1} \right] dt \right| &= \left| f^{(k)}(x) - f^{(k)}(x_i) - \gamma_{k+1}(x - x_i) \right| \\
&= (S_{ki} - \gamma_{k+1}) \left| x - x_i \right|.
\end{aligned}$$

Thus,

$$\begin{aligned}&\left|f(x)-L_n(x)-\omega_n(x)\sum_{j=1}^{k}\frac{(-1)^j}{j!}E_j(0)g_j\left[x_0;x_1;\ldots;x_n\right]\right|\\&\le\frac{1}{k!}\sum_{i=0}^{n}\left|p_{ni}(x)\right|\left|x-x_i\right|^{k+1}\left(S_{ki}-\gamma_{k+1}\right)\frac{2k!\sqrt{\left|B_{2k+2}\right|}}{\sqrt{(2k+2)!}}\sqrt{2^{2k+2}-1}\\&=2\frac{\left|\omega_n(x)\right|}{\sqrt{(2k+2)!}}\sqrt{2^{2k+2}-1}\sqrt{\left|B_{2k+2}\right|}F_k(x).\end{aligned}$$

The second inequality is proved. The third inequality holds by a similar proof. □

Lemma 3.1. *Let* $\pi=\{x_0=a<x_1<\cdots<x_n=b\}$ *be a given uniform subdivision of the interval* $[a,b]$, *i.e.,* $x_i=x_0+ih$, $h=(b-a)/n$, $i=0,1,2,\ldots,n$. *If* $x\in(x_{j-1},x_j)$, *for some* $j\in\{1,2,\ldots,n\}$, *then*

$$|\omega_n(x)|\le j!(n-j+1)!h^{n+1}, \tag{3.6}$$

$$C_k(x)\le\frac{2^n}{n!}\left\{\frac{1}{2}\left[n+1+|n-2j+1|\right]\right\}^k h^{k-n}, \tag{3.7}$$

and

$$C_k(x)\left|\omega_n(x)\right|\le\alpha_{jnk}\frac{n-j+1}{n}\frac{2^n(b-a)^{k+1}}{\binom{n}{j}}, \tag{3.8}$$

where

$$\alpha_{jnk}=\left[\frac{1}{2n}\left(n+1+|2j-n-1|\right)\right]^k. \tag{3.9}$$

Proof. For $i<j$ we have

$$|x-x_i|\le(j-i)h$$

and for $i\ge j$ we have

$$|x-x_i|\le(i-j+1)h$$

such that

$$\begin{aligned}|(x-x_0)\cdots(x-x_{j-1})(x-x_j)\cdots(x-x_n)|&\le j!h^j(n-j+1)!h^{n-j+1}\\&=j!(n-j+1)!h^{n+1}.\end{aligned}$$

We have

$$\begin{aligned}C_k(x)&\le\sum_{i=0}^{j-1}\frac{(j-i)^kh^k}{i!(n-i)!h^n}+\sum_{i=j}^{n}\frac{(i-j+1)^kh^k}{i!(n-i)!h^n}\\&\le\sum_{i=0}^{j-1}\frac{j^kh^k}{i!(n-i)!h^n}+\sum_{i=j}^{n}\frac{(n-j+1)^kh^k}{i!(n-i)!h^n}\\&\le h^{k-n}\max\left\{j^k,(n-j+1)^k\right\}\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\\&=\frac{2^n}{n!}\left\{\frac{1}{2}\left[n+1+|n-2j+1|\right]\right\}^k h^{k-n},\end{aligned}$$

since

$$\max\left[j, n-j+1\right] = \frac{1}{2}\left(n+1+|2j-n-1|\right)$$

such that it is not difficult to show that (3.7) holds.

Finally, from the above relations we find that (3.8) holds, too. □

Remark 3.2. Note that

$$\alpha_{jnk} \leq 1$$

and $\alpha_{jnk} = 1$ if and only if $j = 1$ or $j = n$. If we choose $x \in [x_j, x_{j+1}]$, $j = 0, 1, \ldots, n-1$, then we get the factor $(j+1)/n$ instead of the factor $(n-j+1)/n$ in (3.8).

Theorem 3.3. *Under the assumptions of Lemma 3.1 and Theorem 3.2 we have*

$$\left| f(x) - L_n(x) - \omega_n(x) \sum_{j=1}^{k} \frac{(-1)^j E_j(0)}{j!} g_j\left[x_0; x_1; \ldots; x_n\right] \right| \leq \left[\Gamma_{k+1} - \gamma_{k+1}\right] \delta_k \alpha_{jnk} \frac{n-j+1}{n} \frac{2^n (b-a)^{k+1}}{\binom{n}{j}}.$$

Proof. The proof follows immediately from Theorem 3.2 and Lemma 3.1. □

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