New York Journal of Mathematics

New York J. Math. **7** (2001) 117–148.

Multiblock Problems for Almost Periodic Matrix Functions of Several Variables

Leiba Rodman, Ilya M. Spitkovsky, and Hugo J. Woerdeman

Abstract. In this paper we solve positive and contractive multiblock problems in the Wiener algebra of almost periodic functions of several variables. We thus generalize the classical four block problem that appears in robust control in many ways. The necessary and sufficient conditions are in terms of appropriate Toeplitz (positive case) and Hankel operators (contractive case) on Besikovitch space. In addition, a model matching interpretation is given, and some more general patterns are treated as well.

Contents

1. Introduction

The suboptimal four block problem can be stated as follows. We let L_∞ and H_∞ denote the Lebesgue space and Hardy space, with respect to the essential supremum norm $\|\cdot\|_\infty$, on $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, respectively. We denote by $X^{p\times q}$ the set of $p \times q$ matrices with entries in the set X. Let $f_{11} \in L_\infty^{n_1\times m_1}$, $f_{12} \in L_\infty^{n_1\times m_2}$, $f_{21} \in L_\infty^{n_2\times m_1}$, and $f_{22} \in L_\infty^{n_2\times m_2}$ be given. Find, if possible, $\phi \in H_\infty^{n_1\times m_2}$ so that

$$\sup_{|z|=1} \left\| \begin{bmatrix} f_{11}(z) & f_{12}(z)+\phi(z) \\ f_{21}(z) & f_{22}(z) \end{bmatrix} \right\| < 1. \tag{1.1}$$

Received March 29, 2001.

Mathematics Subject Classification. 42A75, 15A54, 47A56, 47A57, 42A82, 47B35, 93B28.

Key words and phrases. Positive multiblock extensions, contractive multiblock extensions, band method, almost periodic matrix functions, Besikovitch spaces, Toeplitz operators, Hankel operators, model matching.

The research of all authors is partially supported by NSF Grant DMS 9988579. HJW is also partially supported by a Research Grant from the College of William and Mary.

ISSN 1076-9803/01

Here $\|\cdot\|$ is the operator norm (=the largest singular value) of a matrix.

This problem, as well as many of its variations, including the optimal four block problem (allowing equality in (1.1)) is ubiquitous in robust, or H_∞, control. Its solution is typically given in terms of appropriately defined Hankel-type operators. The mathematical and engineering literature on the four block problem is extensive, and we quote here only a representative sample of books [6], [13], [2], [7], [8], where further information and references can be found.

The present paper grew out of the authors' desire to provide a solution to a multivariable almost periodic analogue of the four block problem. In the multivariable setting we impose a linear order on $\mathbb{R}^k$ through the use of a halfspace. This idea goes back to the seminal papers [11], [12] in which it was recognized that several of the results in one-variable Hardy space theory carry over to the setting of Hardy spaces on linearly ordered groups.

A natural starting point for solving the four block problem in our setting is to consider a related positive definite extension problem. While we treat the problem in great generality in subsequent sections, it may be instructive to state here a new problem that we solve which is most closely related to (1.1). This concerns 4×4 block matrix valued functions with entries in the Wiener algebra

$$\mathcal{W} = \left\{ f(z) = \sum_{i=-\infty}^{\infty} f_i z^i : \sum_{i=-\infty}^{\infty} \|f_i\| < \infty \right\}$$

on the unit circle. Throughout the paper, a (bounded linear) Hilbert space operator $T : \mathcal{B} \to \mathcal{B}$ is called *positive definite* (notation: $T > 0$) if $\langle Tx, x\rangle \geq \epsilon\langle x, x\rangle$ for every $x \in \mathcal{B}$, where $\epsilon > 0$ is independent of x, and where $\langle x, y\rangle$ is the inner product in the Hilbert space $\mathcal{B}$.

Theorem 1.1. *Let $k_{ij} \in \mathcal{W}^{n_i \times n_j}$, $i, j = 1, 2, 3, 4$ be given. There exist $\phi \in \mathcal{W}^{n_1 \times n_4}$ with $\phi(z) = \sum_{i=0}^{\infty} \phi_i z^i$ so that*

$$k_{\text{ext}}(z) = \begin{bmatrix} k_{11}(z) & k_{12}(z) & k_{13}(z) & k_{14}(z) + \phi(z) \\ k_{21}(z) & k_{22}(z) & k_{23}(z) & k_{24}(z) \\ k_{31}(z) & k_{32}(z) & k_{33}(z) & k_{34}(z) \\ k_{41}(z) + \phi(z)^* & k_{42}(z) & k_{43}(z) & k_{44}(z) \end{bmatrix} > 0, \ |z| = 1,$$

if and only if the following Toeplitz-like operator $T : \mathcal{B}_2 \to \mathcal{B}_2$ is positive definite. Here the Hilbert space $\mathcal{B}_2$ is given by

$$\mathcal{B}_2 = \begin{bmatrix} H_2^{n_1 \times n_1} & L_2^{n_1 \times n_2} & L_2^{n_1 \times n_3} & L_2^{n_1 \times n_4} \ominus H_2^{n_1 \times n_4} \\ 0 & H_2^{n_2 \times n_2} & L_2^{n_2 \times n_3} & L_2^{n_2 \times n_4} \\ 0 & 0 & H_2^{n_3 \times n_3} & L_2^{n_3 \times n_4} \\ 0 & 0 & 0 & H_2^{n_4 \times n_4} \end{bmatrix}$$

where L_2 and H_2 are the Lebesgue and Hardy space on $\mathbb{T}$, respectively, with respect to the Hilbert space norm,

$$T(g) = P_{\mathcal{B}_2}\left((k_{ij})_{i,j=1}^4 g\right), \quad g = (g_{ij})_{i,j=1}^4 \in \mathcal{B}_2,$$

and $P_{\mathcal{B}_2}$ is the orthogonal projection of $L_2^{(n_1+n_2+n_3+n_4)\times(n_1+n_2+n_3+n_4)}$ onto $\mathcal{B}_2$.

The Hilbert space structure in $\mathcal{B}_2$ is the standard one: If $\mathcal{X}_{i,j}$, $i = 1, \dots, p$, $j = 1, \dots, q$ are Hilbert spaces, then the matrix $\mathcal{X} = [\mathcal{X}_{i,j}]_{i,j=1}^{p,q}$ is a Hilbert space with the inner product

$$\langle [x_{i,j}]_{i,j=1}^{p,q}, [y_{i,j}]_{i,j=1}^{p,q} \rangle = \sum_{i=1}^{p} \sum_{j=1}^{q} \langle x_{i,j}, y_{i,j} \rangle_{\mathcal{X}_{i,j}}, \quad x_{i,j}, y_{i,j} \in \mathcal{X}_{i,j},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{X}_{i,j}}$ is the inner product in $\mathcal{X}_{i,j}$.

The above result is a special case of Theorem 2.3 corresponding to $p = q = 3$ and periodic functions defined on $\mathbb{R}$ (as opposed to almost periodic functions defined on $\mathbb{R}^k$). Observe also that Theorem 2.3 deals with arbitrary representations of $\mathbb{R}^k$ as unions of two halfspaces that overlap in the origin; for $k = 1$ there is essentially one such representation: $\mathbb{R} = [0, \infty) \cup (-\infty, 0]$. Theorem 2.3 also gives (i) a construction of a specific $k_{\text{ext}}(z)$, that enjoys a maximum entropy; and (ii) a linear fractional description of the set of all solutions.

Using Theorem 1.1 one may solve a four block problem by letting

$$k_{ii}(z) = I_{n_i}, \; i = 1, 2, 3, 4,$$
$$k_{12}(z) = k_{21}(z) = k_{34}(z) = k_{43}(z) = 0,$$
$$k_{13}(z) = k_{31}(z)^* = f_{11}(z), \quad k_{14}(z) = k_{41}(z)^* = f_{12}(z),$$
$$k_{23}(z) = k_{32}(z)^* = f_{21}(z), \quad k_{24}(z) = k_{42}(z)^* = f_{22}(z).$$

Note that this particular four block problem is in the Wiener algebra rather than in L_∞.

Our main results are stated and proved in the framework of algebras of almost periodic matrix functions of several variables with Fourier spectrum in a given subgroup (see the next section for definitions and basic properties). This generality allows us to treat at once many particular situations, including the periodic case. Related problems of factorizations, positive and contractive extensions (which may be termed "one-block" problems in the context of the present paper) have been studied and solved in [26], [24], [25] for almost periodic matrix functions of several variables, and in earlier papers [28], [23] for almost periodic scalar and matrix functions of one variable. This development was largely motivated by recent applications of almost periodic functions in convolution equations on finite intervals, inverse scattering, and stochastic processes. We use in the present paper several key results proved in [26], [24] and [25].

Abstract band methods served as key technical tools in [28], [23], [26], [24], [25]. A standard exposition of the abstract band method is found in Part IX of [10]. Another version of the abstract band method was developed and used in [25]. In the present paper the abstract band method plays a key role as well. In Sections 2 and 3 we use the standard abstract band method (see, e.g., Part IX of [10]), while in Section 5 we use the more general version constructed in [15].

The paper is organized as follows. In Section 2 we formulate and solve a general banded almost periodic several variables positive extension problem of which Theorem 1.1 is a very particular case. In Section 3 we apply the results of Section 2 to obtain solutions to a general contractive extension problem. The relation of the results of Section 3 to those of Section 2 are based on the same principle that relates the four block problem to the setting of Theorem 1.1. A class of problems of model matching type is studied in Section 4. In Section 5 we shall solve positive extensions problems for a non-banded pattern, namely, a specific example of a whole class of positive extension problems for almost periodic functions of several variables.

We use the standard notation $\mathbb{C}, \mathbb{R}, \mathbb{T}, \mathbb{Z}$, and $\mathbb{N}$ for the sets of complex numbers, of real numbers, of unimodular complex numbers, of integers, and of positive integers, respectively.

2. The positive multiblock problem

We first introduce the concepts and notation concerning almost periodic functions.

We let (AP^k) denote the algebra of complex valued almost periodic functions of k real variables, i.e., the closed subalgebra of $L^\infty(\mathbb{R}^k)$ (with respect to the standard Lebesgue measure) generated by all the functions $e_\lambda(t) = e^{i\langle\lambda,t\rangle}$, where $\lambda = (\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k$. Here the variable $t = (t_1, \cdots, t_k) \in \mathbb{R}^k$, and

$$\langle\lambda, t\rangle = \sum_{j=1}^{k} \lambda_j t_j$$

is the standard inner product of λ and t. The norm in (AP^k) will be denoted by $\|\cdot\|_\infty$. Recall that for any $f \in (AP^k)$ its *Fourier series* is defined by the formal sum

$$\sum_{\lambda} f_\lambda e^{i\langle\lambda,t\rangle}, \tag{2.1}$$

where

$$f_\lambda = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} e^{-i\langle\lambda,t\rangle} f(t)dt,\ \lambda \in \mathbb{R}^k, \tag{2.2}$$

and the sum in (2.1) is taken over the set $\sigma(f) = \{\lambda \in \mathbb{R}^k : \ f_\lambda \neq 0\}$, called the *Fourier spectrum* of f. The Fourier spectrum of every $f \in (AP^k)$ is at most a countable set. The *mean* $M\{f\}$ of $f \in (AP^k)$ is defined by $M\{f\} = f_0 = \lim_{T\to\infty} \frac{1}{(2T)^k} \int_{[-T,T]^k} f(t)dt$. The *Wiener algebra* (APW^k) is defined as the set of all $f \in (AP^k)$ such that the Fourier series of f converges absolutely. The Wiener algebra is a Banach $*$-algebra with respect to the *Wiener norm* $\|f\|_W = \sum_{\lambda\in\mathbb{R}^k} |f_\lambda|$ (the multiplication in (APW^k) and the involution are defined pointwise). Note that (APW^k) is dense in (AP^k).

Denote by $(AP^k)^{m\times n}$ (resp., $(APW^k)^{m\times n}$) the set, which is an algebra if $m = n$, of $m \times n$ matrices with entries in (AP^k) (resp., (APW^k)). The infinity norm in $(AP^k)^{m\times n}$ is

$$\|f\|_\infty = \sup_{t\in\mathbb{R}^k} \|f(t)\|, \quad f \in (AP^k)^{m\times n},$$

where $\|\cdot\|$ is the operator norm. The mean $M\{F\}$ for $F = [F_{p,q}]_{p=1,q=1}^{m,n} \in (AP^k)^{m\times n}$ is defined by $M\{F\} = [M\{F_{p,q}\}]_{p=1,q=1}^{m,n}$. The Fourier spectrum $\sigma(F)$ of $F \in (AP^k)^{m\times n}$ is, by definition, the set of all $\lambda \in \mathbb{R}^k$ such that $M\{e_{-\lambda}F\} \neq 0$. In other words, $\sigma([F_{p,q}]_{p=1,q=1}^{m,n}) = \cup_{p=1,q=1}^{m,n} \sigma(F_{p,q})$.

A matrix function $f \in (AP^k)^{n\times n}$ is called *positive definite* if $f(t)$ is Hermitian for every $t \in \mathbb{R}^k$, and there exists an $\epsilon > 0$ such that $f(t) \geq \epsilon I_n$ for all $t \in \mathbb{R}^k$, where I_n is the $n \times n$ identity matrix. For the general theory of almost periodic functions of one and several variables we refer the reader to the books [3, 19, 20] and to Chapter 1 in [22].

Let Δ be a non-empty subset of $\mathbb{R}^k$. Denote

(2.3)
$$(AP^k)_\Delta = \{f \in (AP^k) : \ \sigma(f) \subseteq \Delta\}, \ (APW^k)_\Delta = \{f \in (APW^k) : \ \sigma(f) \subseteq \Delta\}.$$

If Δ is an additive subgroup of $\mathbb{R}^k$, then $(AP^k)_\Delta$ (resp. $(APW^k)_\Delta$) is a unital subalgebra of (AP^k) (resp. (APW^k)).

Introduce an inner product on (AP^k) by the formula

$$\langle f, g\rangle = M\{fg^*\}, \ f, g \in (AP^k). \tag{2.4}$$

Here and elsewhere we denote by g^* the function $g^*(t) = \overline{g(t)}$; if $g \in (AP^k)^{m\times n}$, then $g^* \in (AP^k)^{n\times m}$ is defined by $g^*(t) = \overline{g(t)^T}$, where ^{T} designates the transposed matrix. The completion of (AP^k) with respect to inner product (2.4) is called the *Besikovitch space* and is denoted by (B^k). Thus (B^k) is a Hilbert space. For a nonempty set $\Lambda \subseteq \mathbb{R}^k$, define the projection

$$P_\Lambda\left(\sum_{\lambda\in\sigma(f)} f_\lambda e^{i\langle\lambda,t\rangle}\right) = \sum_{\lambda\in\sigma(f)\cap\Lambda} f_\lambda e^{i\langle\lambda,t\rangle},$$

where $f \in (APW^k)$. The projection P_Λ extends by continuity to the orthogonal projection (also denoted P_Λ) on (B^k). We denote by $(B^k)_\Lambda$ the range of P_Λ, or, equivalently, the completion of $(AP^k)_\Lambda$ with respect to the inner product (2.4). The matrix valued Besikovitch space $(B^k)^{n\times m}$ consists of $n \times m$ matrices with components in (B^k), with the standard Hilbert space structure:

$$\langle (f_{i,j})_{i=1,j=1}^{n,m}, (g_{i,j})_{i=1,j=1}^{n,m}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m}\langle f_{i,j}, g_{i,j}\rangle. \tag{2.5}$$

Similarly, $(B^k)_\Lambda^{n\times m}$ is the Hilbert space of $n \times m$ matrices with components in $(B^k)_\Lambda$. In the periodic case ($\Lambda = \mathbb{Z}^k$) we may identify $(B^k)_\Lambda$ with $L_2(\mathbb{T}^k)$.

A subset S of $\mathbb{R}^k$ is called a *halfspace* if it has the following properties:

(i) $\mathbb{R}^k = S \cup (-S)$;
(ii) $S \cap (-S) = \{0\}$;
(iii) if $x, y \in S$ then $x + y \in S$;
(iv) if $x \in S$ and α is a nonnegative real number, then $\alpha x \in S$.

Note that conditions (iii) and (iv) mean that S is a *cone*, and conditions (ii), (iii), and (iv) together mean that S is a *pointed cone*. A standard example of a halfspace is given by

$$E_k = \{(x_1, \cdots, x_k)^T \in \mathbb{R}^k \setminus \{0\} \ : \\ x_1 = x_2 = \cdots = x_{j-1} = 0, \ x_j \neq 0 \ \Rightarrow x_j > 0\} \cup \{0\}.$$

(The vectors in $\mathbb{R}^k$ are understood as column vectors.) One can show using basic results on linearly ordered real vector spaces (see [5], Section IV.5 in [9]) that a set $S \subseteq \mathbb{R}^k$ is a halfspace if and only if there exists a real invertible $k \times k$ matrix A such that

$$S = AE_k \stackrel{\text{def}}{=} \{Ax \ : \ x \in E_k\}. \tag{2.6}$$

See Section 2 of [26] for more details.

In this section we consider the following *positive extension problem*. Let S be a halfspace in $\mathbb{R}^k$ and Λ an additive subgroup of $\mathbb{R}^k$. Further, fix

$$p \in \mathbb{N}, \quad q \in \{1, \dots, p\}, \quad \text{and} \quad n_0, n_1, \dots, n_p \in \mathbb{N}.$$

Given are:

$$f_{ij} = f_{ji}^* \in (APW^k)_\Lambda^{n_i \times n_j} \quad \text{for} \quad |j - i| < q,$$

and

$$f_{ij}^- = (f_{ji}^+)^* \in (APW^k)_{\Lambda \cap (-S)}^{n_i \times n_j}, \quad \text{for} \quad j - i = q.$$

Find, if possible,

$$f_{ij}^+ = (f_{ji}^-)^* \in (APW^k)_{\Lambda \cap (S \setminus \{0\})}^{n_i \times n_j}, \quad \text{for} \quad j - i = q,$$

and

$$f_{ij} = f_{ji}^* \in (APW^k)_\Lambda^{n_i \times n_j} \quad \text{for} \quad |j - i| > q,$$

so that

$$(f_{ij})_{i,j=0}^p > 0, \tag{2.7}$$

where

$$f_{ij} = f_{ji}^* := f_{ij}^- + f_{ij}^+ \quad \text{for} \quad j - i = q.$$

The inequality in (2.7) is interpreted as follows:

$$(f_{ij})_{i,j=0}^p(t) \geq \epsilon I, \quad \text{for every} \quad t \in \mathbb{R}^k,$$

where $\epsilon > 0$ is independent of t. An analogous interpretation is given to inequalities of the form $G > 0$ in the sequel. The necessary and sufficient condition for existence of a solution (2.7) will be the positive definiteness of an appropriately defined Toeplitz operator. In addition, in that case we shall construct a solution to the problem that has a certain maximality property, and we shall construct a linear fractional description for the set of solutions as well.

Before we can state and prove our theorems we do have to introduce some additional notation and derive some auxiliary results. We start by quoting a result from [26] (Corollary 5.3) that will be used extensively in the sequel.

Theorem 2.1. *Let $G \in (APW^k)^{n \times n}$ and assume that the matrix $G(t)$ is positive definite for every $t \in \mathbb{R}^k$, and $\det G(t) \geq \epsilon$ for every $t \in \mathbb{R}^k$, where $\epsilon > 0$ is independent of t. Let also Λ' be the minimal additive subgroup of $\mathbb{R}^k$ which contains $\sigma(G)$. Then $G(t)$ admits canonical factorizations of the forms*

$$G(t) = A_+(t) \cdot (A_+(t))^* = (\widetilde{A}_+(t))^* \cdot \widetilde{A}_+(t), \tag{2.8}$$

where $A_+^{\pm 1}, \widetilde{A}_+^{\pm 1} \in (APW^k)_{S \cap \Lambda'}^{n \times n}$. The factors $A_+^{\pm 1}, \widetilde{A}_+^{\pm 1}$ are defined up to a right/left constant unitary multiple.

Let

$$\mathcal{M} = (APW^k)_\Lambda^{N \times N}, \quad N = n_0 + \dots + n_p,$$

$$\mathcal{M}_1 = \mathcal{M}_4^* = \Big\{ (f_{ij})_{i,j=0}^p \in \mathcal{M} : \ f_{ij} = 0, \ j - i < q \quad \text{and } f_{ij} \in (APW^k)_{\Lambda \cap (S \setminus \{0\})}^{n_i \times n_j}, \ j - i = q \Big\},$$

$$\mathcal{M}_2^0 = (\mathcal{M}_3^0)^* = \Big\{ (f_{ij})_{i,j=0}^p \in \mathcal{M} : \ f_{ij} = 0, \ j-i<0; \ f_{ii} \in (APW^k)^{n_i \times n_i}_{\Lambda \cap (S \setminus \{0\})},$$

$$i = 0, \dots, p; f_{ij} \in (APW^k)^{n_i \times n_j}_{\Lambda \cap (-S)}, \ j-i=q; \text{ and } f_{ij}=0, \ j-i>q \Big\},$$

and

$$\mathcal{M}_d = \mathbb{C}^{n_0 \times n_0} \oplus \cdots \oplus \mathbb{C}^{n_p \times n_p}.$$

We identify here $\mathbb{C}^{n_j \times n_j}$ with $(APW^k)^{n_j \times n_j}_{\{0\}}$. Then

$$e := \bigoplus_{i=0}^{p} I_{n_i \times n_i} \in \mathcal{M}_d,$$

$$\mathcal{M} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \tag{2.9}$$

and the following multiplication table holds:

(2.10)

$\cdot$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$

,

where

$$\begin{aligned} \mathcal{M}_+^0 &= (\mathcal{M}_-^0)^* = \mathcal{M}_2^0 \dot{+} \mathcal{M}_1, \\ \mathcal{M}_c &= \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0, \\ \mathcal{M}_+ &= \mathcal{M}_-^* = \mathcal{M}_+^0 \dot{+} \mathcal{M}_d. \end{aligned}$$

For example, $\mathcal{M}_2^0 \mathcal{M}_4 \subseteq \mathcal{M}_-^0$. Note that $\mathcal{M}_-^0$ and $\mathcal{M}_+^0$ are independent of q. We shall use the letter e to also refer to identity matrices of smaller size as well, e.g., $e = \bigoplus_{i=0}^{p-1} I_{n_i}$. The appropriate interpretation should be clear from the context. Notice that we may restate the positive extension problem formulated in this section as follows: Given $k = k^* \in \mathcal{M}_c$, find, if possible, $m_1 \in \mathcal{M}_1$ so that

$$k + m_1 + m_1^* > 0.$$

The element $k + m_1 + m_1^* > 0$ is called a *positive extension* of k, i.e., f is a positive extension of k if and only if $f > 0$ and $f - k \in \mathcal{M}_1 \dot{+} \mathcal{M}_4$. It should be noted that the problem is now completely formulated as in the general band method (see Chapter XXXIV of [10] or [29]), except that we have to show that the notion of positivity $f > 0$ coincides with the notion of positivity as required in the general band method. This follows easily as a consequence of the next lemma.

Lemma 2.2. *Let $f \in \mathcal{M}$ be so that $f > 0$. Then there exist $f_+ \in \mathcal{M}_+^0$ and $f_d \in \mathcal{M}_d$ so that*

$$f = (e + f_+)^* f_d (e + f_+), \tag{2.11}$$

$f_d > 0$, $f_d \in \mathcal{M}_d$, $e + f_+$ is invertible in $\mathcal{M}$ and $(e+f_+)^{-1} - e \in \mathcal{M}_+^0$. Moreover, this factorization is unique. In addition, f also allows the factorization

$$f = (e + f_-)^* \widetilde{f}_d (e + f_-), \tag{2.12}$$

with $\widetilde{f}_d > 0$, $\widetilde{f}_d \in \mathcal{M}_d$ and f_-, $(e+f_-)^{-1} - e \in \mathcal{M}_-^0$. Again, factorization (2.12) *is unique.*

The factorizations (2.11) and (2.12) are referred to as *the right and the left canonical factorizations* of f, respectively, with respect to the decomposition $\mathcal{M} = \mathcal{M}_-^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_+^0$.

Proof of Lemma 2.2. We do this by induction on the number of row and column blocks p. Note that the statement of this lemma also makes sense when $p = 0$ (in contrast with other parts of this section). In this case we interpret

$$\mathcal{M}_-^0 = (APW^k)^{n_0 \times n_0}_{\Lambda \cap (-S \backslash \{0\})}, \quad \mathcal{M}_d = \mathbb{C}^{n_0 \times n_0}, \quad \mathcal{M}_+^0 = (APW^k)^{n_0 \times n_0}_{\Lambda \cap (S \backslash \{0\})}.$$

It will be most convenient for the proof to include the case $p = 0$.

When $p = 0$, the result of Lemma 2.2 follows from Theorem 2.1. Now assume the result has been established up to $p - 1$. Letting $f = (f_{ij})_{i,j=0}^p$ and writing

$$(f_{ij})_{i,j=0}^{p-1} = (e + f_+^{(p-1)})^* f_d^{(p-1)} (e + f_+^{(p-1)}),$$

we get that the right canonical factors of f in (2.11) are given by

$$e + f_+ \ := \begin{bmatrix} e + f_+^{(p-1)} & 0 \\ 0 & e + g_+ \end{bmatrix} \begin{bmatrix} e & h \\ 0 & e \end{bmatrix},$$

$$f_d \ := \begin{bmatrix} f_d^{(p-1)} & 0 \\ 0 & g_d \end{bmatrix}.$$

Here $h = [(f_{ij})_{i,j=0}^{p-1}]^{-1} (f_{ip})_{i=0}^{p-1}$, and

$$f_{pp} - (f_{pi})_{i=0}^{p-1} [(f_{ij})_{i,j=0}^{p-1}]^{-1} (f_{ip})_{i=0}^{p-1} = (e + g_+)^* g_d (e + g_+)$$

is a right canonical factorization with respect to the decomposition

$$(APW^k)^{n_p \times n_p}_{\Lambda \cap (-S \backslash \{0\})} \dot{+} \mathbb{C}^{n_p \times n_p} \dot{+} (APW^k)^{n_p \times n_p}_{\Lambda \cap (S \backslash \{0\})}.$$

Notice that

$$f_{pp} - (f_{pi})_{i=0}^{p-1} [(f_{ij})_{i,j=0}^{p-1}]^{-1} (f_{ip})_{i=0}^{p-1} > 0.$$

The uniqueness of the factorization follows using standard arguments (see, e.g., [26]). The proof of (2.12) is analogous. □

We shall formulate our necessary and sufficient condition in terms of a Hilbert space operator. We thus start with introducing the appropriate Hilbert spaces. Let $\mathcal{B} = (B^k)^{N \times N}$, and let

$$\mathcal{B}_1,\ \mathcal{B}_2^0,\ \mathcal{B}_2,\ \mathcal{B}_d,\ \mathcal{B}_3^0,\ \mathcal{B}_3,\ \mathcal{B}_4,\ \mathcal{B}_-^0,\ \mathcal{B}_+^0,\ \mathcal{B}_-,\ \mathcal{B}_+,\ \mathcal{B}_d$$

be the closures of

$$\mathcal{M}_1, \mathcal{M}_2^0, \mathcal{M}_2 = \mathcal{M}_2^0 \dot{+} \mathcal{M}_d, \mathcal{M}_d, \mathcal{M}_3^0, \mathcal{M}_3 = \mathcal{M}_3^0 \dot{+} \mathcal{M}_d,$$
$$\mathcal{M}_4, \mathcal{M}_-^0, \mathcal{M}_+^0, \mathcal{M}_-, \mathcal{M}_+, \mathcal{M}_d$$

in $\mathcal{B}$, respectively. Alternatively, one may for example introduce $\mathcal{B}_1$ in the same way $\mathcal{M}_1$ was defined with APW^k replaced by B^k. Notice that we have the following orthogonal sum decomposition

$$\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2^0 \oplus \mathcal{B}_d \oplus \mathcal{B}_3^0 \oplus \mathcal{B}_4.$$

In addition, we introduce P_1, P_2^0, etc., as the orthogonal projections of $\mathcal{B}$ onto $\mathcal{B}_1$, $\mathcal{B}_2^0$, etc. It is not hard to see that if one of these projections is applied to a member of $\mathcal{M}$ the result ends up in the appropriate subspace of $\mathcal{M}$. So, for instance, if $m \in \mathcal{M}$, then $P_c m \in \mathcal{M}_c$.

We will now state our main result. For $f \in (APW^k)^{m\times m}$ we denote

$$\|f\|_\infty = \sup_{x\in\mathbb{R}^k} \|f(x)\|,$$

where $\|X\|$ is the operator norm of an operator or matrix X.

Theorem 2.3. *Let $k = k^* \in \mathcal{M}_c$ be given. Then the following statements are equivalent:*

(i) *k has a positive extension;*
(ii) *the operator $T : \mathcal{B}_2 \to \mathcal{B}_2$ defined by $T(g) = P_2(kg)$, is positive definite;*
(iii) *the operator $\widetilde{T} : \mathcal{B}_3 \to \mathcal{B}_3$ defined by $\widetilde{T}(g) = P_3(kg)$, is positive definite.*

In that case, let

$$x = T^{-1}(e), \quad y = \widetilde{T}^{-1}(e).$$

Then $x^{-1} \in \mathcal{M}_+$, $y^{-1} \in \mathcal{M}_-$, $P_d x > 0$, $P_d y > 0$, and

$$f_0 := x^{*-1}(P_d x)x^{-1} = y^{*-1}(P_d y)y^{-1}$$

is a positive extension of k. In fact, f_0 is the unique positive extension of k with $f_0^{-1} \in \mathcal{M}_c$. Moreover, if we let $u = x(P_d x)^{-\frac{1}{2}}$ and $v = y(P_d y)^{-\frac{1}{2}}$, then f is a positive extension of k if and only if

$$f = (u + vg)^{*-1}(e - g^* g)(u + vg)^{-1},$$

for some (unique!) $g \in \mathcal{M}_1$ with $\|g\|_\infty < 1$. Also, f is a positive extension of k if and only if

$$f = (v + uh)^{*-1}(e - h^* h)(v + uh)$$

for some (unique!) $h \in \mathcal{M}_4$ with $\|h\|_\infty < 1$.

Lastly, if f is a positive extension of k with canonical factorizations

$$f = (e + f_+)^* f_d (e + f_+) = (e + f_-)^* \widetilde{f}_d (e + f_-),$$

then

$$f_d \le (P_d x)^{-1} \text{ and } \widetilde{f}_d \le (P_d y)^{-1}, \tag{2.13}$$

i.e., the matrices $(P_d x)^{-1} - f_d$ and $(P_d y)^{-1} - \widetilde{f}_d$ are positive semidefinite, and equality holds in at least one of the two inequalities in (2.13) if and only if $f = f_0$ (and thus both inequalities are equalities).

The "maximizing" property of f_0 exhibited in (2.13) may also be presented as follows. Let $f \in (APW^k)^{N\times N}$ be positive definite. From Proposition 2.3 in [26], it follows that $\log(\det f)$ belongs to (APW^k). The number

$$\Delta(f) := M\{\log(\det f)\} \tag{2.14}$$

will be referred to as the *entropy* of f.

Proposition 2.4. *Let $k = k^* \in \mathcal{M}_c$ be given and suppose that k has a positive extension. Let f_0 be defined as in Theorem* 2.3. *If f is a positive extension of k then*

$$\Delta(f) \leq \Delta(f_0),$$

and equality holds if and only if $f = f_0$.

The main effort in proving Theorem 2.3 lies in showing that $x^{-1} \in \mathcal{M}_+$ and $y^{-1} \in \mathcal{M}_-$ provided $T > 0$ and $\widetilde{T} > 0$. The remainder of the proof is a standard application of the band method.

For the reader's convenience, we state a result (that includes a particular case of Theorem 2.1 of [24] and remarks thereafter) that will be used in the proof of Theorem 2.3.

For $g \in (APW^k)^{m\times n}_{\Lambda'}$, where Λ' is an additive subgroup of $\mathbb{R}^k$, and for $p \in \mathbb{Z}$ define the *generalized Hankel* operator

$$\mathbf{H}(g)_{\Lambda'} \colon (B^k)^{n\times p}_{S\cap\Lambda'} \to (B^k)^{m\times p}_{(-S)\cap\Lambda'}$$

by

$$\mathbf{H}(g)_{\Lambda'} h = P_{-S}(gh), \quad h \in (B^k)^{n\times p}_{S\cap\Lambda'}. \tag{2.15}$$

We suppress the dependence of $\mathbf{H}(g)_{\Lambda'}$ on S in our notation. It is not hard to see that the norm of $\mathbf{H}(g)_{\Lambda'}$ is independent of the choice of the positive integer p.

Theorem 2.5. [24] *Let $f \in (APW^k)^{m\times n}_{(-S)\cap\Lambda'}$ be given, where Λ' is an additive subgroup of $\mathbb{R}^k$. Then the following two statements are equivalent*:

(i) *f has a strictly contractive extension $h \in (APW^k)^{m\times n}_{\Lambda'}$, i.e.,*

$$\|h\|_\infty := \sup_{t\in\mathbb{R}^k} \|h(t)\| < 1, \quad \text{and} \quad h_\lambda = f_\lambda, \quad \text{for} \quad \lambda \in (-S)\cap\Lambda'.$$

(ii) *The generalized Hankel operator $\mathbf{H}(f)_{\Lambda'}$ is a strict contraction.*

When one (*and thus both*) *of* (i)–(ii) *is satisfied, put*

$$\begin{aligned}
\widehat{\alpha}(t) &= \left[I - \mathbf{H}(f)_{\Lambda'}(\mathbf{H}(f)_{\Lambda'})^*\right]^{-1}(e),\\
\widehat{\beta}(t) &= \mathbf{H}(f)_{\Lambda'}\left[I - (\mathbf{H}(f)_{\Lambda'})^*\mathbf{H}(f)_{\Lambda'}\right]^{-1}(e),\\
\widehat{\gamma}(t) &= (\mathbf{H}(f)_{\Lambda'})^*\left[I - \mathbf{H}(f)_{\Lambda'}(\mathbf{H}(f)_{\Lambda'})^*\right]^{-1}(e),\\
\widehat{\delta}(t) &= \left[I - (\mathbf{H}(f)_{\Lambda'})^*\mathbf{H}(f)_{\Lambda'}\right]^{-1}(e),
\end{aligned}$$

where e stands for matrix function on $\mathbb{R}^k$ that takes the constant identity matrix value. Then $\widehat{\alpha}$ is invertible in $(APW^k)^{m\times m}_{\Lambda'}$ and $M\{\widehat{\alpha}\}$ is positive definite. Similarly, $\widehat{\delta}$ is invertible in $(APW^k)^{n\times n}_{\Lambda'}$ and $M\{\widehat{\delta}\} > 0$. Also, $\widehat{\alpha}^{-1} \in (APW^k)^{m\times m}_{(-S)\cap\Lambda'}$ and $\widehat{\delta}^{-1} \in (APW^k)^{n\times n}_{S\cap\Lambda'}$.

Further, let

$$\alpha(t) = \widehat{\alpha}(t)M\{\widehat{\alpha}\}^{-\frac{1}{2}}, \quad \beta(t) = \widehat{\beta}(t)M\{\widehat{\delta}\}^{-\frac{1}{2}}; \tag{2.16}$$

$$\gamma(t) = \widehat{\gamma}(t)M\{\widehat{\alpha}\}^{-\frac{1}{2}}, \quad \delta(t) = \widehat{\delta}(t)M\{\widehat{\delta}\}^{-\frac{1}{2}}. \tag{2.17}$$

Then the function

$$h_0(t) = \beta(t)\delta(t)^{-1} = [\alpha(t)^*]^{-1}\gamma(t)^*,\ t \in \mathbb{R}^k, \tag{2.18}$$

is a strictly contractive extension in $(APW^k)^{m\times n}_{\Lambda'}$ *of* f.

Proposition 2.6. *Let* $T : \mathcal{B}_2 \to \mathcal{B}_2$ *be as in Theorem* 2.3 *and suppose that* $T > 0$. *Then* $x := T^{-1}(e)$ *is an invertible element of* $\mathcal{M}_+$.

Proof. Since $x \in \mathcal{B}_2$ it has the form $x = (x_{ij})_{i,j=0}^p$ with $x_{ij} = 0$ for $j - i < 0$; $x_{ii} \in (B^k)^{n_i\times n_j}_{\Lambda\cap S}$ for $i = 0, \ldots, p$; $x_{ij} \in (B^k)^{n_i\times n_j}_{\Lambda}$ for $0 < j-i < q$; $x_{ij} \in (B^k)^{n_i\times n_j}_{\Lambda\cap(-S)}$ for $j - i = q$; and $x_{ij} = 0$ for $j - i > q$. It therefore suffices to show that $x_{ii}^{\pm1} \in (APW^k)^{n_i\times n_i}_{\Lambda\cap S}$ and $x_{ij} \in (APW^k)^{n_i\times n_j}_{\Lambda}$ for $j - i > 0$.

Denote $k = (k_{ij})_{i,j=0}^p$. First observe that by applying T to elements $g = (g_{ij})_{i,j=0}^p \in \mathcal{B}_2$ with $g_{ij} = 0$ for $(i,j) \neq (0,0)$ we get that the operator $T_{00} : (B^k)^{n_0\times n_0}_{\Lambda\cap S} \to (B^k)^{n_0\times n_0}_{\Lambda\cap S}$ defined by

$$T_{00}(g_{00}) = P_S(k_{00}g_{00})$$

is positive definite. Here P_S is the orthogonal projection of $(B^k)^{m\times m}_{\Lambda}$ onto $(B^k)^{m\times m}_{\Lambda\cap S}$. It follows from Section 5 in [26] that $k_{00} > 0$, and therefore has a right canonical factorization (Theorem 2.1)

$$k_{00} = (e + k_{00,+})^* k_{00,d}(e + k_{00,+}).$$

One easily checks that the equation $x = T^{-1}(e)$, or equivalently $P_2(kx) = e$, implies that $x_{00} = (e + k_{00,+})^{-1}k_{00,d}^{-1}$, and therefore $x_{00}^{\pm1} \in (APW^k)^{n_0\times n_0}_{\Lambda\cap S}$.

Next (assuming $q \geq 2$, otherwise this step is not necessary), observe that by applying T to elements $g = (g_{ij})_{i,j=0}^p \in \mathcal{B}_2$ with $g_{ij} = 0$ for $j \neq 1$ we obtain positive definiteness of the operator

$$\begin{bmatrix} M_{00} & M_{10}^* \\ M_{10} & T_{11} \end{bmatrix}$$

on $(B^k)^{n_0\times n_1}_{\Lambda} \oplus (B^k)^{n_1\times n_1}_{\Lambda\cap S}$. Here

$$M_{00}(g_0) = k_{00}g_0,\ M_{10}(g_0) = P_S(k_{10}g_0),\ g_0 \in (B^k)^{n_0\times n_1}_{\Lambda},$$

and

$$T_{11}(g_1) = P_S(k_{11}g_1),\ g_1 \in (B^k)^{n_1\times n_1}_{\Lambda\cap S}.$$

We had already established that $k_{00} > 0$ and thus $M_{00} > 0$. Consequently, we get that $S_{11} := T_{11} - M_{10}M_{00}^{-1}M_{10}^* > 0$. Observe that S_{11} is the operator

$$S_{11}(g_1) = P_S((k_{11} - k_{10}k_{00}^{-1}k_{10}^*)g_1),$$

and consequently (using again Section 5 in [26]) we obtain

$$k_{11} - k_{10}k_{00}^{-1}k_{10}^* > 0.$$

Performing a right canonical factorization

$$k_{11} - k_{10}k_{00}^{-1}k_{10}^* = (e + s_{11,+})^* s_{11,d}(e + s_{11,+}),$$

it is straightforward to check that

$$x_{11} = (e + s_{11,+})^{-1}s_{11,d}^{-1}, \quad \text{and} \quad x_{01} = -k_{00}^{-1}k_{01}x_{11}.$$

(Note: $k_{10}^* = k_{01}$.) Thus $x_{01} \in (APW^k)_{\Lambda}^{n_0 \times n_1}$ and $x_{11}^{\pm 1} \in (APW^k)_{\Lambda \cap S}^{n_1 \times n_1}$. Repeating this argument we obtain that $x_{ij} \in (APW^k)_{\Lambda}^{n_i \times n_j}$ for $0 \le i < j \le q-1$, and $x_{ii}^{\pm 1} \in (APW^k)_{\Lambda \cap S}^{n_i \times n_i}$ for $i = 0, \dots, q-1$.

Let us now look at the q^{th} column of x. For this, observe that by applying T to elements $g = (g_{ij})_{i,j=0}^{p} \in \mathcal{B}_2$ with $g_{ij} = 0$ for $j \neq q$ we get that

$$\begin{bmatrix} \widetilde{T}_{00} & A & H \\ A^* & M_2 & B^* \\ H^* & B & T_{qq} \end{bmatrix} > 0,$$

where

$$\widetilde{T}_{00}: \quad (B^k)_{\Lambda \cap (-S)}^{n_0 \times n_q} \to (B^k)_{\Lambda \cap (-S)}^{n_0 \times n_q}, \quad \widetilde{T}_{00}(g_{0q}) = P_{-S}(k_{00} g_{0q}),$$

$$A: \quad (B^k)_{\Lambda}^{Q \times n_q} \to (B^k)_{\Lambda \cap (-S)}^{n_0 \times n_q}, \quad A \begin{bmatrix} g_{1q} \\ \vdots \\ g_{q-1,q} \end{bmatrix} = P_{-S} \left(\sum_{r=1}^{q-1} k_{0r} g_{rq} \right),$$

$$H: \quad (B^k)_{\Lambda \cap S}^{n_q \times n_q} \to (B^k)_{\Lambda \cap (-S)}^{n_0 \times n_q}, \quad H g_{qq} = P_{-S}(k_{0q} g_{qq}),$$

$$M_2: \quad (B^k)_{\Lambda}^{Q \times Q} \to (B^k)_{\Lambda}^{Q \times Q}, \quad M_2 \begin{bmatrix} g_{1q} \\ \vdots \\ g_{q-1,q} \end{bmatrix} = [(k_{ij})_{i,j=1}^{q-1}] \begin{bmatrix} g_{1q} \\ \vdots \\ g_{q-1,q} \end{bmatrix},$$

$$B: \quad (B^k)_{\Lambda}^{Q \times n_q} \to (B^k)_{\Lambda \cap S}^{n_q \times n_q}, \quad B \begin{bmatrix} g_{1q} \\ \vdots \\ g_{q-1,q} \end{bmatrix} = P_S \left(\sum_{r=1}^{q-1} k_{qr} g_{rq} \right),$$

$$T_{qq}: \quad (B^k)_{\Lambda \cap S}^{n_q \times n_q} \to (B^k)_{\Lambda \cap S}^{n_q \times n_q}, \quad T_{qq} g_{qq} = P_S(k_{qq} g_{qq}),$$

and where $Q = n_1 + \cdots + n_{q-1}$. Thus $M_2 > 0$ and

$$\begin{bmatrix} \widetilde{T}_{00} - A M_2^{-1} A^* & H - A M_2^{-1} B^* \\ H^* - B M_2^{-1} A^* & T_{qq} - B M_2^{-1} B^* \end{bmatrix} > 0. \tag{2.19}$$

Notice that $\widetilde{T}_{00} - A M_2^{-1} A^*$ is the Toeplitz operator

$$g_{0q} \mapsto P_{-S} \left((k_{00} - (k_{0i})_{i=0}^{q-1} [(k_{ij})_{i,j=1}^{q-1}]^{-1} (k_{i0})_{i=0}^{q-1}) g_{0q} \right),$$

and thus its symbol

$$h_1 = k_{00} - (k_{0i})_{i=0}^{q-1} [(k_{ij})_{i,j=1}^{q-1}]^{-1} (k_{i0})_{i=0}^{q-1}$$

is positive definite: $h_1 > 0$. Likewise

$$h_2 = k_{qq} - (k_{qi})_{i=1}^{q-1} \left[(k_{ij})_{i,j=1}^{q-1} \right] (k_{iq})_{i=1}^{q-1} > 0.$$

Write now the canonical factorizations (which exist by Theorem 2.1)

$$h_1 = (e + h_{1,-})^* h_{1,d}(e + h_{1,-}), \quad h_2 = (e + h_{2,+})^* h_{2,d}(e + h_{2,+}),$$

where

$$h_{1,-}, (e + h_{1,-})^{-1} - e \in (APW^k)^{n_0 \times n_0}_{\Lambda\cap(-S\setminus\{0\})}, \quad h_{1,d} \in \mathbb{C}^{n_0\times n_0}, \quad h_{1,d} > 0,$$

and

$$h_{2,+}, (e + h_{2,+})^{-1} - e \in (APW^k)^{n_q \times n_q}_{\Lambda\cap(S\setminus\{0\})}, \quad h_{2,d} \in \mathbb{C}^{n_1\times n_1}, \quad h_{2,d} > 0.$$

Introducing the operators

$$H_1 : (B^k)^{n_0\times n_q}_{\Lambda\cap(-S)} \to (B^k)^{n_0\times n_q}_{\Lambda\cap(-S)}, \quad H_2 : (B^k)^{n_q\times n_q}_{\Lambda\cap S} \to (B^k)^{n_q\times n_q}_{\Lambda\cap S} \tag{2.20}$$

by

$$H_1 g = h^{\frac{1}{2}}_{1,d}(e + h_{1,-})g, \quad H_2 g = h^{\frac{1}{2}}_{2,d}(e + h_{2,+})g,$$

we obtain the equalities

$$\widetilde{T}_{00} - AM_2^{-1}A^* = H_1^* H_1, \quad T_{qq} - BM_2^{-1}B^* = H_2^* H_2.$$

Observe that

$$H_3 := H_1^{*-1}(H - AM_2^{-1}B^*)H_2^{-1} \ : \ (B^k)^{n_q\times n_q}_{\Lambda\cap S} \ \to \ (B^k)^{n_0\times n_q}_{\Lambda\cap(-S)}$$

is the operator defined by

$$H_3 g = P_{-S}\left((e + h_{1,-})^{*-1} h^{-\frac{1}{2}}_{1,d}\phi(e + h_{2,+})^{-1} h^{-\frac{1}{2}}_{2,d} g\right), \ g \in (B^k)^{n_q\times n_q}_{\Lambda\cap S},$$

where

$$\phi = k_{0q} - (k_{0i})^{q-1}_{i=1}\left[(k_{ij})^{q-1}_{i,j=1}\right]^{-1}(k_{iq})^{q-1}_{i=0}.$$

By (2.19) we get that $\|H_3\| < 1$. Applying now Theorem 2.5, we get that

$$\begin{aligned} \beta &:= H_3(I - H_3^* H_3)^{-1}(e) \in (APW^k)^{n_0\times n_q}_{\Lambda\cap(-S)}, \\ \delta &:= (I - H_3^* H_3)^{-1}(e) \in (APW^k)^{n_q\times n_q}_{\Lambda\cap S}, \end{aligned}$$

$\delta^{-1} \in (APW^k)^{n_q\times n_q}_{\Lambda\cap S}$ and $M\{\delta\} > 0$. It is now straightforward to check the equalities

$$x_{0q} = (e + h_{1,-})^{-1} h^{-\frac{1}{2}}_{1,d}\beta h^{-\frac{1}{2}}_{2,d}, \qquad x_{qq} = (e + h_{2,+})^{-1} h^{-\frac{1}{2}}_{2,d}\delta h^{-\frac{1}{2}}_{2,d}, \tag{2.21}$$

and

$$\begin{aligned} (x_{iq})^{q-1}_{i=1} &= -M_2^{-1}(A^* x_{0q} + B^* x_{qq}) \\ &= -\left[(k_{ij})^{q-1}_{i,j=1}\right]^{-1}\left(\left[(k_{0i})^{q-1}_{i=1}\right]^* x_{0q} + \left[(k_{qi})^{q-1}_{i=1}\right]^* x_{qq}\right). \end{aligned} \tag{2.22}$$

Indeed, to verify (2.21) and (2.22), notice that

$$\begin{bmatrix} \widetilde{T}_{00} & A & H \\ A^* & M_2 & B^* \\ H^* & B & T_{qq} \end{bmatrix} \begin{bmatrix} x_{0q} \\ x_{1q} \\ \vdots \\ x_{qq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e \end{bmatrix}, \tag{2.23}$$

and substitute formulas (2.21) and (2.22) for x_{jq} in (2.23). Now clearly,

$$x_{0q} \in (APW^k)^{n_0\times n_q}_{\Lambda\cap(-S)}, \ x^{\pm 1}_{qq} \in (APW^k)^{n_q\times n_q}_{\Lambda\cap S},$$

$$M\{x_{qq}\} > 0,\ (x_{iq})_{i=1}^{q-1} \in (APW^k)_\Lambda^{Q\times n_q}.$$

This yields that the q^{th} column of x is of the required form. For columns $q+1, \dots, p$ of x one uses the same reasoning with $(k_{ij})_{i,j=0}^{q}$ replaced by $(k_{ij})_{i,j=s}^{q+s}$, $s = 1, \dots, p-q$, respectively. This concludes the proof. □

Analogously to the proof of Proposition 2.6, one may prove the following.

Proposition 2.7. *Let $\widetilde{T} : \mathcal{B}_3 \to \mathcal{B}_3$ be as in Theorem* 2.3 *and suppose $\widetilde{T} > 0$. Then $y := \widetilde{T}^{-1}(e)$ is an invertible element of $\mathcal{M}_-$.*

Now we may prove our main result.

Proof of Theorem 2.3. First assume (i), i.e., k has a positive extension f. Then the multiplication operator $g \mapsto fg$ is a positive definite operator on $\mathcal{B}$. But then its restrictions to $\mathcal{B}_2$ and $\mathcal{B}_3$ are also positive definite. Using the multiplication table (2.10), one obtains easily that for $g \in \mathcal{M}_2$ we have that $T(g) = P_2(kg) = P_2(fg)$. But then the same holds for $g \in \mathcal{B}_2$. Consequently, it follows that $T > 0$. Likewise, one shows that $\widetilde{T} > 0$.

Assume now that (ii) holds. By Proposition 2.6 we have that $x := T^{-1}(e) \in \mathcal{M}_2$, $fx \in e + \mathcal{M}_4 + \mathcal{M}_3^0 + \mathcal{M}_1$, and $x^{-1} \in \mathcal{M}_+$. In addition, it is easy to see that $P_d x > 0$. By Theorem 1.1 in Chapter XXXIV of [10] we obtain that f has a positive extension f_0 (in fact, a so-called "band extension"), which is given by

$$f_0 = x^{*-1}(P_d x)x^{-1}.$$

But then (i) holds, and consequently (iii) holds as well. It follows by Proposition 2.7 and Theorems 1.2 and 1.3 in Chapter XXXIV of [10] that f_0 may also be found via

$$f_0 = y^{*-1}(P_d y)y^{-1}.$$

This shows the first part of the theorem.

Next, observe that in the terminology of Chapter XXXIV of [10], $\mathcal{M}$ is "an algebra with band structure (2.9) in the unital C^*-algebra $\mathcal{B}$". Furthermore, note that Axiom (A) in Chapter XXXIV of [10] is satisfied; in other words, if $F \in \mathcal{M}_+$ is such that $\sup_{t\in\mathbb{R}^k} \|F(t)\| < 1$, then $(I-F)^{-1} \in \mathcal{M}_+$. Indeed, by Proposition 2.3 in [26] (with $f = I - F$ and $\Psi(z) = z^{-1}$) we obtain $(I-F)^{-1} \in (APW)^{N\times N}$; on the other hand, the series $(I-F)^{-1} = I + F + F^2 + \cdots$ converges in $\|\cdot\|_\infty$, hence we have $(I-F)^{-1} \in \mathcal{M}_+$ as required. Since Axiom (A) in Chapter XXXIV of [10] holds, we may apply Theorem 2.1 of Chapter XXXIV in [10], yielding the linear fractional description of the set of all positive extensions.

Next, we need to check that $\mathcal{M}$ with decomposition (2.9) satisfies Axioms (C1) and (C2) in Chapter XXXIV, Section 4 of [10]. Indeed, if $f \in \mathcal{M}$ is positive then $P_d f$ is a positive semi-definite matrix, and $P_d f$ equals zero if and only if $f = 0$. Thus we may apply Theorems 1.3 and 4.2 in Chapter XXXIV in [10] to obtain the last statement in the theorem. □

Proof of Proposition 2.4. Observe that if f has the right spectral factorization (2.11), then

$$\log(\det f) = \log(\det(e + f_+)^*) + \log\det f_d + \log(\det(e + f_+)).$$

Notice: $\log(\det(e+f_+)) \in \mathcal{M}_+$ (see Proposition 3.2 in [24]). Since $(e+f_+)^{\pm 1} \in e+\mathcal{M}_+^0$ we have that $M\{\log(\det(e+f_+))\}=0$. Thus

$$\Delta(f) = \log\det f_d.$$

But then it follows from (2.13) that

$$\Delta(f) = \log\det f_d \le \log\det(P_d x)^{-1} = \Delta(f_0)$$

with equality if and only if $f = f_0$ (since for matrices $A \neq B$ with $A \ge B > 0$ we have $\det A > \det B$; in the terminology of Section XXXIV.4 in [10], the function "log det" is strictly $\mathcal{B}$-monotone). The reasoning may also be applied to the other inequality in (2.13). ☐

It should be noted that similar results may be obtained when the q^{th} diagonal of k does not solely consist of elements in $(APW^k)_{\Lambda\cap(-S)}$ but is per block in any of

$$(APW^k)_{\Lambda\cap(-S)},\ (APW^k)_{\Lambda\cap(-S\setminus\{0\})},$$
$$(APW^k)_{\Lambda},\ \{0\},\ (APW^k)_{\Lambda\cap(\nu-S)},$$

or

$$(APW^k)_{\Lambda\cap(\nu-(S\setminus\{0\}))},$$

where $\nu \in \Lambda$. Each time a mixture of choices is made, the spaces $\mathcal{M}_1$, $\mathcal{M}_2^0$, $\mathcal{M}_d$, $\mathcal{M}_3^0$, $\mathcal{M}_4$ should be appropriately defined and will result in different types of Toeplitz operators, each time yielding an analogue of Theorem 2.3. The proofs of such variations require the same type of simple modifications as the ones outlined in [23, Section 10] and [25, Section 4.5]. We omit further details.

3. The contractive *st*-block problem

Fix $s,t \in \mathbb{N}$, $\max\{t-s,0\} \le q \le t+1$, a halfspace S of $\mathbb{R}^k$ and a subgroup Λ of $\mathbb{R}^k$. Our problem is the following. Let $f = (f_{ij})_{i=1,j=1}^{s,t}$ be given so that $f_{ij} \in (APW^k)_\Lambda^{n_i\times m_j}$, $j-i<q$, and $f_{ij} \in (APW^k)_{\Lambda\cap(-S)}^{n_i\times m_j}$, $j-i=q$. Find, if possible,

$$\widetilde{f} = (\widetilde{f}_{ij})_{i=1,j=1}^{s,t} \in (APW^k)_\Lambda^{(n_1+\cdots+n_s)\times(m_1+\cdots+m_t)},$$

so that

$$\widetilde{f}_{ij} = f_{ij} \quad \text{for} \quad j-i<q,$$
$$\widetilde{f}_{ij} - f_{ij} \in (APW^k)_{\Lambda\cap(S\setminus\{0\})}^{n_i\times m_j} \quad \text{for} \quad j-i=q,$$

and

$$\|\widetilde{f}\|_\infty < 1.$$

Such an $\widetilde{f}$ is called a *strictly contractive extension* of f.

In order to state the results we introduce the following spaces. Let

$$\mathcal{A}_1 = \Big\{ f = (f_{ij})_{i,j=1}^{s} :\ f_{ij} \in (APW^k)_\Lambda^{n_i\times n_j} \quad \text{for} \quad i<j;$$
$$f_{ij} \in (APW^k)_{\Lambda\cap S}^{n_i\times n_j} \quad \text{for} \quad i=j; \quad f_{ij}=0 \quad \text{for} \quad i>j \Big\},$$

$$\mathcal{A}_2 = \Big\{ f = (f_{ij})_{i=1,j=1}^{s,t} : \ f_{ij} \in (APW^k)_{\Lambda}^{n_i \times m_j} \quad \text{for} \quad j - i < q;$$
$$f_{ij} \in (APW^k)_{\Lambda \cap (-S)}^{n_i \times m_j} \quad \text{for} \quad j - i = q; \quad f_{ij} = 0 \quad \text{for} \quad j - i > q \Big\},$$

$$\mathcal{A}_3 = \Big\{ f = (f_{ij})_{i,j=1}^{t} : \ f_{ij} \in (APW^k)_{\Lambda}^{m_i \times m_j} \quad \text{for} \quad i < j;$$
$$f_{ij} \in (APW^k)_{\Lambda \cap S}^{m_i \times m_j} \quad \text{for} \quad i = j; \quad f_{ij} = 0 \quad \text{for} \quad i > j \Big\},$$

$$\mathcal{A}_4 = \Big\{ f = (f_{ij})_{i=1,j=1}^{s,t} : \ f_{ij} = 0 \quad \text{for} \quad j - i < q;$$
$$f_{ij} \in (APW^k)_{\Lambda \cap (S \setminus \{0\})}^{n_i \times m_j} \text{ for } j - i = q; \quad f_{ij} \in (APW^k)_{\Lambda}^{n_i \times m_j} \text{ for } j - i > q \Big\}.$$

Further, let

$$(\mathcal{A}_1^*)_d = \mathbb{C}^{n_1 \times n_1} \oplus \cdots \oplus \mathbb{C}^{n_s \times n_s}, \quad (\mathcal{A}_3)_d = \mathbb{C}^{m_1 \times m_1} \oplus \cdots \oplus \mathbb{C}^{m_t \times m_t}.$$

We let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ denote the closures of $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in $(B^k)_{\Lambda}^{(\Sigma n_i) \times (\Sigma n_i)}$, $(B^k)_{\Lambda}^{(\Sigma n_i) \times (\Sigma m_i)}$, and $(B^k)_{\Lambda}^{(\Sigma m_i) \times (\Sigma m_i)}$, respectively. Also, $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ stand for the closures of $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ in $(B^k)_{\Lambda}^{(\Sigma n_i) \times (\Sigma n_i)}$ and $(B^k)_{\Lambda}^{(\Sigma m_i) \times (\Sigma n_i)}$, respectively. For a closed subspace $\mathcal{C}$ of $(B^k)^{N \times M}$ the symbol $P_{\mathcal{C}}$ stands for the orthogonal projection of $(B^k)^{N \times M}$ onto $\mathcal{C}$.

Theorem 3.1. *Let $f \in \mathcal{A}_2$ be given. Then the following statements are equivalent*:

(i) *f has a strictly contractive extension.*

(ii) *The operator $H : \mathcal{C}_3 \to \mathcal{C}_2$ defined by $H(g) = P_{\mathcal{C}_2}(fg)$, is a strict contraction.*

(iii) *The operator $\widetilde{H} : \mathcal{C}_2^* \to \mathcal{C}_1^*$ defined by $\widetilde{H}(g) = P_{\mathcal{C}_1^*}(fg)$, is a strict contraction.*

In that case, let

$$\widehat{\alpha} = (I - \widetilde{H}\widetilde{H}^*)^{-1}(e), \qquad \widehat{\delta} = (I - H^*H)^{-1}(e).$$

Then $\widehat{\alpha}^{-1} \in \mathcal{A}_1^*$, $\widehat{\delta}^{-1} \in \mathcal{A}_3$, and $P_{(\mathcal{A}_1^*)_d}(\widehat{\alpha}) > 0$ and $P_{(\mathcal{A}_3)_d}(\widehat{\delta}) > 0$. Further, put

$$\alpha = \widehat{\alpha}[P_{(\mathcal{A}_1^*)_d}(\widehat{\alpha})]^{-\frac{1}{2}}, \quad \delta = \widehat{\delta}[P_{(\mathcal{A}_3)_d}(\widehat{\delta})]^{-\frac{1}{2}},$$

$\beta = H(\delta)$ and $\gamma = \widetilde{H}^*(\alpha)$. Then

$$h_0 := \beta\delta^{-1} = \alpha^{*-1}\gamma^*$$

is a strictly contractive extension of f. In fact, h_0 is the unique strictly contractive extension with $h_0(e - h_0^* h_0)^{-1} \in \mathcal{A}_2$. Moreover, h is a strictly contractive extension of f if and only if

$$h = (\alpha g + \beta)(\gamma g + \delta)^{-1},$$

for some (unique!) $g \in \mathcal{A}_4$ with $\|g\|_\infty < 1$. Also, h is a strictly contractive extension of f if and only if

$$h = (\alpha^* + g\beta^*)^{-1}(\gamma^* + g\delta^*),$$

for some (unique!) $g \in \mathcal{A}_4^*$ with $\|g\|_\infty < 1$.

In addition, let h be a strictly contractive extension of f, and perform factorizations

$$I - h^*h = (e + h_+)^* h_{1,d}(e + h_+), \tag{3.1}$$

and

$$I - hh^* = (e + h_-)^* h_{2,d}(e + h_-), \tag{3.2}$$

with $h_+, (e + h_+)^{-1} \in \mathcal{A}_3$*;* $P_{(\mathcal{A}_3)_d}(h_+) = 0$*;* $h_{1,d} \in (\mathcal{A}_3)_d$*;* $h_-, (e + h_-)^{-1} \in \mathcal{A}_1^*$*;* $P_{(\mathcal{A}_1^*)_d}(h_-) = 0$*, and* $h_{2,d} \in (\mathcal{A}_1)_d$*. Then*

$$h_{1,d} \le P_{(\mathcal{A}_3)_d}(\widehat{\delta}), \; h_{2,d} \le P_{(\mathcal{A}_1)_d}(\widehat{\alpha}),$$

and equality occurs in one of the inequalities if and only if $h = h_0$ (*and thus both inequalities are equalities*).

Finally,

$$\|h_0\|_{B^k} \le \frac{\|f\|_{B^k}}{\sqrt{1 - \|H\|^2}} = \frac{\|f\|_{B^k}}{\sqrt{1 - \|\widetilde{H}\|^2}}, \tag{3.3}$$

where

$$\|g\|_{B^k} = [\operatorname{trace} M\{g^*g\}]^{1/2}, \quad g \in (B^k)^{(\sum n_j)\times(\sum m_j)} \tag{3.4}$$

is the Besikovitch *norm.*

Proof. It is straightforward that (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) hold. Indeed, if f_{ext} is a strictly contractive extension of f, then the map

$$g \mapsto f_{ext}g$$

is a strictly contractive operator acting $(B^k)_\Lambda^{(\Sigma m_i)\times\ell} \to (B^k)_\Lambda^{(\Sigma n_i)\times\ell}$, where $\ell \in \mathbb{N}$ (in particular, we shall use it for the cases when $\ell = \Sigma m_i$ or $\ell = \Sigma n_i$). But then so are any restrictions of this multiplication operator. In particular, it follows that $\|H\| < 1$ and $\|\widetilde{H}\| < 1$.

Let us show (ii) $\Rightarrow$ (i). So assume that $\|H\| < 1$. Let $Q = n_1 + \ldots + n_s$ and $R = m_1 + \ldots + m_t$. Put

$$\mathcal{M} = (APW^k)_\Lambda^{(Q+R)\times(Q+R)} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \tag{3.5}$$

where

$$\mathcal{M}_1 = \begin{bmatrix} 0 & \mathcal{A}_4 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{M}_2^0 = \begin{bmatrix} \mathcal{A}_1^0 & \mathcal{A}_2 \\ 0 & \mathcal{A}_3^0 \end{bmatrix},$$

$$\mathcal{M}_d = \begin{bmatrix} (\mathcal{A}_1^*)_d & 0 \\ 0 & (\mathcal{A}_3)_d \end{bmatrix}, \quad \mathcal{M}_3^0 = (\mathcal{M}_2^0)^*, \quad \mathcal{M}_4 = (\mathcal{M}_1)^*,$$

with

$$\mathcal{A}_r^0 = \{f = (f_{jk})_{j,k} \in \mathcal{A}_r \ : \ M\{f_{ii}\} = 0 \text{ for all } i\}, \quad r = 1, 3.$$

Furthermore, let

$$k = \begin{bmatrix} e & f \\ f^* & e \end{bmatrix}.$$

Now we are in the setting of Theorem 2.3. Moreover, with k as defined above, we find that the operator T in Theorem 2.3(ii) corresponds to

$$T = \begin{bmatrix} I & H \\ H^* & I \end{bmatrix}.$$

Thus we apply Theorem 2.3(ii) and obtain that k has a positive extension

$$k_{ext} = \begin{bmatrix} e & f_{ext} \\ f_{ext}^* & e \end{bmatrix},$$

where $f_{ext} - f \in \mathcal{A}_4$. But then it follows that f has a strictly contractive extension f_{ext}. This shows (ii) $\Rightarrow$ (i).

In the same manner one may show (iii) $\Rightarrow$ (i) using Theorem 2.3.

In particular, the equivalence of (ii) and (iii) shows that $\|H\| = \|\widetilde{H}\|$.

Next we observe that with T and $\widetilde{T}$ as in Theorem 2.3 with the above choice of k, we have that

$$T \begin{bmatrix} e & -\widehat{\beta} \\ 0 & \widehat{\delta} \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}, \quad \widetilde{T} \begin{bmatrix} \widehat{\alpha} & 0 \\ -\widehat{\gamma} & e \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix},$$

where $\widehat{\beta} = H(\widehat{\delta})$ and $\widehat{\gamma} = \widetilde{H}^*(\widehat{\alpha})$. Thus when we apply Theorem 2.3 with the above choice of k, we find that x and y defined in Theorem 2.3 correspond to

$$x = \begin{bmatrix} e & -\widehat{\beta} \\ 0 & \widehat{\delta} \end{bmatrix}, \quad y = \begin{bmatrix} \widehat{\alpha} & 0 \\ -\widehat{\gamma} & e \end{bmatrix}.$$

Thus we obtain from the properties of x and y that $\widehat{\alpha}^{-1} \in \mathcal{A}_1^*$, $\widehat{\delta}^{-1} \in \mathcal{A}_3$, and $P_{(\mathcal{A}_1^*)_d}(\widehat{\alpha}) > 0$ and $P_{(\mathcal{A}_3)_d}(\widehat{\delta}) > 0$. Next, when we introduce u and v as in Theorem 2.3, and let

$$g = \begin{bmatrix} 0 & \widetilde{g} \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_1,$$

then we get that

$$(u+vg)^{*-1}(e-g^*g)(u+vg)^{-1} = \begin{bmatrix} e & (\alpha\widetilde{g}+\beta)(\gamma\widetilde{g}+\delta)^{-1} \\ [(\alpha\widetilde{g}+\beta)(\gamma\widetilde{g}+\delta)^{-1}]^* & e \end{bmatrix}.$$

This yields the first linear fractional description of the set of all solutions. Likewise, using the linear fractional description $(v+uh)^{*-1}(e-h^*h)(v+uh)^{-1}$ of Theorem 2.3 yields the second linear fractional description.

Note also that it follows from Theorem 2.3 that

$$f_0^{-1} = \begin{bmatrix} I & h_0 \\ h_0^* & I \end{bmatrix}^{-1} = \begin{bmatrix} * & h_0(I - h_0^* h_0)^{-1} \\ * & * \end{bmatrix} \in \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0.$$

Thus $h_0(I - h_0^* h_0)^{-1} \in \mathcal{A}_2$. In addition, the statement regarding the inequalities

$$h_{1,d} \le P_{(\mathcal{A}_3)_d}(\widehat{\delta}), \;\; h_{2,d} \le P_{(\mathcal{A}_1)_d}(\widehat{\alpha}),$$

follows directly by applying (2.13).

Finally for the last statement, we use a similar type of argument as in the proof of Theorem 7.1 in [24]. Indeed, without loss of generality we may assume that $\|H\| \neq 0$, excluding the trivial case $f = 0$. Let $\epsilon = \frac{1}{\|H\|}$. Let now $\widetilde{f}(t) = \beta(t)\delta(t)^{-1} - f(t)$. Using the easily derived inequality

$$\operatorname{trace} M^*M \le -\log\det(I - M^*M),$$

which holds for every $M \in \mathbb{C}^{Q\times R}$ with $\|M\| < 1$, we have:

$$\begin{aligned}\|f+\widetilde{f}\|^2_{B^k} &= \lim_{T\to\infty} \frac{1}{(2T)^k}\int_{[-T,T]^k} \text{trace}\left[\left((f+\widetilde{f})(t)\right)^*(f+\widetilde{f})(t)\right] dt \\ &\le -\lim_{T\to\infty} \frac{1}{(2T)^k}\int_{[-T,T]^k} \log\det\left(I-((f+\widetilde{f})(t))^*(f+\widetilde{f})(t)\right) dt \\ &= -\lim_{T\to\infty} \frac{1}{(2T)^k}\int_{[-T,T]^k} \log\det\left(I-(\beta(t)\delta(t)^{-1})^*\beta(t)\delta(t)^{-1}\right) dt.\end{aligned}$$

Since $\left(\delta(t)P_{(\mathcal{A}_3)_d}(\delta)^{-1}\right)^{\pm 1} \in e + \mathcal{A}_3^0$ we get the equality

$$M\left\{\log\det\left(\delta(t)P_{(\mathcal{A}_3)_d}(\delta)^{-1}\right)\right\} = 0.$$

Also

$$I-(\beta(t)\delta(t)^{-1})^*\beta(t)\delta(t)^{-1} = \delta(t)^{-1*}(\delta(t)^*\delta(t)-\beta(t)^*\beta(t))\delta(t)^{-1} = \delta(t)^{-1*}\delta(t)^{-1}.$$

Using the last two equalities, we obtain

(3.6)

$$\|f+\widetilde{f}\|^2_{B^k} \le M\left\{\log\det\left(\delta(t)\delta(t)^*\right)\right\} = \log\det(P_{(\mathcal{A}_3)_d}(\widehat{\delta})) = \text{trace}(\log P_{(\mathcal{A}_3)_d}(\widehat{\delta})).$$

Note that

$$P_{(\mathcal{A}_3)_d}(\widehat{\delta}) = P_{(\mathcal{A}_3)_d}((I-H^*H)^{-1}(I_{\sum m_i})) = P_{(\mathcal{A}_3)_d}(I+H^*(I-HH^*)^{-1}H)(I_{\sum m_i})).$$

From the inequality $\log(1+r) \le r$ valid for $r \ge 0$ we get that

$$\log\det P_{(\mathcal{A}_3)_d}(\widehat{\delta}) \le \text{trace}\left[P_{(\mathcal{A}_3)_d}(H^*(I-HH^*)^{-1}H)\right]. \tag{3.7}$$

Since $\|H\| = \epsilon$, it follows from (3.6) and (3.7) that

$$\|f+\widetilde{f}\|^2_{B^k} \le \frac{1}{1-\epsilon^{-2}}\,\text{trace}\, M\left\{(H^*H)\left(I_{\sum m_i}\right)\right\} = \frac{\epsilon^2}{\epsilon^2-1}\|f\|^2_{B^k}.$$

This proves the inequality in (3.3). The equality there follows because $\|H\| = \|\widetilde{H}\|$. □

Also in the contractive setting we may state an entropy result along the lines of Proposition 2.4. Its proof is based on the same observations and we will omit it.

Proposition 3.2. *Let $f \in \mathcal{A}_2$ be given and suppose that f has a strictly contractive extension. Let h_0 be defined as in Theorem* 3.1*. If h is a strictly contractive extension of f then*

$$\Delta(I-h^*h) \le \Delta(I-h_0^*h_0),$$

and equality holds if and only if $h = h_0$.

Analogously to the observation made at the end of Section 2, the same methods apply for other situations in which the elements of f in the q^{th} diagonal, per block, are taken from one of the sets

$$(APW^k)_{\Lambda\cap(-S)},\ (APW^k)_{\Lambda\cap(-S\setminus\{0\})},$$

$$(APW^k)_{\Lambda},\ \{0\},\ (APW^k)_{\Lambda\cap(-S+\nu)},$$

$$(APW^k)_{\Lambda\cap((-S\setminus\{0\})+\nu)},$$

where $\nu \in \Lambda$. For each selection of one of the above sets for every block on the q-th diagonal, a theorem results which is analogous to Theorem 3.1. The corresponding analogs of Proposition 3.2 hold true as well.

We state as an example the variation where the elements of f in the qth diagonal are all taken from $(APW^k)_{\Lambda\cap(-S\setminus\{0\})}$. This variation will be used in the next section.

Introduce

$$\widehat{\mathcal{A}}_2 = \Big\{ f = (f_{ij})_{i=1,j=1}^{s,t} : \ f_{ij} \in (APW^k)_{\Lambda}^{n_i\times m_j} \quad \text{for} \quad j-i<q; \\ f_{ij} \in (APW^k)_{\Lambda\cap(-S\setminus\{0\})}^{n_i\times m_j} \quad \text{for} \quad j-i=q; \quad f_{ij}=0 \quad \text{for} \quad j-i>q \Big\},$$

$$\widehat{\mathcal{A}}_4 = \Big\{ f = (f_{ij})_{i=1,j=1}^{s,t} : \ f_{ij} = 0 \quad \text{for} \quad j-i<q; \\ f_{ij} \in (APW^k)_{\Lambda\cap S}^{n_i\times m_j} \ \text{for} \ j-i=q; \quad f_{ij} \in (APW^k)_{\Lambda}^{n_i\times m_j} \ \text{for} \ j-i>q \Big\},$$

and let $\widehat{\mathcal{C}}_2$ and $\widehat{\mathcal{C}}_2^*$ denote the closures in $(B^k)_{\Lambda}^{(\Sigma n_i)\times(\Sigma m_i)}$ and $(B^k)_{\Lambda}^{(\Sigma m_i)\times(\Sigma n_i)}$ of $\widehat{\mathcal{A}}_2$ and $\widehat{\mathcal{A}}_2^*$, respectively. The spaces $\mathcal{A}_3$, $\mathcal{C}_3$, $\mathcal{A}_1^*$, and $\mathcal{C}_1^*$ are defined as before.

Theorem 3.3. *Let $f \in \widehat{\mathcal{A}}_2$ be given. Then the following statements are equivalent:*

(i) *f has a strictly contractive extension, i.e., there exists an*
$$\widetilde{f} \in (APW^k)_{\Lambda}^{(n_1+\cdots+n_s)\times(m_1+\cdots+m_t)},$$
such that $P_{\widehat{\mathcal{A}}_2}(\widetilde{f}) = f$ and $\|\widetilde{f}\|_\infty < 1$.

(ii) *The operator $H_0 : \mathcal{C}_3 \to \widehat{\mathcal{C}}_2$ defined by $H_0(g) = P_{\widehat{\mathcal{C}}_2}(fg)$, is a strict contraction.*

(iii) *The operator $\widetilde{H}_0 : \widehat{\mathcal{C}}_2^* \to \mathcal{C}_1^*$ defined by $\widetilde{H}_0(g) = P_{\mathcal{C}_1^*}(fg)$, is a strict contraction.*

In that case, let
$$\widehat{\alpha} = (I - \widetilde{H}_0\widetilde{H}_0^*)^{-1}(e), \qquad \widehat{\delta} = (I - H_0^*H_0)^{-1}(e).$$
Then $\widehat{\alpha}^{-1} \in \mathcal{A}_1^*$, $\widehat{\delta}^{-1} \in \mathcal{A}_3$, and $P_{(\mathcal{A}_1^*)_d}(\widehat{\alpha}) > 0$ and $P_{(\mathcal{A}_3)_d}(\widehat{\delta}) > 0$. Further, put
$$\alpha = \widehat{\alpha}[P_{(\mathcal{A}_1^*)_d}(\widehat{\alpha})]^{-\frac{1}{2}}, \quad \delta = \widehat{\delta}[P_{(\mathcal{A}_3)_d}(\widehat{\delta})]^{-\frac{1}{2}},$$
$\beta = H_0(\delta)$ and $\gamma = \widetilde{H}_0^*(\alpha)$. Then
$$h_0 := \beta\delta^{-1} = \alpha^{*-1}\gamma^*$$
is a strictly contractive extension of f. In fact, h_0 is the unique strictly contractive extension with $h_0(e - h_0^*h_0)^{-1} \in \widehat{\mathcal{A}}_2$. Moreover, h is a strictly contractive extension of f if and only if
$$h = (\alpha g + \beta)(\gamma g + \delta)^{-1},$$
for some (unique!) $g \in \widehat{\mathcal{A}}_4$ with $\|g\|_\infty < 1$. Also, h is a strictly contractive extension of f if and only if
$$h = (\alpha^* + g\beta^*)^{-1}(\gamma^* + g\delta^*),$$
for some (unique!) $g \in \widehat{\mathcal{A}}_4^*$ with $\|g\|_\infty < 1$.

In addition, let h *be a strictly contractive extension, and perform factorizations* (3.1) *and* (3.2), *with* $h_+, (e+h_+)^{-1} \in \mathcal{A}_3$*;* $P_{(\mathcal{A}_3)_d}(h_+) = 0$*;* $h_{1,d} \in (\mathcal{A}_3)_d$*;* $h_-, (e+h_-)^{-1} \in \mathcal{A}_1^*$*;* $P_{(\mathcal{A}_1^*)_d}(h_-) = 0$*, and* $h_{2,d} \in (\mathcal{A}_1)_d$*, as in Theorem* 3.1*. Then*

$$h_{1,d} \le P_{(\mathcal{A}_3)_d}(\widehat{\delta}), \ h_{2,d} \le P_{(\mathcal{A}_1)_d}(\widehat{\alpha}),$$

and equality occurs in one of the inequalities if and only if $h = h_0$ (*and thus both inequalities are equalities*)*. Finally,*

$$\|h_0\|_{B^k} \le \frac{\|f\|_{B^k}}{\sqrt{1-\|H_0\|^2}} = \frac{\|f\|_{B^k}}{\sqrt{1-\|\widetilde{H}_0\|^2}},$$

where $\|\cdot\|_{B^k}$ *is given by* (3.4).

The proof of Theorem 3.3 requires a modification of the proof of Theorem 3.1 (and thus implicitly, of the proof of Theorem 2.1). These modifications are of the same type as the ones in [23, Section 10], [24, Section 4], and [25, Section 4.5]. We omit further details.

4. Model matching

It is well-known that solutions of the standard four block (as well as one-block and two-block) problems lead, when using frequency domain approach, to results concerning model matching, a key problem in control systems. This approach to model matching has been extensively studied in the the engineering literature (see, for example, [4], [7], and references there), especially for one-dimensional systems, and see [27] for some recent results in this direction for multidimensional systems. In this section we provide an interpretation of Theorem 3.3 in the context of model matching. We consider filters acting on square summable sequences indexed by an additive group in $\mathbb{R}^k$. The case of the group $\mathbb{Z}$ in $\mathbb{R}$ is the familiar case, treated extensively in the literature (see, e. g., [21]).

Let Λ be an additive subgroup of $\mathbb{R}^k$. For $\Delta \subseteq \Lambda$ we let $\ell_2^N(\Delta)$ denote the Hilbert space of sequences $(v_\lambda)_{\lambda\in\Delta}$ where at most countably many $v_\lambda \in \mathbb{C}^N$ are nonzero and which are square summable in norm, i.e., $\sum_{\lambda\in\Delta} \|v_\lambda\|^2 < \infty$. By $\ell_1^{N\times M}(\Delta)$ we denote the Banach space of sequences $(f_\lambda)_{\lambda\in\Delta}$ where at most countably many $f_\lambda \in \mathbb{C}^{N\times M}$ are nonzero and which are summable in norm, i.e., $\sum_{\lambda\in\Delta} \|f_\lambda\| < \infty$.

Fix a halfspace S of $\mathbb{R}^k$. With S we associate an ordering $\le_S$ on Λ by $q \le_S p$ if and only if $p-q \in S$. We shall use the interval notation with the usual conventions. So, for instance, $S \cap \Lambda = [0,\infty)$. With an element $f \in \ell_1^{N\times M}([0,\infty))$, we associate a filter $\Sigma_f : \ell_2^M([0,\infty)) \to \ell_2^N([0,\infty))$, defined by

$$\Sigma_f((u_\lambda)_{\lambda\in[0,\infty)}) = (y_\lambda)_{\lambda\in[0,\infty)}, \ y_\lambda = \sum_{\alpha\in[0,\lambda]} f_\alpha u_{\lambda-\alpha}.$$

We shall depict the filter as

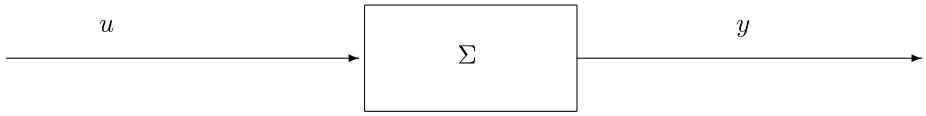

FIGURE 1

and call $(u_\lambda)_\lambda$ the *input* and $(y_\lambda)_\lambda$ the *output* of the filter. The concatenation of two filters results in the *product filter* $\Sigma_h\Sigma_f$. The difference filter $\Sigma_f - \Sigma_h$ may be depicted as in Figure 2.

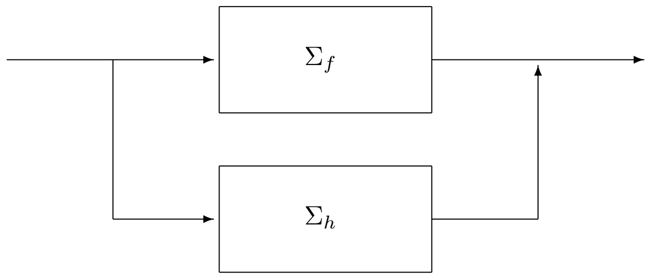

FIGURE 2

With an element $f = (f_\lambda)_{\lambda\in[0,\infty)} \in \ell_1^{N\times M}([0,\infty))$ we may associate a member of $(APW^k)^{N\times M}_{\Lambda\cap S}$, which with a slight abuse of notation we shall also denote by f, and which is defined via

$$f(t) = \sum_{\lambda\in[0,\infty)} f_\lambda e_\lambda(t), t \in \mathbb{R}^k.$$

Note that $\Sigma_h\Sigma_f = \Sigma_{hf}$ and $\Sigma_h - \Sigma_f = \Sigma_{h-f}$. For a filter Σ_f we define its norm by

$$\|\Sigma_f\| = \sup_{u\neq 0} \frac{\|\Sigma_f(u)\|}{\|u\|}.$$

It is not hard to see that $\|\Sigma_f\| = \|f\|_\infty := \sup_{t\in\mathbb{R}^k} \|f(t)\|$.

The *model matching problem* for linear filters is the following. Given are filters Σ_{f_1}, Σ_{f_2}, Σ_{f_3}, find a filter Σ_h so that the filter $\Sigma_{f_1} - \Sigma_{f_2}\Sigma_h\Sigma_{f_3}$ depicted in Figure 3 has minimal possible norm.

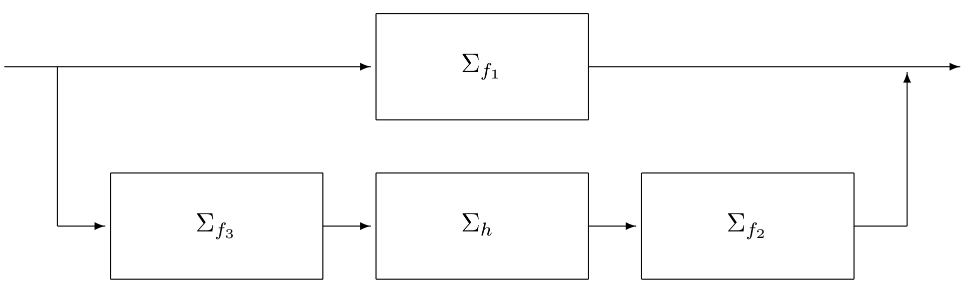

FIGURE 3

Equivalently, given $f_1 \in (APW^k)^{N\times M}_{\Lambda\cap S}$, $f_2 \in (APW^k)^{N\times P}_{\Lambda\cap S}$ and $f_3 \in (APW^k)^{Q\times M}_{\Lambda\cap S}$, find $h \in (APW^k)^{P\times Q}_{\Lambda\cap S}$ so that $\|f_1 - f_2hf_3\|_\infty$ is as small as possible.

We shall assume that f_2 and f_3 allow factorizations

$$f_2 = H_2 \begin{bmatrix} G_2 & 0 \\ 0 & 0 \end{bmatrix} K_2, \quad f_3 = H_3 \begin{bmatrix} G_3 & 0 \\ 0 & 0 \end{bmatrix} K_3, \tag{4.1}$$

where $H_2 \in (APW^k)^{N\times N}_{\Lambda\cap S}$, $H_2^{-1} \in (APW^k)^{N\times N}_{\Lambda}$, $K_2^{\pm 1} \in (APW^k)^{P\times P}_{\Lambda\cap S}$, $H_3^{\pm 1} \in (APW^k)^{Q\times Q}_{\Lambda\cap S}$, $K_3 \in (APW^k)^{M\times M}_{\Lambda\cap S}$, $K_3^{-1} \in (APW^k)^{M\times M}_{\Lambda}$, and $G_2 \in (APW^k)^{k_2\times k_2}_{\Lambda\cap S}$, $G_2^{-1} \in (APW^k)^{k_2\times k_2}_{\Lambda}$, $G_3 \in (APW^k)^{k_3\times k_3}_{\Lambda\cap S}$, $G_3^{-1} \in (APW^k)^{k_3\times k_3}_{\Lambda}$, for some $k_2, k_3 \in \mathbb{N}$. It follows in particular that $f_2(t)$ and $f_3(t)$ have constant ranks k_2 and k_3, respectively, for every $t \in \mathbb{R}^k$.

Under these assumptions we shall provide a solution to the *suboptimal problem*: Let

$$\nu > \inf_h \|f_1 - f_2 h f_3\|_\infty,$$

construct one/all $h \in (APW^k)^{P\times Q}_{\Lambda\cap S}$ such that

$$\|f_1 - f_2 h f_3\|_\infty < \nu. \tag{4.2}$$

We emphasize that our setting of the suboptimal problem, including factorizations (4.1), involves almost periodic functions with Fourier spectrum in Λ only. A suboptimal problem of the above type, but under the more restrictive hypotheses that f_2 and f_3 are square size and invertible, was solved in [24]. We solve the above suboptimal problem by reducing it to a contractive four block problem and subsequently applying the results of Section 3. For the reduction we follow closely the ideas of [8].

We first need the following auxiliary result.

Proposition 4.1. *Let f_2 and f_3 satisfy the condition* (4.1)*. Then there exist*

$$f_{2,i} \in (APW^k)^{N\times k_2}_{\Lambda\cap S}, \quad f_{2,o} \in (APW^k)^{k_2\times P}_{\Lambda\cap S},$$
$$f_{3,ci} \in (APW^k)^{Q\times k_3}_{\Lambda\cap S}, \quad f_{3,co} \in (APW^k)^{M\times k_3}_{\Lambda\cap S},$$

such that

$$f_2 = f_{2,i} f_{2,o}, \quad f_3 = f_{3,co} f_{3,ci}, \quad f_{2,i}^* f_{2,i} \equiv I, \quad f_{3,ci} f_{3,ci}^* \equiv I, \tag{4.3}$$

$f_{2,o}$ has a right inverse $f_{2,o}^\dagger \in (APW^k)^{P\times k_2}_{\Lambda\cap S}$, and $f_{3,co}$ has a left inverse $f_{3,co}^\dagger \in (APW^k)^{k_3\times Q}_{\Lambda\cap S}$.

Factorizations (4.3) of f_2 an f_3 are to be understood as *inner/outer* factorizations, where the inner factor is "energy conserving" and the outer factor is "stably invertible".

Proof. We shall prove the statement regarding f_2. Applying then this result to $f_3(-t)^*$ will yield the factorization for f_3.

Let

$$F = H_2 \begin{bmatrix} G_2 \\ 0 \end{bmatrix}.$$

Then F^*F is a positive definite element of $(APW^k)^{k_2\times k_2}_{\Lambda}$. By Theorem 2.1 there exists an invertible element g of $(APW^k)^{k_2\times k_2}_{\Lambda\cap S}$ so that $F^*F = g^*g$. Let

$$f_{2,i} = Fg^{-1}, \quad f_{2,o} = \begin{bmatrix} g & 0 \end{bmatrix} K_2, \quad f_{2,o}^\dagger = K_2^{-1} \begin{bmatrix} g^{-1} \\ 0 \end{bmatrix}.$$

It is straightforward to check that $f_{2,i}$, $f_{2,o}$, and $f_{2,o}^\dagger$ have the required properties. □

Theorem 4.2. *Let* $f_1 \in (APW^k)^{N\times M}_{\Lambda\cap S}$, $f_2 \in (APW^k)^{N\times P}_{\Lambda\cap S}$ *and* $f_3 \in (APW^k)^{Q\times M}_{\Lambda\cap S}$ *be given so that* (4.1) *is satisfied. Introduce* $f_{2,o}, f_{3,co}, f_{2,i}$ *and* $f_{3,ci}$ *as in Proposition* 4.1. *Also, let* $\nu > 0$. *Write*

$$\frac{1}{\nu}\begin{bmatrix} f_{2,i}^* \\ I - f_{2,i}f_{2,i}^* \end{bmatrix} f_1 \begin{bmatrix} I - f_{3,ci}^* f_{3,ci} & f_{3,ci}^* \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix},$$

and let $s = t = 2$ *and* $q = 1$. *Then the suboptimal model matching problem* (4.2) *is solvable if and only if*

$$\begin{bmatrix} f_{11} & P_{-S\setminus\{0\}}(f_{12}) \\ f_{21} & f_{22} \end{bmatrix} \tag{4.4}$$

has a strictly contractive extension. In that case, for every strictly contractive extension

$$\begin{bmatrix} f_{11} & \widetilde{f}_{12} \\ f_{21} & f_{22} \end{bmatrix} \tag{4.5}$$

of (4.4) *we have that*

$$h := \nu f_{2,o}^{\dagger}(f_{12} - \widetilde{f}_{12}) f_{3,co}^{\dagger}$$

is a solution to the suboptimal model matching problem (4.2). *Conversely, if* h *is a solution to the suboptimal model matching problem* (4.2) *then*

$$\begin{bmatrix} f_{11} & f_{12} - \frac{1}{\nu} f_{2,o} h f_{3,co} \\ f_{21} & f_{22} \end{bmatrix}$$

is a strictly contractive extension of (4.4).

Proof. First let (4.5) be a strictly contractive extension of (4.4), i.e.,

$$\left\| \begin{bmatrix} f_{11} & \widetilde{f}_{12} \\ f_{21} & f_{22} \end{bmatrix} \right\|_\infty < 1$$

and $P_{-S\setminus\{0\}}(\widetilde{f}_{12}) = P_{-S\setminus\{0\}}(f_{12})$. Letting $h := \nu f_{2,o}^{\dagger}(f_{12} - \widetilde{f}_{12}) f_{3,co}^{\dagger}$, we get that $h \in (APW^k)^{P\times Q}_{\Lambda\cap S}$, and

$$\|f_1 - f_2 h f_3\|_\infty = \left\| \begin{bmatrix} f_{2,i}^* \\ I - f_{2,i}f_{2,i}^* \end{bmatrix} (f_1 - f_2 h f_3) \begin{bmatrix} I - f_{3,ci}^* f_{3,ci} & f_{3,ci}^* \end{bmatrix} \right\|_\infty;$$

the equality follows because $V := \begin{bmatrix} f_{2i}^* \\ I - f_{2i}f_{2i}^* \end{bmatrix}$ is an isometry, i.e., $V^*V = I$, and $\begin{bmatrix} I - f_{3,ci}^* f_{3,ci} & f_{3,ci}^* \end{bmatrix}$ is a co-isometry. Thus,

$$\begin{aligned} \|f_1 - f_2 h f_3\|_\infty &= \left\| \nu \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} - \begin{bmatrix} 0 & f_{2,o} h f_{3,co} \\ 0 & 0 \end{bmatrix} \right\|_\infty \\ &= \nu \left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} - \begin{bmatrix} 0 & f_{12} - \widetilde{f}_{12} \\ 0 & 0 \end{bmatrix} \right\|_\infty < \nu. \end{aligned}$$

For the converse, let h be a solution to the suboptimal model matching problem (4.2), and let $\widetilde{f}_{12} = f_{12} - \frac{1}{\nu} f_{2,o} h f_{3,co}$. Then $P_{-S\setminus\{0\}}(\widetilde{f}_{12}) = P_{-S\setminus\{0\}}(f_{12})$ and

$$\left\| \begin{bmatrix} f_{11} & f_{12} + (\widetilde{f}_{12} - f_{12}) \\ f_{21} & f_{22} \end{bmatrix} \right\|_\infty$$
$$= \frac{1}{\nu} \left\| f_1 - \begin{bmatrix} f_{2,i} & I - f_{2,i} f_{2,i}^* \end{bmatrix} \begin{bmatrix} 0 & f_{2,o} h f_{3,co} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I - f_{3,ci}^* f_{3,ci} \\ f_{3,ci} \end{bmatrix} \right\|_\infty$$
$$= \frac{1}{\nu} \| f_1 - f_2 h f_3 \|_\infty < 1.$$

□

By using Theorem 3.3 it is now straightforward to obtain a full solution to the suboptimal model matching problem in terms of Hankel type operators on Besikovitch space. The solution to the classical model matching problem ($k = 1$, $\Lambda = \mathbb{Z}$) may be found in [8, Section 8.1] for rational matrix functions.

Note that Theorem 4.2 together with Theorem 3.3 provide frequency–domain formulas for solutions of the suboptimal problem. These formulas may be not computationally practical. One might anticipate that a conversion to state space formulas, if possible, would give more practical formulas. However, this must remain a subject for future research.

One may wonder whether there exist systems in time domain giving rise to transfer functions that are of the type $f \in (APW^k)^{N\times M}_{\Lambda\cap S}$ as above. It has been known since the 80s that delay systems give rise to almost periodic transfer functions (see, e.g., [18]). Another source are multidimensional systems; [1] is a general reference. We mention here also the following example (see [16], [17]), which is relevant to the periodic case ($\Lambda = \mathbb{Z}^k$); for notational simplicity, we let $k = 2$:

$$\Sigma : \begin{cases} x(n_1, n_2) = A_1 x(n_1 - 1, n_2) + A_2 x(n_1, n_2 - 1) + \\ \qquad + B_1 u(n_1 - 1, n_2) + B_2 u(n_1, n_2 - 1) \\ \\ y(n_1, n_2) = C_1 x(n_1 - 1, n_2) + C_2 x(n_1, n_2 - 1) + \\ \qquad + D_1 u(n_1 - 1, n_2) + D_2 u(n_1, n_2 - 1) \\ \\ x(n_1, n_2) = 0 \quad \text{for} \quad (n_1, n_2) \in \mathbb{Z}^2 \quad \text{such that} \quad n_1 \le 0, \text{ and } n_1 + n_2 = 0; \\ x(n_1, n_2) = 0 \quad \text{for} \quad (n_1, n_2) \in \mathbb{Z}^2 \quad \text{such that} \quad n_1 \ge 1, \text{ and } n_1 + n_2 = 1 \end{cases}$$

Here A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 are constant matrices of appropriate sizes. It should be noted that the initial conditions on x are slightly different from those in [16]. The vectors $u(n), x(n)$ and $y(n)$, where $n = (n_1, n_2)$, are usually referred to as the *input*, *state* and *output* vectors, respectively. We now apply the z-transform, as follows. Let S be the halfspace

$$S = \{(v_1, v_2) \in \mathbb{R}^2 : \text{either} \quad v_1 + v_2 > 0 \quad \text{or} \quad v_1 \le 0 \quad \text{and} \quad v_1 + v_2 = 0\}.$$

Letting

$$\widehat{x}(z_1, z_2) = \sum_{(n_1,n_2)\in S\cap\mathbb{Z}^2} x(n_1, n_2) z_1^{n_1} z_2^{n_2},$$
$$\widehat{y}(z_1, z_2) = \sum_{(n_1,n_2)\in S\cap\mathbb{Z}^2+(0,1)} y(n_1, n_2) z_1^{n_1} z_2^{n_2},$$

$$\widehat{u}(z_1, z_2) = \sum_{(n_1,n_2)\in S\cap\mathbb{Z}^2} u(n_1, n_2) z_1^{n_1} z_2^{n_2},$$

and solving for $\widehat{y}$ in terms of $\widehat{u}$, we obtain that

$$\widehat{y}(z_1, z_2) = f(z_1, z_2)\widehat{u}(z_1, z_2), \tag{4.6}$$

where

$$f(z_1, z_2) = z_1 D_1 + z_2 D_2 + (z_1 C_1 + z_2 C_2)(I - z_1 A_1 - z_2 A_2)^{-1}(z_1 B_1 + z_2 B_2).$$

(To make (4.6) precise, one has to assume that $u(0, 0) = 0$.) The function f is known as the *transfer function* of the system Σ. Assuming that A_1 and A_2 are such that the spectral radius of $z_1 A_1 + z_2 A_2$ is less than one for every $z_1, z_2 \in \mathbb{T}$, upon the substitution $z_j = e^{it_j}$, $j = 1, 2$, we obtain that $f \in (APW^2)^{N\times M}_{\mathbb{Z}^2\cap S}$.

5. Multiblocks having more general patterns: An example

Using the semi-band structure version of the band method developed in [15] instead of the standard band method we used in previous sections, we are able to treat more general positive extension problems. In this section we shall treat one example illustrating the main ideas. Though far more general results can be stated and proved (the theory developed in [14] gives an indication how to produce the most general setup), the notational complexity of doing this is so overwhelming that we restrict ourselves to the following situation only.

Fix a halfspace $S \subset \mathbb{R}^k$, an additive subgroup $\Lambda \subseteq \mathbb{R}^k$ and a vector $\nu \in S$. Let now

$$\Delta_+ = (\nu - S) \cap S \cap \Lambda, \quad \text{and} \quad \Delta = \Delta_+ \cup (-\Delta_+).$$

A positive extension problem can be stated as follows: *Given*

$$k_{11} \in (APW^k)^{n_1\times n_1}_{\Lambda},\ k^-_{12} = (k^+_{21})^* \in (APW^k)^{n_1\times n_2}_{\Lambda\cap(-S)},\ k_{22} \in (APW^k)^{n_2\times n_2}_{\Lambda},$$

$$k^-_{24} = (k^+_{42})^* \in (APW^k)^{n_2\times n_4}_{\Lambda\cap(-S)},\ k^c_{33} \in (APW^k)^{n_3\times n_3}_{\Delta},\ k_{34} = k^*_{43} \in (APW^k)^{n_3\times n_4}_{\Lambda},$$

$$k_{44} \in (APW^k)^{n_4\times n_4}_{\Lambda},$$

find

$$k = (k_{ij})^4_{i,j=1} \in (APW^k)^{(n_1+\cdots+n_4)\times(n_1+\cdots+n_4)}_{\Lambda},$$

so that

$$k > 0, \quad P_{-S}(k_{12}) = k^-_{12}, \quad P_{-S}(k_{24}) = k^-_{24}, \quad P_{\Delta}(k_{33}) = k^c_{33}.$$

We call k a *positive extension* of

$$k_c = \begin{bmatrix} k_{11} & k^-_{12} & 0 & 0 \\ k^+_{21} & k_{22} & 0 & k^-_{24} \\ 0 & 0 & k^c_{33} & k_{34} \\ 0 & k^+_{42} & k_{43} & k_{44} \end{bmatrix}.$$

In order to use the framework of [15] we need to introduce several subspaces of $\mathcal{A} := (APW^k)^{Q\times Q}_{\Lambda}$, where $Q = n_1 + \cdots + n_4$. In the following formulas we shall omit the subscript Λ (for example, $(APW^k)_{\Delta_+\setminus\{0\}}$ is to be understood as $(APW^k)_{(\Delta_+\setminus\{0\})\cap\Lambda}$), as well as the sizes of the individual blocks as they are clear from the block's position in a matrix.

Let

$$\mathcal{A}_+^0 = (\mathcal{A}_-^0)^* = \begin{bmatrix} (APW^k)_{S\setminus\{0\}} & (APW^k) & (APW^k) & (APW^k) \\ 0 & (APW^k)_{S\setminus\{0\}} & (APW^k) & (APW^k) \\ 0 & 0 & (APW^k)_{S\setminus\{0\}} & 0 \\ 0 & 0 & (APW^k) & (APW^k)_{S\setminus\{0\}} \end{bmatrix},$$

$$\widetilde{\mathcal{A}}_d = \mathcal{A}_d = \begin{bmatrix} (APW^k)_{\{0\}} & & 0 \\ & \ddots & \\ 0 & & (APW^k)_{\{0\}} \end{bmatrix},$$

$$\mathcal{A}_2^0 = \begin{bmatrix} (APW^k)_{S\setminus\{0\}} & (APW^k)_{-S} & 0 & 0 \\ 0 & (APW^k)_{S\setminus\{0\}} & 0 & (APW^k)_{-S} \\ 0 & 0 & (APW^k)_{\Delta_+\setminus\{0\}} & 0 \\ 0 & 0 & (APW^k) & (APW^k)_{S\setminus\{0\}} \end{bmatrix},$$

$$\widetilde{\mathcal{A}}_4^* = \mathcal{A}_1 = \begin{bmatrix} 0 & (APW^k)_{S\setminus\{0\}} & (APW^k) & (APW^k) \\ 0 & 0 & (APW^k) & (APW^k)_{S\setminus\{0\}} \\ 0 & 0 & (APW^k)_{S\setminus\Delta_+} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{\mathcal{A}}_-^0 = (\widetilde{\mathcal{A}}_+^0)^* = \begin{bmatrix} (APW^k)_{-S\setminus\{0\}} & 0 & 0 & 0 \\ (APW^k) & (APW^k)_{-S\setminus\{0\}} & 0 & 0 \\ (APW^k) & (APW^k) & (APW^k)_{-S\setminus\{0\}} & 0 \\ (APW^k) & (APW^k) & (APW^k) & (APW^k)_{-S\setminus\{0\}} \end{bmatrix},$$

$$\widetilde{\mathcal{A}}_3^0 = \begin{bmatrix} (APW^k)_{-S\setminus\{0\}} & 0 & 0 & 0 \\ (APW^k)_S & (APW^k)_{-S\setminus\{0\}} & 0 & 0 \\ 0 & 0 & (APW^k)_{-\Delta_+\setminus\{0\}} & 0 \\ 0 & (APW^k)_S & (APW^k) & (APW^k)_{-S\setminus\{0\}} \end{bmatrix},$$

$$\mathcal{A}_2 = \mathcal{A}_2^0 \dot{+} \mathcal{A}_d,\ \widetilde{\mathcal{A}}_3 = \widetilde{\mathcal{A}}_3^0 \dot{+} \widetilde{\mathcal{A}}_d,$$
$$\mathcal{A}_-^* = \mathcal{A}_+ = \mathcal{A}_+^0 \dot{+} \mathcal{A}_d,\ \widetilde{\mathcal{A}}_- = \widetilde{\mathcal{A}}_+^* = \widetilde{\mathcal{A}}_-^0 \dot{+} \widetilde{\mathcal{A}}_d.$$

It is straightforward to check that the following multiplication tables hold:

(5.1)

$\cdot$	$\mathcal{A}_-^0$	$\mathcal{A}_d$	$\mathcal{A}_+^0$
$\mathcal{A}_-^0$	$\mathcal{A}_-^0$	$\mathcal{A}_-^0$	$\mathcal{A}$
$\mathcal{A}_d$	$\mathcal{A}_-^0$	$\mathcal{A}_d$	$\mathcal{A}_+^0$
$\mathcal{A}_+^0$	$\mathcal{A}$	$\mathcal{A}_+^0$	$\mathcal{A}_+^0$

$\cdot$	$\widetilde{\mathcal{A}}_-^0$	$\widetilde{\mathcal{A}}_d$	$\widetilde{\mathcal{A}}_+^0$
$\widetilde{\mathcal{A}}_-^0$	$\widetilde{\mathcal{A}}_-^0$	$\widetilde{\mathcal{A}}_-^0$	$\widetilde{\mathcal{A}}$
$\widetilde{\mathcal{A}}_d$	$\widetilde{\mathcal{A}}_-^0$	$\widetilde{\mathcal{A}}_d$	$\widetilde{\mathcal{A}}_+^0$
$\widetilde{\mathcal{A}}_+^0$	$\widetilde{\mathcal{A}}$	$\widetilde{\mathcal{A}}_+^0$	$\widetilde{\mathcal{A}}_+^0$

.

Moreover, we have:

$$\mathcal{A}_+ = \mathcal{A}_1 \dot{+} \mathcal{A}_2, \quad \widetilde{\mathcal{A}}_- = \widetilde{\mathcal{A}}_4 \dot{+} \widetilde{\mathcal{A}}_3,$$

and

$$\mathcal{A}_1 \mathcal{A}_+ \subseteq \mathcal{A}_1, \quad \mathcal{A}_2 \mathcal{A}_d \subseteq \mathcal{A}_2, \quad \mathcal{A}_1^* \mathcal{A}_2 \subseteq \mathcal{A}_-^0,$$
$$\widetilde{\mathcal{A}}_4 \widetilde{\mathcal{A}}_- \subseteq \widetilde{\mathcal{A}}_4, \quad \widetilde{\mathcal{A}}_3 \widetilde{\mathcal{A}}_d \subseteq \widetilde{\mathcal{A}}_3, \quad \widetilde{\mathcal{A}}_4^* \widetilde{\mathcal{A}}_3 \subseteq \widetilde{\mathcal{A}}_+^0,$$
$$\widetilde{\mathcal{A}}_- \mathcal{A}_2 \subseteq \mathcal{A}_-^0 \dot{+} \mathcal{A}_2, \quad \mathcal{A}_2 \mathcal{A}_- \subseteq \mathcal{A}_-^0 \dot{+} \mathcal{A}_2, \quad \widetilde{\mathcal{A}}_3^* \mathcal{A}_- \subseteq \mathcal{A}_-^0 \dot{+} \mathcal{A}_2.$$

We shall let $\mathcal{B}_1$, $\mathcal{B}_2^0$, etc., denote the closures of $\mathcal{A}_1$, $\mathcal{A}_2^0$, etc., in $\mathcal{B} = (B^k)_\Lambda^{Q\times Q}$. In addition, P_1, P_2^0, etc., are the orthogonal projections of $\mathcal{B}$ onto $\mathcal{B}_1$, $\mathcal{B}_2^0$, etc., respectively. We observe that when $k > 0$ is in $\mathcal{A}$, then we have that k allows the factorizations

$$k = (e + k_+)^* k_d (e + k_+), \tag{5.2}$$

where $k_+, (e + k_+)^{-1} - e \in \mathcal{A}_+^0$ and $k_d \in \mathcal{A}_d$. Indeed, applying a constant permutation matrix that interchanges the third and fourth blocks, the factorization (5.2) is reduced to that of Lemma 2.2.

We are now ready to state the main result in this section.

Theorem 5.1. *Let* $k_c = k_c^* \in \mathcal{A}_2^0 \dot{+} \mathcal{A}_d \dot{+} \mathcal{A}_2^{0*} = \widetilde{\mathcal{A}}_3^0 \dot{+} \widetilde{\mathcal{A}}_d \dot{+} \widetilde{\mathcal{A}}_3^{0*}$ *be given. Then the following are equivalent*:

(i) k_c *has a positive extension.*

(ii) *The operator* $T : \mathcal{B}_2 \to \mathcal{B}_2$ *defined by* $T(g) = P_2(k_c g)$ *is positive definite.*

(iii) *The operator* $\widetilde{T} : \widetilde{\mathcal{B}}_3 \to \widetilde{\mathcal{B}}_3$ *defined by* $\widetilde{T}(g) = P_3(k_c g)$ *is positive definite.*

In that case, let

$$x = T^{-1}(e), \quad y = T^{-1}(e).$$

Then

$$k_0 = x^{*-1} P_d(x) x^{-1} = y^{*-1} \widetilde{P}_d(y) y^{-1}$$

is a positive extension of k_c. *Moreover, if we let* $u = x P_d(x)^{-\frac{1}{2}}$ *and* $v = y \widetilde{P}_d(y)^{-\frac{1}{2}}$, *then* k *is a positive extension of* k_c *if and only if*

$$k = (vg + u)^{*-1} (e - g^* g)(vg + u)^{-1}$$

for some (unique!) $g \in \mathcal{A}_1$ *with* $\|g\|_\infty < 1$. *Lastly, if* k *is a positive extension of* k_c *with factorization* (5.2), *then*

$$k_d \le P_d(x)^{-1},$$

and equality holds if and only if $k = k_0$.

Proof. As before, the main effort goes into showing that $x^{-1} \in \mathcal{A}_+$ and $y^{-1} \in \widetilde{\mathcal{A}}_-$. We proceed analogously to the proof of Proposition 2.6.

Observing that $T > 0$ when applied to $g = (g_{ij})_{i,j=1}^4 \in \mathcal{B}_2$ with $g_{ij} = 0$, $j \neq 1$, yields that $k_{11} > 0$, and one checks that $x_{11} = (e + k_{11,+})^{-1} k_{11,d}^{-1}$, where

$$k_{11} = (e + k_{11,+})^* k_{11,d} (e + k_{11,+})$$

is a right canonical factorization.

Next observe that by applying T to elements $g = (g_{ij})_{i,j=1}^4 \in \mathcal{B}_2$ with $g_{ij} = 0$ for $j \neq 2$, we get that the operator

$$\begin{bmatrix} \widetilde{T}_{11} & H_{12} \\ H_{12}^* & T_{22} \end{bmatrix}, \tag{5.3}$$

on $(\mathcal{B}^k)_{\Lambda\cap(-S)}^{n_1\times n_2} \oplus (\mathcal{B}^k)_{\Lambda\cap S}^{n_2\times n_2}$ is positive definite. Here

$$\widetilde{T}_{11}(g_1) = P_{-S}(k_{11} g_1), \quad g_1 \in (B^k)_{\Lambda\cap(-S)}^{n_1\times n_2},$$

and

$$H_{12}(g_2) = P_{-S}(k_{12}^- g_2), \quad T_{22}(g_2) = P_S(k_{22} g_2), \quad g_2 \in (B^k)_{\Lambda\cap S}^{n_2\times n_2}.$$

Since $\widetilde{T}_{11} > 0$ and $T_{22} > 0$, it follows that their symbols k_{11} and k_{22} are positive definite. Write the canonical factorizations (which exist by Theorem 2.1)

$$k_{11} = (e + k_{1,-})^* k_{1,d} (e + k_{1,-}), \quad k_{22} = (e + k_{2,+})^* k_{2,d} (e + k_{2,+}),$$

where

$$k_{1,-}, (e + k_{1,-})^{-1} - e \in (APW^k)_{\Lambda\cap(-S\setminus\{0\})}^{n_1\times n_1}, \ k_{1,d} \in \mathbb{C}^{n_1\times n_1}, \quad k_{1,d} > 0,$$

and

$$k_{2,+}, (e + k_{2,+})^{-1} - e \in (APW^k)_{\Lambda\cap(S\setminus\{0\})}^{n_2\times n_2}, \ k_{2,d} \in \mathbb{C}^{n_2\times n_2}, \quad k_{2,d} > 0.$$

Introducing the operators

$$H_1 : (B^k)_{\Lambda\cap(-S)}^{n_1\times n_2} \to (B^k)_{\Lambda\cap(-S)}^{n_1\times n_2} \quad \text{and} \quad H_2 : (B^k)_{\Lambda\cap S}^{n_2\times n_2} \to (B^k)_{\Lambda\cap S}^{n_2\times n_2}$$

by the formulas

$$H_1 g = k_{1,d}^{\frac{1}{2}} (e + k_{1,-}) g, \quad H_2 g = k_{2,d}^{\frac{1}{2}} (e + k_{2,+}) g,$$

we get that

$$\widetilde{T}_{11} = H_1^* H_1, \quad T_{22} = H_2^* H_2.$$

Consider the operator

$$H_3 := H_1^{*-1} H_{12} H_2^{-1} \ : \ (B^k)_{\Lambda\cap S}^{n_2\times n_2} \ \to \ (B^k)_{\Lambda\cap(-S)}^{n_1\times n_2}.$$

We have

$$H_3 g = P_{-S}\left((e + k_{1,-})^{*-1} k_{1,d}^{-\frac{1}{2}} k_{12}^- (e + k_{2,+})^{-1} k_{2,d}^{-\frac{1}{2}} g \right), \quad g \in (B^k)_{\Lambda\cap S}^{n_2\times n_2}.$$

Since the operator (5.3) is positive definite, it follows that $\|H_3\| < 1$. Applying Theorem 2.5, we get that

$$\begin{aligned} \beta \ &:= \ H_3 (I - H_3^* H_3)^{-1}(e) \in (APW^k)_{\Lambda\cap(-S)}^{n_1\times n_2}, \\ \delta \ &:= \ (I - H_3^* H_3)^{-1}(e) \in (APW^k)_{\Lambda\cap S}^{n_2\times n_2}, \end{aligned}$$

$\delta^{-1} \in (APW^k)^{n_2\times n_2}_{\Lambda\cap S}$ and $M\{\delta\} > 0$. It is now straightforward to check, analogously to (2.21) and (2.22), that

$$x_{12} = (e+k_{1,-})^{-1}k_{1,d}^{-\frac{1}{2}}\beta k_{2,d}^{-\frac{1}{2}}, \quad x_{22} = (e+k_{2,+})^{-1}k_{2,d}^{-\frac{1}{2}}\delta k_{2,d}^{-\frac{1}{2}},$$

and moreover

$$x_{12} \in (APW^k)^{n_1\times n_2}_{\Lambda\cap(-S)}, \quad x_{22}^{\pm1} \in (APW^k)^{n_2\times n_2}_{\Lambda\cap S}, \quad M\{x_{22}\} > 0.$$

In order to obtain that x_{33} is of the desired form, observe that T applied to $(g_{ij})_{i,j=1}^4 \in \mathcal{B}_2$ with $g_{ij} = 0$, $j \neq 3$, yields that

$$\begin{bmatrix} T_{33} & M_{43}^* \\ M_{43} & M_{44} \end{bmatrix} > 0,$$

where

$$T_{33}g_3 = P_{\Delta_+}(k_{33}^c g_3), \; M_{43}g_3 = k_{43}g_3, \; g_3 \in (B^k)^{n_3\times n_3}_{\Delta_+},$$
$$M_{44}g_4 = k_{44}g_4, \; g_4 \in (B^k)^{n_4\times n_3}.$$

Notice that operator $T_{33} - M_{43}^*M_{44}^{-1}M_{43}$ is positive definite, and

$$\left(T_{33} - M_{43}^*M_{44}^{-1}M_{43}\right)g_3 = P_{\Delta_+}\left((k_{33}^c - k_{34}k_{44}^{-1}k_{43})g\right), \quad g_3 \in (B^k)^{n_3\times n_3}_{\Delta_+}.$$

It follows from (the proof of) Theorem 4.4.1 in [25] that

$$x_{33} = (T_{33} - M_{43}^*M_{44}^{-1}M_{43})^{-1}(I_{n_3}),$$

has the property that $x_{33} \in (APW^k)^{n_3\times n_3}_{\Delta_+}$, $x_{33}^{-1} \in (APW^k)^{n_3\times n_3}_{\Lambda\cap S}$ and $M\{x_{33}\} > 0$. In addition,

$$x_{43} = -k_{44}^{-1}k_{43}x_{33} \in (APW^k)^{n_4\times n_3}_{\Lambda}.$$

For the fourth column of x observe that T applied to $(g_{ij})_{i,j=1}^4 \in \mathcal{B}_2$ with $g_{ij} = 0$, $j \neq 4$, yields that

$$\begin{bmatrix} \widetilde{T}_{22} & H_{24} \\ H_{24}^* & T_{44} \end{bmatrix} > 0,$$

where

$$\widetilde{T}_{22}(g_2) = P_{-S}(k_{22}g_2), \quad g_2 \in (B^k)^{n_2\times n_4}_{\Lambda\cap(-S)},$$
$$H_{24}(g_4) = P_{-S}(k_{24}^-g_4), \quad T_{44}(g_4) = P_S(k_{44}g_4), \quad g_4 \in (B^k)^{n_4\times n_4}_{\Lambda\cap S}.$$

The same reasoning as for the second column of x yields that x_{24} and x_{44} are of the required form.

We are now in a position to complete the proof of Theorem 5.1, by applying Theorems 1.1, 1.3, 1.4, 1.6, 1.10 in [15] to the current setting. The only additional observation that needs to be made is the following equality (in the notation of Theorem 1.6 of [15]):

$$\begin{aligned} &(-c^*vg+cu)(vg+u)^{-1} + (vg+u)^{*-1}(-c^*vg+cu)^* \\ &= (vg+u)^{*-1}[-g^*v^*c^*vg + g^*v^*cu - u^*c^*vg + u^*cu \\ &\quad -g^*v^*cvg + u^*c^*vg - g^*v^*cu + u^*c^*u](vg+u)^{-1} \\ &= (vg+u)^{*-1}(-g^*g+e)(vg+u)^{-1}, \end{aligned}$$

where we used that $c + c^* = u^{*-1}u^{-1} = v^{*-1}v^{-1}$. □

We may state an analog of Proposition 2.4 in the current setting as well.

Proposition 5.2. *Let $k_c = k_c^* \in \mathcal{M}_c$ be given and suppose that k_c has a positive extension. Let k_0 be defined as in Theorem* 5.1. *If k is a positive extension of k_c then*

$$\Delta(k) \leq \Delta(k_0),$$

and equality holds if and only if $k = k_0$.

The proof of this proposition is analogous to the proof of Proposition 2.4, and is left to the reader.

References

[1] N. K. Bose, J. P. Guiver, E. W. Kamen, H. M. Valenzuela, and B. Buchberger, *Multidimensional Systems Theory. Progress, Directions and Open Problems in Multidimensional Systems*, Mathematics and its Applications, 16. D. Reidel Publishing Co., Dordrecht, 1985, MR 87m:93017.

[2] T. Constantinescu, *Schur Parameters, Factorization and Dilation Problems*, OT82, Birkhäuser, Basel, 1996, MR 97g:47012, Zbl 872.47008.

[3] C. Corduneanu, *Almost Periodic Functions*, J. Wiley & Sons, 1968, MR 58 #2006, Zbl 175.09101.

[4] J. C. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Theory*, Macmillan Publishing Company, New York, 1992, MR 93k:93002.

[5] J. Erdös, *On the structure of ordered real vector spaces*, Publ. Math. Debrecen **4** (1956), 334–343, MR 18,137b, Zbl 070.33601.

[6] C. Foiaş and A. E. Frazho, *The Commutant Lifting Approach to Interpolation Problems*, OT44, Birkhäuser, Basel, 1990, MR 92k:47033, Zbl 718.47010.

[7] C. Foiaş, H. Özbay, and A. Tannenbaum, *Robust Control of Infinite-dimensional Systems. Frequency Domain Methods*, Lecture Notes in Control and Information Sciences, 209, Springer Verlag, London, 1996, MR 96m:93001, Zbl 839.93003.

[8] B. A. Francis, *A Course in H^∞ Control Theory*, Lecture Notes Control and Information Science, Springer, New York, 1987, MR 89i:93002, Zbl 624.93003.

[9] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963, MR 30 #2090, Zbl 137.02001.

[10] I. Gohberg, S. Goldberg, and M. A. Kaashoek, *Classes of Linear Operators,* II, OT63, Birkhäuser, Basel, 1993, MR 95a:47001, Zbl 789.47001.

[11] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables*, Acta Math. **99** (1958), 165–202, MR 20 #4155, Zbl 082.28201.

[12] H. Helson and D. Lowdenslager, *Prediction theory and Fourier series in several variables.* II, Acta Math. **106** (1961), 175–213, MR 31 #562, Zbl 102.34802.

[13] V. Ionescu, C. Oară, and M. Weiss, *Generalized Riccati Theory and Robust Control. A Popov Function Approach*, John Wiley and Sons, Chichester, 1999, MR 2000f:93003, Zbl 915.34024.

[14] M. A. Kaashoek and H. J. Woerdeman, *Positive extensions and diagonally connected patterns*, Operator Theory: Advances and Applications **124** (2001), 287–305, CMP 1 839 841.

[15] M. A. Kaashoek and C. G. Zeinstra, *The band method and generalized Carathéodory-Toeplitz interpolation at operator points*, Integral Equations and Operator Theory **33** (1999), 175–210, MR 99k:47035, Zbl 929.47006.

[16] D. S. Kalyuzhniy, *Multiparametric dissipative linear stationary dynamical scattering systems: Discrete case*, J. Operator Theory **43** (2000), no. 2, 427–460, MR 2001d:47019a.

[17] D. S. Kalyuzhniy, *Multiparametric dissipative linear stationary dynamical scattering systems: Discrete case,* II: *Existence of conservative dilations*, Integral Equations Operator Theory **36** (2000), no.1, 107–120, MR 2001d:47019b, Zbl 948.47013.

[18] E. W. Kamen, P. P. Khargonekar, and A. Tannenbaum, *Pointwise stability and feedback control of linear systems with noncommensurate time delays*, Acta Appl. Math. **2** (1984), no. 2, 159–184, MR 85m:93034, Zbl 561.93045.

[19] B. M. Levitan, *Almost Periodic Functions*, GITTL, Moscow, 1953. (Russian), MR 15,700a.

[20] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, 1982, MR 84g:34004, Zbl 499.43005.

[21] A. V. Oppenheim and A. S. Willsky (with S. H. Nawab), *Signals and Systems* (2nd ed.), Prentice Hall, Upper Saddle River, New Jersey, 1997.

[22] A. A. Pankov, *Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations*, Kluwer, Dordrecht/Boston/London, 1990, MR 92f:35002, Zbl 712.34001.

[23] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Carathéodory-Toeplitz and Nehari problems for matrix valued almost periodic functions*, Trans. Amer. Math. Soc. **350** (1998), 2185–2227, MR 98h:47023, Zbl 896.42005.

[24] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Contractive extension problems for matrix valued almost periodic functions of several variables*, J. Operator Theory, to appear.

[25] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Abstract band method via factorization, positive and band extensions of multivariable almost periodic matrix functions, and spectral estimation*, Memoirs of the AMS, to appear.

[26] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Factorization of almost periodic matrix functions of several variables and Toeplitz operators*, Operator Theory: Advances and Applications, **122** (2001), 385–416.

[27] E. Rogers and P. Rocha (editors), *Recent Progress in Multidimensional Control Theory and Applications*, Multidimensional Systems and Signal Processing **11** (2000), 1–182, MR 2001c:93009.

[28] I. M. Spitkovsky and H. J. Woerdeman, *The Carathéodory-Toeplitz Problem for Almost Periodic Functions*, J. Funct. Anal. **115** (1993), 281–293, MR 94f:47020, Zbl 796.42009.

[29] H. J. Woerdeman, *Matrix and Operator Extensions*, CWI Tract 68. Centre for Mathematics and Computer Science, Amsterdam, The Netherlands, 1989, MR 91d:47001.

Department of Mathematics, P. O. Box 8795, The College of William and Mary, Williamsburg VA 23187-8795
lxrodm@math.wm.edu http://www.math.wm.edu/~lxrodm/

Department of Mathematics, P. O. Box 8795, The College of William and Mary, Williamsburg VA 23187-8795
ilya@math.wm.edu http://www.math.wm.edu/~ilya/

Department of Mathematics, P. O. Box 8795, The College of William and Mary, Williamsburg VA 23187-8795
hugo@math.wm.edu http://www.math.wm.edu/~hugo/

This paper is available via http://nyjm.albany.edu:8000/j/2001/7-9.html.