New York Journal of Mathematics

New York J. Math. **6** (2000) 55–71.

The Homology of Peiffer Products of Groups

W. A. Bogley and N. D. Gilbert

ABSTRACT. The Peiffer product of groups first arose in work of J.H.C. Whitehead on the structure of relative homotopy groups, and is closely related to problems of asphericity for two-complexes. We develop algebraic methods for computing the second integral homology of a Peiffer product. We show that a Peiffer product of superperfect groups is superperfect, and determine when a Peiffer product of cyclic groups has trivial second homology. We also introduce a double wreath product as a Peiffer product.

CONTENTS

Introduction

Given two groups acting on each other by automorphisms, it is natural to ask whether these groups can be embedded in an overgroup in such a way that the given actions are realized by conjugation. If the actions are trivial, this can be done simply by forming the direct product of the two groups. In general, the question has a negative answer.

One is led to the following construction. Let G and H be groups and suppose we are given fixed actions $(g, h) \mapsto g^h$ and $(h, g) \mapsto h^g$ of each group on the other.

Received October 1, 1999.

Mathematics Subject Classification. 20J05, 20E22, 20F05.

Key words and phrases. homology, Peiffer product, asphericity, two-complex, double wreath product.

W. A. Bogley would like to thank the Department of Mathematics at Heriot-Watt University, Edinburgh, for its gracious hospitality while this work was in progress. Bogley was supported by a U. K. Engineering and Physical Sciences Research Council Visiting Fellowship (GR/L49932).

ISSN 1076-9803/00

(These are assumed to be right actions, so that $(g^h)^{h'} = g^{hh'}$, for example.) Let Π denote the normal closure in the free product $G * H$ of all elements

$$g^{-1}h^{-1}gh^g\,,h^{-1}g^{-1}hg^h\ : g \in G, h \in H.$$

The quotient $G \bowtie H := (G * H)/\Pi$ is the *Peiffer product* of G and H with the given actions. For example, when G and H act trivially on each other, the Peiffer product is just the direct product $G \bowtie H \cong G \times H$, and Π is the Cartesian subgroup $\Pi = G \,\square\, H = \ker(G * H \to G \times H)$. When G and H are infinite cyclic with nontrivial actions, it is shown in [8] that the Peiffer product is the quaternion group of order eight: $\mathbb{Z} \bowtie \mathbb{Z} \cong Q_8$. The question that was alluded to in the opening paragraph can now be posed as the following

Embedding Question. When are the natural maps $G \to G \bowtie H \leftarrow H$ injective?

The Peiffer product of groups (so named in [8]) first arose in a topological setting in the work of J.H.C. Whitehead [14] on the structure of relative homotopy groups. Suppose that a connected two-complex Z is a union of connected subcomplexes X and Y that intersect in the common one-skeleton $X \cap Y = Z^1$. One consequence of Whithehead's work in [14] is that the relative homotopy group $\pi_2(Z, Z^1)$ can be decomposed as the Peiffer product of the relative groups $\pi_2(X, Z^1)$ and $\pi_2(Y, Z^1)$:

$$\pi_2(Z, Z^1) \cong \pi_2(X, Z^1) \bowtie \pi_2(Y, Z^1). \tag{1}$$

(The actions arise via the homotopy action of $\pi_1(Z^1)$.) As an application, Whitehead proposed his notorious

Asphericity Question. Are subcomplexes of aspherical two-complexes themselves aspherical?

The point here is that by comparing the homotopy sequences of the pairs (X, Z^1) and (Z, Z^1), one sees that if Z is aspherical (i.e., $\pi_2(Z) = 0$), then X is aspherical if and only if $\pi_2(X, Z^1)$ embeds in $\pi_2(Z, Z^1)$. The longstanding unresolved status of Whitehead's Asphericity Question therefore stands as testimony to the subtlety of the Embedding Question for Peiffer products.

The Peiffer product has been applied to algebraic problems (see for example its implicit role in [10]) and to the topological setting in which it first appeared, the calculation of low dimensional homotopy and homology groups [3, 5, 7, 9]. The following theorem of M. N. Dyer [6] connects the vanishing of the second homotopy group of a two-complex to the vanishing of the second homology group of its second relative homotopy group.

Theorem. [6] *Let X be a connected two-complex with one-skeleton X^1. If X does not have the homotopy type of the two-sphere, then X is aspherical if and only if $H_2(\pi_2(X, X^1)) = 0$.*

In this paper we consider the purely algebraic problem of determining the second integral homology $H_2(G \bowtie H)$ of a Peiffer product $G \bowtie H$ in terms of information about the factors G and H. As in [5], we exploit the description of a Peiffer product via semidirect products. By way of general results, we show that any Peiffer product of superperfect groups is superperfect (Corollary 2.3) and we give a very short proof of the Künneth formula for the second homology of direct products (§4.1).

Our main results support a systematic approach to the problem of H_2 calculations for Peiffer products. We draw the reader's attention to Theorem 3.2 and

Corollary 3.3, which are definitive technical results within this context. We illustrate the effectivenes of our approach by showing how to determine whether a Peiffer product of cyclic groups has trivial second homology (Theorem 4.4). This in turn is related to the Embedding Question for these Peiffer products. (See Corollary 4.5 and the succeeding examples.) We also introduce (§4.4) a double wreath product construction as a Peiffer product and we investigate its second homology.

Notation. When a group H acts on a group G on the right, $[G;H]$ will denote the subgroup of G generated by all elements $g^{-1}g^h$, $g \in G, h \in H$. Then $[G;H]$ is normal in G, and we denote the quotient $G/[G;H]$ by G_H. We shall use this subscript notation for any quotient group obtained by killing an action. For example, if H normalizes G in a common overgroup, then H acts on G by conjugation: $g^h = h^{-1}gh$ and $[G;H] = [G,H]$ is the subgroup generated by the commutators $[g,h] = g^{-1}h^{-1}gh = g^{-1}g^h$. Thus $G_H = G/[G,H]$ and if $G = H$, then $G_G = G/[G,G]$ is the abelianized group G^{ab}.

Now suppose that G and H are groups acting on each other on the right. We form the following normal subgroups in the free product $G * H$:

$$
\begin{aligned}
S &= \langle\langle g^{-1}h^{-1}gh^g : g \in G, h \in H\rangle\rangle \\
T &= \langle\langle h^{-1}g^{-1}hg^h : g \in G, h \in H\rangle\rangle
\end{aligned}
\tag{2}
$$

and we set $\Pi = ST$. Note that $(G*H)/S$ and $(G*H)/T$ are the semidirect products $G \ltimes H$ and $G \rtimes H$ respectively. The quotient group $(G*H)/\Pi = G \bowtie H$ is the Peiffer product of G and H with the given actions. The images of the natural maps $G \to G \bowtie H \leftarrow H$ will be denoted by $\bar{G}$ and $\bar{H}$ respectively.

1. The low-dimensional homology of products of subgroups

R. Brown's five-term exact sequence [5] for the homology of a group P, equal to the product of two normal subgroups M and N, is

$$
\begin{aligned}
H_2(P) \to H_2(P/M) \oplus H_2(P/N) &\to (M \cap N)/[M,N] \\
&\to H_1(P) \to H_1(P/M) \oplus H_1(P/N) \to 0.
\end{aligned}
\tag{3}
$$

We shall be interested in the setting in which $M \cap N = [M,N]$, in which case the group P decomposes as a Peiffer product of M and N.

Proposition 1.1. *In a Peiffer product $G \bowtie H$, the subgroups $\bar{G}$ and $\bar{H}$ are normal subgroups satisfying $G \bowtie H = \bar{G}\bar{H}$ and $\bar{G} \cap \bar{H} = [\bar{G},\bar{H}]$. Conversely, if P is a group with normal subgroups M,N satisfying $P = MN$ and $M \cap N = [M,N]$, then $P \cong M \bowtie N$.*

Proof. It is immediate from the defining relations of the Peiffer product that $\bar{G}$ and $\bar{H}$ are normal subgroups of $G \bowtie H$ and that $\bar{G}\bar{H} = G \bowtie H$. Next,

$$(G \bowtie H)/\bar{G} \cong H_G \quad \text{and} \quad (G \bowtie H)/\bar{H} \cong G_H.$$

As in [3, 7], we note that the quotient $(G \bowtie H)/[\bar{G},\bar{H}]$ is isomorphic to $G_H \times H_G$ and elements of $\bar{G} \cap \bar{H}$ lie in the kernel of the quotient map $G \bowtie H \to G_H \times H_G$. Hence $\bar{G} \cap \bar{H} \subseteq [\bar{G},\bar{H}]$.

Conversely, suppose that M,N are normal subgroups of a group P with $P = MN$. Following Brown [5], we form the Peiffer product $M \bowtie N$ using the conjugation actions in P. By identifying $M \bowtie N$ as a quotient of the semidirect product

$M \rtimes N$, Brown obtains a short exact sequence

$$1 \to (M \cap N)/[M,N] \to M \bowtie N \to P \to 1.$$

Hence if $M \cap N = [M,N]$ then $M \bowtie N \cong P$. □

With Proposition 1.1, the Brown homology sequence (3) for the group $G \bowtie H$ and its normal subgroups $\bar{G}$ and $\bar{H}$ shows that the maps $G_H \leftarrow G \bowtie H \to H_G$ induce homomorphisms

$$H_1(G \bowtie H) \xrightarrow{\alpha_1} H_1(G_H) \oplus H_1(H_G)$$
$$H_2(G \bowtie H) \xrightarrow{\alpha_2} H_2(G_H) \oplus H_2(H_G)$$

where α_1 is an isomorphism and α_2 is surjective. For later convenience, we note the following immediate corollary.

Corollary 1.2. *Let G and H be perfect groups (i.e., $H_1(G) = H_1(H) = 0$). Then for any actions of G and H on each other, the Peiffer product $G \bowtie H$ is perfect.*

Our next aim is to investigate the kernel of α_2. The maps $G \leftarrow G * H \to H$ induce a homomorphism

$$\Pi_{G*H} \xrightarrow{\beta} [G;H]_G \oplus [H;G]_H$$

that is obviously surjective.

Proposition 1.3. *There is a surjective homomorphism* $\ker \alpha_2 \to \ker \beta$. *If the maps $H_2(G) \to H_2(G_H)$ and $H_2(H) \to H_2(H_G)$ are each injective, then* $\ker \alpha_2 \cong \ker \beta$.

Proof. The standard five-term exact sequences for the extensions

$$1 \to \Pi \to G * H \xrightarrow{\pi} G \bowtie H \to 1$$
$$1 \to [G;H] \to G \to G_H \to 1$$
$$1 \to [H;G] \to H \to H_G \to 1$$

combine into a commutative diagram involving the maps α_1, α_2 and β as shown in Figure 1.1. A diagram chase gives a surjection $\ker \alpha_2 \to \ker \beta$. If π denotes the quotient map $G * H \to G \bowtie H$ then we obtain an extension of abelian groups

$$0 \to \ker \alpha_2 \cap \operatorname{im} H_2(\pi) \to \ker \alpha_2 \to \ker \beta \to 0.$$

Now each of $H_2(G) \to H_2(G_H)$ and $H_2(H) \to H_2(H_G)$ factors through $H_2(\pi)$ and it follows that if $H_2(G) \to H_2(G_H)$ and $H_2(H) \to H_2(H_G)$ are each injective then $\ker \alpha_2 \to \ker \beta$ is an isomorphism. □

The converse of Proposition 1.3 is false, as the following example shows.

Example. Let $A = \langle a, b \mid [a,b] \rangle$ be free abelian of rank 2 and let $V = \{1, x, y, xy\}$ be a Klein 4–group. We let x act on A by inversion and let y act trivially. Define an action of A on V by $x^a = y$, $x^b = xy$, $y^a = xy$, $y^b = x$. In $A \bowtie V$ we have $a^{-1}ya = xy$ and $y^{-1}ay = a$: hence $y = xy$ and $x = 1$. Then $a^{-1}xa = y$ implies $y = 1$, so that $\bar{V} = 1$ and

$$A \bowtie V = (A \bowtie V)/\bar{V} = A_V \cong V,$$

$$
\begin{tikzcd}
H_2(G) \oplus H_2(H) \arrow[r, equal] \arrow[d, "H_2(\pi)"] & H_2(G) \oplus H_2(H) \arrow[d] \\
H_2(G \bowtie H) \arrow[r, "\alpha_2"] \arrow[d] & H_2(G_H) \oplus H_2(H_G) \arrow[d] \\
\Pi_{G*H} \arrow[r, "\beta"] \arrow[d] & [G;H]_G \oplus [H;G]_H \arrow[d] \\
H_1(G) \oplus H_1(H) \arrow[r, equal] \arrow[d] & H_1(G) \oplus H_1(H) \arrow[d] \\
H_1(G \bowtie H) \arrow[r, "\alpha_1"] & H_1(G_H) \oplus H_1(H_G)
\end{tikzcd}
$$

FIGURE 1.1. Five term sequences

and $H_2(A \bowtie V) = \mathbb{Z}_2$, $H_2(A_V) = \mathbb{Z}_2$ and $H_2(V_A) = 0$. We see that α_i is an isomorphism ($i \geq 1$), and so β is an isomorphism: $\Pi/[\Pi, A * V] \cong \mathbb{Z}^2 \oplus V$. However $\mathbb{Z} = H_2(A) \to H_2(A_V) = \mathbb{Z}_2$ is not injective.

The structure of $H_2(G \bowtie H)$ is summarised in the Hasse diagram in Figure 1.2.

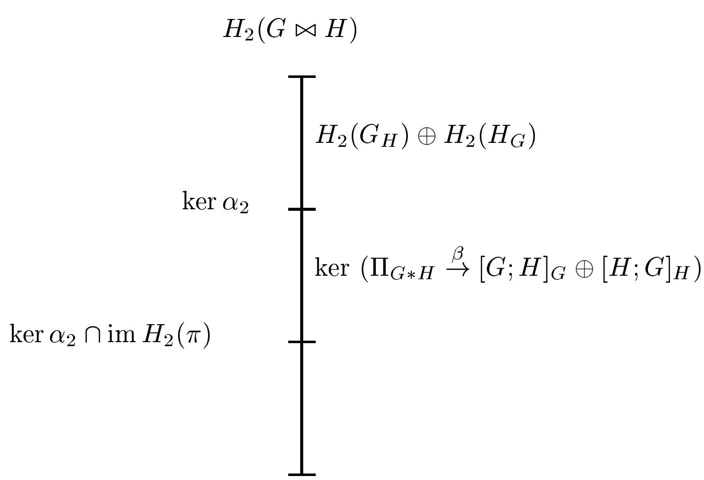

FIGURE 1.2. Structure of $H_2(G \bowtie H)$

2. Twisted bilinear relations

Recall that $\Pi = ST$ where S denotes the normal closure in $G * H$ of all elements $g^{-1}h^{-1}gh^g$, $g \in G$, $h \in H$ and T is the normal closure of all elements $h^{-1}g^{-1}hg^h$, $g \in G$, $h \in H$. The inclusions of S and T in Π induce a surjective homomorphism

$$S_{G*H} \oplus T_{G*H} \to \Pi_{G*H}.$$

In this and the following section, we completely describe the structure of S_{G*H}, with analogous remarks holding for T_{G*H}. The image in S_{G*H} of the normal generator $g^{-1}h^{-1}gh^g$ for S will be denoted $\langle g, h\rangle$. Thus,

$$\langle g, h\rangle = g^{-1}h^{-1}gh^g[S, G * H] \in S_{G*H} = S/[S, G * H].$$

Lemma 2.1. *The abelian group S_{G*H} is generated by the elements $\langle g, h\rangle$ $(g \in G, h \in H)$ and the following relations hold:*

(i) $\langle gg', h\rangle = \langle g', h^g\rangle\langle g, h\rangle$,
(ii) $\langle g, hh'\rangle = \langle g, h\rangle\langle g, h'\rangle$,

These relations imply that $\langle g, 1\rangle = \langle 1, h\rangle = 1$ in S_{G*H} and that $\langle g, h\rangle^{-1} = \langle g, h^{-1}\rangle = \langle g^{-1}, h^g\rangle$. Furthermore, if $\mathbf{x}$ (resp. $\mathbf{y}$) is a generating set for G (resp. H), then S_{G*H} is generated by the elements $\langle x, y\rangle$, $x \in \mathbf{x}$, $y \in \mathbf{y}$.

Proof. The elements $\langle g, h\rangle$ generate S_{G*H} since S is the normal closure in $G * H$ of all the elements $g^{-1}h^{-1}gh^g$. The validity of the relations (i) and (ii) may be checked directly: for example, working modulo $[S, G * H]$,

$$\begin{aligned}
\langle gg', h\rangle &= g'^{-1}g^{-1}h^{-1}gg'h^{gg'} \\
&= g'^{-1}h^{-g}g'h^{gg'}h^{-gg'}g'^{-1}h^g g^{-1}h^{-1}gg'h^{gg'} \\
&= \langle g', h^g\rangle h^{-gg'}g'^{-1}h^g g^{-1}h^{-1}gh^g h^{-g}g'h^{gg'} \\
&= \langle g', h^g\rangle\langle g, h\rangle.
\end{aligned}$$

The remaining assertions of the lemma follow directly from the relations (i) and (ii). □

Denote the augmentation ideal in $\mathbb{Z}G$ by $\mathfrak{g}$ and write $H_1(H) = H^{ab} = H/[H, H]$. Here is a restatement of Lemma 2.1. The elementary proof is left to the reader.

Corollary 2.2. *There is a surjective homomorphism $\sigma : H^{ab} \otimes_G \mathfrak{g} \to S_{G*H}$ given by $\sigma(h[H, H] \otimes (g - 1)) = \langle g, h\rangle$.*

Corollary 2.3. *If G and H are perfect (i.e., $G^{ab} = H^{ab} = 0$), then the natural map $H_2(G) \oplus H_2(H) \to H_2(G \bowtie H)$ is surjective. If G and H are both superperfect (i.e., $H_1 = H_2 = 0$), then so is $G \bowtie H$.*

Proof. When H is perfect we have $S_{G*H} = 0$ by Corollary 2.2. When both G and H are perfect, this implies that $\Pi_{G*H} = 0$. The first statement follows from the five term homology sequence for $G \bowtie H = (G * H)/\Pi$. (See Figure 1.1.) The second statement follows immediately. □

3. The structure of S_{G*H}

In this section we show that the surjection $\sigma : H^{ab} \otimes_G \mathfrak{g} \to S_{G*H}$ is actually an isomorphism. Of course, the discussion also applies to show that $T_{G*H} \cong G^{ab} \otimes_H \mathfrak{h}$, where $\mathfrak{h}$ denotes the augmentation ideal in $\mathbb{Z}H$. Following Brown [5], we rely on the fact that the Peiffer product $G \bowtie H$ is a quotient of the semi-direct product $G \ltimes H = (G*H)/S$. We begin by building a standard two-complex with fundamental group $G \ltimes H$.

We have an action of G by automorphisms on H. Let K (resp. L) be the two-complex modeled on a presentation $\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$ (resp. $\mathcal{Q} = \langle \mathbf{y} : \mathbf{s} \rangle$) for G (resp. H). For each $(x, y) \in \mathbf{x} \times \mathbf{y}$, we can choose a reduced word $v_{x,y}$ in the free group on $\mathbf{y}$ so that the relation $y^x = v_{x,y}$ holds in $G \ltimes H$. If we set $t_{x,y} = x^{-1}y^{-1}xv_{x,y}$ and $\mathbf{t} = \{t_{x,y} : (x, y) \in \mathbf{x} \times \mathbf{y}\}$, then

$$\mathcal{R} = \langle \mathbf{x}, \mathbf{y} : \mathbf{r}, \mathbf{s}, \mathbf{t} \rangle$$

is a presentation for the split extension $G \ltimes H$. Let M denote the two-complex modeled on $\mathcal{R}$ and let $p : \widetilde{M} \to M$ be the universal covering projection. The complex M contains the one-point union $K \vee L$ as a subcomplex and the pre-image $p^{-1}(K \vee L) = \overline{K \vee L}$ has fundamental group $S = \ker(G * H \to G \ltimes H)$. The homology sequence for the pair $(\widetilde{M}, \overline{K \vee L})$ determines an exact sequence of $\mathbb{Z}(G \ltimes H)$-modules

$$\pi_2(M) \to H_2(\widetilde{M}, \overline{K \vee L}) \to S^{ab} \to 0 \tag{4}$$

in which the second term is the free $\mathbb{Z}(G \ltimes H)$-module with basis elements $\tilde{e}_{x,y}$, $(x, y) \in \mathbf{x} \times \mathbf{y}$, corresponding to the two-cells of $M - (K \vee L)$. The basis element $\tilde{e}_{x,y} \in H_2(\widetilde{M}, \overline{K \vee L})$ is mapped to the coset $t_{x,y}[S, S] \in S^{ab}$, where $t_{x,y}$ is viewed in $G * H$.

Lemma 3.1. *Given two-complexes K and L modeled on presentations $(\mathbf{x} : \mathbf{r})$ for G and $(\mathbf{y} : \mathbf{s})$ for H and with M constructed as above, there is an exact sequence*

$$\pi_2(M) \to H_2(M, K \vee L) \to S_{G*H} \to 0 \tag{5}$$

of abelian groups. Here, the second term is the free abelian group with basis consisting of the two-cells $e_{x,y}$, $(x, y) \in \mathbf{x} \times \mathbf{y}$, of $M - (K \vee L)$ and the basis element $e_{x,y}$ is mapped to $\langle x, y \rangle \in S_{G*H}$. The first map in the sequence factors as $\pi_2(M) \xrightarrow{h} H_2(M) \to H_2(M, K \vee L)$ where h is the Hurewicz homomorphism.

Proof. The result follows upon killing the action of $G \ltimes H$ (i.e., of $G * H$) in the sequence (4). □

Module generators for $\pi_2(M)$ have been described by Y. G. Baik and S. J. Pride [1] (see [2]). In practice it is a simple matter to determine the images of these generators under the map $\pi_2(M) \to H_2(M, L \vee K)$ and thus to work out a presentation for S_{G*H} in terms of the generators $\langle x, y \rangle$, $x \in \mathbf{x}$, $y \in \mathbf{y}$. We briefly describe the Baik-Pride π_2 generators for the reader's convenience.

Recall that K (resp. L) is modeled on a presentation $\mathcal{P} = (\mathbf{x} : \mathbf{r})$ (resp. $\mathcal{Q} = (\mathbf{y} : \mathbf{s})$) for G (resp. H) and that M is is modeled on a presentation $\mathcal{R} = (\mathbf{x}, \mathbf{y} : \mathbf{r}, \mathbf{s}, \mathbf{t})$ where $\mathbf{t}$ consists of relations of the form $t_{x,y} = x^{-1}y^{-1}xv_{x,y}$, $x \in \mathbf{x}$, $y \in \mathbf{y}$, that realize the action of G on H. Baik and Pride describe generators for $\pi_2(M)$ in terms of spherical pictures. (See [4] for a general treatment of spherical pictures.)

In addition to the collection of all spherical pictures over $\mathcal{P}$ and $\mathcal{Q}$, two additional families of spherical pictures are sufficient to generate $\pi_2(M)$ as a $\mathbb{Z}(G \ltimes H)$-module.

Given $(x, s) \in \mathbf{x} \times \mathbf{s}$, construct a disc picture $A_{x,s}$ over the presentation $(\mathbf{x}, \mathbf{y} : \mathbf{s}, \mathbf{t})$ consisting of a single positively oriented s-disc surrounded by a series of $t_{x,y}$-discs according to the occurences of the letters $y^{\pm 1}$ in the relator s. The boundary of each $t_{x,y}$-disc has two oppositely oriented occurences of arcs labeled by x; these x-arcs are joined so as to form an annulus of $t_{x,y}$-discs surrounding the original s-disc. Since the generator x, when viewed as an element of G determines an automorphism of H, the boundary of $A_{x,s}$ supports a word in $\mathbf{y} \cup \mathbf{y}^{-1}$ that determines the identity element of H. Thus we can choose a disc picture $B_{x,s}$ over $\mathcal{Q} = (\mathbf{y} : \mathbf{s})$ whose boundary word is the same as that of $A_{x,s}$. Gluing these two pictures together along their common boundary, we obtain a spherical picture $\mathbb{P}_{x,s}$ over $\mathcal{R}$. Constructing one such picture for each $(x, s) \in \mathbf{x} \times \mathbf{s}$, we refer to the resulting family as the pictures of Type I.

Given $(r, y) \in \mathbf{r} \times \mathbf{y}$, choose a disc picture $C_{r,y}$ over $(\mathbf{x}, \mathbf{y} : \mathbf{s}, \mathbf{t})$ with boundary word $r^{-1}y^{-1}ry$. This is possible because G acts on H and so r acts trivially on y. Now attach two oppositely oriented r-discs to match the occurences of r and r^{-1} in the boundary of $C_{r,y}$ and then close up the remaining oppositely oriented y-arcs to obtain a spherical picture $\mathbb{Q}_{r,y}$ over $\mathcal{R}$. Constructing one such picture for each $(r, y) \in \mathbf{r} \times \mathbf{y}$, we refer to the resulting family as the pictures of Type II.

Each element of $\pi_2(M)$ can be represented by a spherical picture over $\mathcal{R}$. Baik and Pride showed that $\pi_2(M)$ is generated as a $\mathbb{Z}(G \ltimes H)$-module by the homotopy elements represented by spherical pictures over $\mathcal{P}$ and $\mathcal{Q}$, together with the selected pictures of Type I and Type II. The image of a homotopy element under the map $\pi_2(M) \to H_2(M, K \vee L)$ is computed by simply counting with multiplicity the occurences of $t_{x,y}$-discs in a representative picture. In particular, all pictures over $\mathcal{P}$ and $\mathcal{Q}$ have trivial image. This process and the construction of Type I and Type II pictures will be illustrated in the proof of Theorem 3.2 below.

Theorem 3.2. *The map* $\sigma : H^{ab} \otimes_G \mathfrak{g} \to S_{G*H}$ *is an isomorphism.* (*And similarly, there is an isomorphism* $G^{ab} \otimes_H \mathfrak{h} \cong T_{G*H}$.)

Proof. Use the multiplication tables to construct presentations $\mathcal{P}$ for G and $\mathcal{Q}$ for H. Thus the presentation $\mathcal{P} = (\mathbf{x} : \mathbf{r})$ has generators $\mathbf{x} = \{(g) : g \in G\}$ and defining relations $\mathbf{r} = \{(g)(g')(gg')^{-1} : g, g' \in G\}$. The presentation $\mathcal{Q} = (\mathbf{y} : \mathbf{s})$ is constructed in the same way. We can then build K, L, and M as described above. The free abelian homology group $H_2(M, K \vee L)$ has basis consisting of the two-cells $e_{(g),(h)}$, $g \in G$, $h \in H$ with boundary word reading $t_{(g),(h)} = (g)^{-1}(h)^{-1}(g)(h^g)$. So $e_{(g),(h)} \in H_2(M, K \vee L)$ maps to $\langle g, h \rangle \in S_{G*H}$. In order to determine the image of $\pi_2(M) \to H_2(M, K \vee L)$, let $g, g' \in G$ and $h, h' \in H$. It suffices to examine the occurences of $\mathbf{t}$-discs in the pictures $A_{(g),s}$ and $C_{r,(h)}$ where $r = (g)(g')(gg')^{-1} \in \mathbf{r}$ and $s = (h)(h')(hh')^{-1} \in \mathbf{s}$. These pictures are displayed in Figure 3.1.

Examining the black discs in Figure 3.1, we find that the Type I picture $\mathbb{P}_{(g),s}$ has image $e_{(g),(h)} + e_{(g),(h')} - e_{(g),(hh')} \in H_2(M, K \vee L)$ and that the Type II picture $\mathbb{Q}_{r,(h)}$ has image $e_{(g),(h)} + e_{(g'),(h^g)} - e_{(gg'),(h)} \in H_2(M, K \vee L)$. Passing to S_{G*H}, this means that the relations (i) and (ii) from Lemma 2.1 are actually defining relations for the generators $\langle g, h \rangle$ of the abelian group S_{G*H}. From this it is straightforward to show that the assignment $\langle g, h \rangle \mapsto h[H, H] \otimes (g - 1)$ defines an inverse to the map σ. $\square$

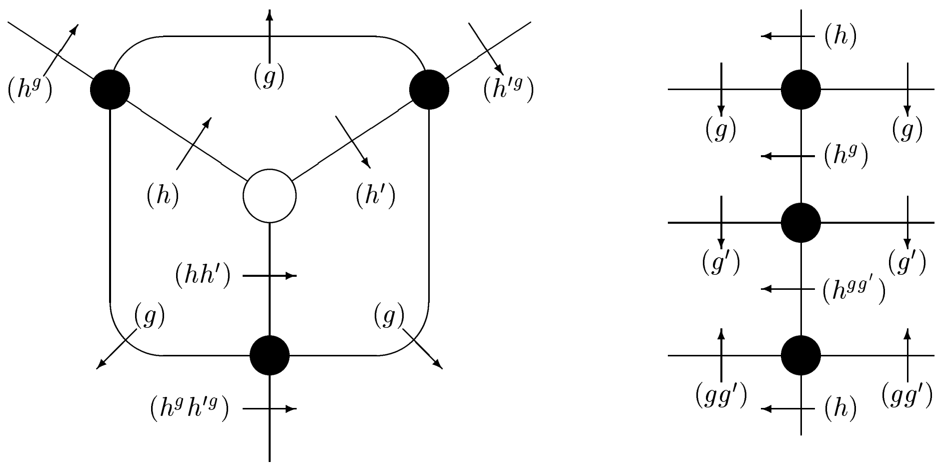

FIGURE 3.1. $A_{(g),s}$ and $C_{r,(h)}$

Corollary 3.3. *There is an isomorphism $S_{G*H} \cong G^{ab} \otimes H^{ab}$ if either*
(a) *G is a free group or*
(b) *G acts trivially on H^{ab}.*

Proof. When G is a free group, the augmentation ideal $\mathfrak{g}$ (resp. abelianization G^{ab}) is the free $\mathbb{Z}G$-module (resp. free abelian group) with basis in one-to-one correspondence with a basis for G. The result (a) follows easily. The result (b) follows from the fact that $G^{ab} \cong \mathfrak{g}/\mathfrak{g}^2$. □

The setting of Corollary 3.3(a) is of interest in low dimensional homotopy theory. When a two-complex Z is a union of aspherical subcomplexes X and Y with intersection $X \cap Y = Z^1$, Whitehead's result (1) shows that the relative homotopy group $\pi_2(Z, Z^1)$ is the Peiffer product $G \bowtie H$ where $G = \pi_2(X, Z^1)$ and $H = \pi_2(Y, Z^1)$. Since X and Y are aspherical, the groups G and H are free.

4. Computations

4.1. **Trivial actions: The Künneth formula.** When G and H act trivially on each other, the Peiffer product is simply the direct product: $G \bowtie H \cong G \times H$. Inspection of the normal generators for S and T reveals that $S = T = \Pi$ is the Cartesian subgroup $\Pi = G \,\square\, H = \ker(G * H \to G \times H)$. Corollary 3.3(b) implies that $\Pi_{G*H} \cong G^{ab} \otimes H^{ab}$. This result was first proved by MacHenry [11]. When we examine the five-term homology sequence for $G \times H = (G * H)/\Pi$ (see Figure 1.1), it is clear that the map $H_1(G) \oplus H_1(H) \to H_1(G \times H)$ is an isomorphism and so the sequence

$$H_2(G) \oplus H_2(H) \to H_2(G \times H) \to G^{ab} \otimes H^{ab} \to 0$$

is exact. Finally, the map $H_2(G) \oplus H_2(H) \to H_2(G \times H)$ is readily seen to be split injective, so we recover the Künneth formula for the second homology of direct products:

$$H_2(G \times H) \cong H_2(G) \oplus H_2(H) \oplus (G^{ab} \otimes H^{ab}).$$

4.2. **Conjugation action.** Let G act on itself by conjugation. As in [3, 8] we have $G \bowtie G \cong G^{ab} \times G$ with the two canonical maps $G \to G \bowtie G$ taking $g \mapsto (1, g)$ and $g \mapsto (\bar{g}, g)$. There is an isomorphism

$$H_2(G \bowtie G) \cong H_2(G) \oplus H_2(G^{ab}) \oplus (G^{ab} \otimes G^{ab}).$$

Corollary 3.3(b) shows that the maps $S_{G*H} \to [H; G]_H$ and $T_{G*H} \to [G; H]_G$ can each be identified with the commutator pairing $G^{ab} \otimes G^{ab} \to [G, G]/[G, [G, G]] = \gamma_2(G)/\gamma_3(G)$. It follows from the commutative diagram in Figure 1.1 that

$$\Pi_{G*G} \cong (G^{ab} \otimes G^{ab}) \oplus \gamma_2(G)/\gamma_3(G)$$

and that $\beta : \Pi_{G*G} \to [G; G]_G \oplus [G; G]_G$ is the map

$$(G^{ab} \otimes G^{ab}) \oplus \gamma_2(G)/\gamma_3(G) \to (\gamma_2(G)/\gamma_3(G))^{\oplus 2}$$

which is the sum of the identity on the second summand and the diagonal commutator pairing

$$\chi \oplus \chi : \bar{g} \otimes \bar{h} \mapsto ([g, h]\gamma_3(G), [g, h]\gamma_3(G)).$$

Therefore $\ker \beta \cong \ker(\chi : G^{ab} \otimes G^{ab} \to \gamma_2(G)/\gamma_3(G))$.

4.3. **Peiffer products of cyclic groups.** Consider a Peiffer product $G \bowtie H$ where G and H are cyclic groups generated by x and y respectively. Proposition 1.3 shows that $H_2(G \bowtie H)$ is isomorphic to the kernel of the map $\beta : \Pi_{G*H} \to [G; H]_G \oplus [H; G]_H$. Suppose that the actions are given by

$$x^y = x^{a+1} \text{ and } y^x = y^{b+1}. \tag{6}$$

where a and b are integers. Note that if a or b is zero, then $G \bowtie H$ is abelian. Given the orders of G and H, it is a simple matter to work out the structure of $G \bowtie H$ and of $H_2(G \bowtie H)$ in this case. We therefore assume that a and b are nonzero.

With the given actions (6), the Peiffer product $G \bowtie H$ is a quotient of the group $P(a, b)$ with presentation

$$P(a, b) = \langle x, y : x^{-1}y^{-1}xy^{b+1}, y^{-1}x^{-1}yx^{a+1} \rangle$$

and we begin by examining some relations in $P(a, b)$ and its quotients.

Lemma 4.1. *In the group $P(a, b)$ and all of its quotient groups,*

(a) $x^a = [x, y] = y^{-b}$ *is central and*
(b) $x^{a^2} = x^{ab} = y^{b^2} = y^{ab} = 1$.

Proof. Working in $P(a, b)$, we have $1 = x^{-1}y^{-1}xy^{b+1} = x^{-1}x^{a+1}y^b = x^a y^b$, which proves (a). Since x^a is central we have $x^a = y^{-1}x^a y = (x^{a+1})^a = x^{a^2+a}$ so that $x^{a^2} = 1$. In the same way we have $y^{b^2} = 1$. Since x^a and y^b are central, we further have $1 = (x^a y^b)^a = y^{ab}$ and similarly $x^{ab} = 1$. This proves (b). ☐

In a Peiffer product of cyclic groups with actions given by (6) where $ab \neq 0$, we may as well assume that the factors G and H are cyclic of finite orders m and n, respectively. We introduce the group $P(a, b; m, n)$ with presentation

$$P(a, b; m, n) = \langle x, y : x^m, y^n, x^{-1}y^{-1}xy^{b+1}, y^{-1}x^{-1}yx^{a+1} \rangle.$$

This group decomposes as a Peiffer product $P(a,b;m,n) = \mathbb{Z}/m \bowtie \mathbb{Z}/n$ whenever the congruence relations

$$(7) \qquad \begin{aligned} (a+1)^n &\equiv 1 \mod m \\ (b+1)^m &\equiv 1 \mod n \end{aligned}$$

are satisfied. (These are necessary and sufficient ensure that the factors $G = \mathbb{Z}/m$ and $H = \mathbb{Z}/n$ act on each other via (6).) For convenience, we introduce the following notation:

$$\mu = \gcd(a,m) \quad \text{and} \quad \nu = \gcd(b,n).$$

In the presence of the congruence relations (7), we can examine the structure of $P(a,b;m,n)$ using Proposition 1.1 as follows:

$$(8) \qquad P(a,b;m,n)/\langle x\rangle \cong \mathbb{Z}/\nu$$

$$(9) \qquad \langle x\rangle/(\langle x\rangle \cap \langle y\rangle) \cong P(a,b;m,n)/\langle y\rangle \cong \mathbb{Z}/\mu$$

$$(10) \qquad \langle x\rangle \cap \langle y\rangle = [\langle x\rangle, \langle y\rangle] = \langle [x,y]\rangle = \langle x^a\rangle$$

Lemma 4.2. *Assume that the parameters a, b, m, and n satisfy the congruence relations (7). If $H_2(P(a,b;m,n)) = 0$, then the subgroup $\langle x^a\rangle$ of $P(a,b;m,n)$ has order $\gcd(\mu,\nu)$ and the group $P(a,b;m,n)$ has order $\mu\nu\gcd(\mu,\nu)$.*

Proof. Let $P = P(a,b;m,n)$ and consider the quotient group $P/\langle x^a\rangle \cong \mathbb{Z}/\mu \oplus \mathbb{Z}/\nu$. The five-term sequence associated to the central extension

$$0 \to \langle x^a\rangle \to P \to \mathbb{Z}/\mu \oplus \mathbb{Z}/\nu \to 1$$

takes the form

$$0 = H_2(P) \to \mathbb{Z}/\mu \otimes \mathbb{Z}/\nu \to \langle x^a\rangle \to H_1(P) \xrightarrow{\cong} \mathbb{Z}/\mu \oplus \mathbb{Z}/\nu \to 0.$$

From this we conclude that the subgroup $\langle x^a\rangle$ of P has order $\gcd(\mu,\nu)$. Computation of the order of P is enabled by (8), (9), and (10). □

As an example, note that $P(-2,-2) \cong P(-2,-2;4,4)$ is the Peiffer product $\mathbb{Z} \bowtie \mathbb{Z}$ with nontrivial actions by the infinite cyclic factors. Since $P(-2,-2)$ is a finite group with a balanced presentation, Lemmas 4.1 and 4.2 show that $x^2 = y^2 = [x,y]$ is a central element of order two and that $\mathbb{Z} \bowtie \mathbb{Z}$ has order eight. As seen in [8], $\mathbb{Z} \bowtie \mathbb{Z}$ is the quaternion group of order eight.

We now examine $H_2(P(a,b;m,n))$ under the assumption that the parameters a, b, m, and n satisfy the congruence relations (7). Using Lemma 4.1, note that

$$x^{an/\nu} = y^{bm/\mu} = 1$$

in $P(a,b;m,n)$. The group $P(a,b;m,n)$ is therefore unchanged if we replace m and n by $\gcd(a^2, ab, m, an/\nu)$ and $\gcd(b^2, ab, n, bm/\mu)$, respectively. The values of μ and ν are unchanged and one can use the binomial theorem to show that the congruence relations (7) are still satisfied. These observations show that we can restrict our attention to the case where $P = G \bowtie H$ where $G = \mathbb{Z}/m = \langle x\rangle$ and $H = \mathbb{Z}/n = \langle y\rangle$, and where the parameters m, n, a, b satisfy the divisor relations

$$(11) \qquad m \mid a^2, ab, \frac{an}{\nu} \quad \text{and} \quad n \mid b^2, ab, \frac{bm}{\mu}.$$

Recall that S (respectively T) is the normal closure of $x^{-1}y^{-1}xy^{b+1}$ (respectively $y^{-1}x^{-1}yx^{a+1}$) in the free product $G * H$, and that $\Pi = ST$. Let Σ (resp. Θ) denote the image of S_{G*H} (resp. T_{G*H}) in Π_{G*H}. Thus $\Pi_{G*H} = \Sigma + \Theta$. Note that the intersection $\Sigma \cap \Theta$ is contained in the kernel of $\beta : \Pi_{G*H} \to [G;H]_G \oplus [H;G]_H$. We obtain explicit information about the kernel of β by studying the sequence of surjections

$$S_{G*H} \to \Sigma \to \Sigma/(\Sigma \cap \Theta) \to [H;G]_H.$$

Lemma 4.3. *Assume that the parameters a, b, m, and n satisfy the divisor relations* (11).

(a) *The group S_{G*H} is cyclic of order* $\gcd(m,n)$, *generated by $\langle x,y\rangle$.*
(b) *The order of the cyclic group Σ is a common divisor of a, b, m, and n. Thus the order of Σ divides* $\gcd(\mu,\nu)$.
(c) *The order of the cyclic group $\Sigma/(\Sigma\cap\Theta)$ is the least common multiple of m/μ and n/ν.*
(d) *The group $[H;G]_H$ is cyclic of order n/ν.*

Analogous results hold for the groups in the sequence of surjections

$$T_{G*H} \to \Theta \to \Theta/(\Sigma \cap \Theta) \to [G;H]_G.$$

Proof. (a) Lemma 2.1 shows that S_{G*H} is cyclic, generated by $\langle x,y\rangle$.The structure of S_{G*H} can be worked out from Theorem 3.2. For example, using the fact that the augmentation ideal $\mathfrak{g}$ in $\mathbb{Z}G$ is given by $\mathfrak{g} \cong \mathbb{Z}G/\langle \sum_{i=0}^{m-1} x^i\rangle$, the group $S_{G*H} \cong H^{ab} \otimes_G \mathfrak{g}$ is cyclic of order $\gcd(n, 1+(b+1)\cdots+(b+1)^{m-1})$. Using the binomial theorem and the divisor relations (11), one can show that $1+(b+1)\cdots+(b+1)^{m-1}$ is congruent to m modulo n and so S_{G*H} is cyclic of order $\gcd(m,n)$.

(b) Consider $\langle x,y\rangle^b = \langle x,y^b\rangle = x^{-1}y^{-b}x(y^b)^x[S,G*H]$. We have

$$x^{-1}y^{-b}x(y^b)^x = x^{-1}y^{-b}xy^{b^2+b} = x^{-1}y^{-b}xy^b = [x,y^b].$$

On the other hand, since $x^ay^b \in ST$, we have $[x,y^b] = [x,x^ay^b] \in [ST,G*H]$. This shows that $\langle x,y\rangle^b$ lies in the kernel of $S_{G*H} \to \Sigma \subseteq \Pi_{G*H}$. In the same way, $\langle x^a,y\rangle = x^{-a}y^{-1}x^ay^{x^a}[S,G*H] = [x^a,y][S,G*H]$ is in the kernel of $S_{G*H} \to \Sigma$ since $[x^a,y] = [y^bx^a,y] \in [ST,G*H]$. Using the twisted bilinear relations from Lemma 2.1 and working modulo the kernel of $S_{G*H} \to \Sigma$, for any positive integer k we have

$$\begin{aligned}
\langle x^k,y\rangle &= \langle x,y^{x^{k-1}}\rangle\langle x^{k-1},y\rangle\\
&= \langle x,y^{(b+1)^{k-1}}\rangle\langle x^{k-1},y\rangle\\
&\equiv \langle x,y\rangle^{(k-1)b+1}\langle x^{k-1},y\rangle\\
&\equiv \langle x,y\rangle\langle x^{k-1},y\rangle.
\end{aligned}$$

It follows that $\langle x^a,y\rangle \equiv \langle x,y\rangle^a$ modulo the kernel of $S_{G*H} \to \Sigma$ and so $\langle x,y\rangle^a$ itself lies in this kernel. Together with part (a), this shows that the order of Σ is a common divisor of m, n, a, and b, as claimed.

(c) Information on the intersection $\Sigma\cap\Theta$ is obtained by reducing elements of S modulo T. For this, we first use the divisor relations (11) to show that the elements x^a and y^b are central in the semi-direct product $G \rtimes H = (G*H)/T$. Working modulo T we have

$$y^{-1}x^ay = (x^y)^a = x^{a^2+a} = x^a$$

since $m \mid a^2$. In addition,

$$y^{-b}xy^b = x^{(a+1)^b} = x^{ab+1} = x$$

since $m \mid \gcd(a^2, ab)$. Using this, we show that $[S, G * H] \subseteq T$. Given $u, w \in G * H$ and working modulo T, we have

$$[w^{-1}x^{-1}y^{-1}xy^{b+1}w, u] \equiv [w^{-1}x^a y^b w, u] \equiv [x^a y^b, u] \equiv 1.$$

This shows that T contains a generating set for $[S, G * H]$.

Now $\Sigma/(\Sigma \cap \Theta) \cong (\Sigma + \Theta)/\Theta = ST/T[S, G * H] = ST/T$ embeds in $(G * H)/T \cong G \rtimes H$ by the map

$$\langle x, y\rangle + (\Sigma \cap \Theta) \mapsto x^a y^b T.$$

Once again using the fact that x^a and y^b are central modulo T and that $(G * H)/T$ is the semidirect product $G \rtimes H = \mathbb{Z}/m \rtimes \mathbb{Z}/n$, we are able to conclude that $\langle x, y\rangle^k \in \Sigma \cap \Theta$ if and only if $m \mid ak$ and $n \mid bk$. The result follows easily.

(d) The group H_G is cyclic of order ν so $[H; G]_H = [H; G]$ is cyclic of order n/ν, generated by y^b. $\square$

Theorem 4.4. *Let $G = \mathbb{Z}/m$ and $H = \mathbb{Z}/n$, generated by x and y and with actions given by* (6)*. If we assume that the parameters a, b, m, and n satisfy the divisor relations* (11)*, then $H_2(G \bowtie H) = 0$ if and only if $m/\mu = n/\nu = \gcd(\mu, \nu)$.*

Proof. Let $P = G \bowtie H$. We use the conclusions of Lemma 4.3 without reference throughout the proof of the theorem. Suppose first that $m/\mu = n/\nu = \gcd(\mu, \nu)$. Since the order of Σ is a divisor of $\gcd(a, b, m, n) = \gcd(\mu, \nu)$ and the order of $[H; G]_H$ is n/ν, the fact that $\gcd(\mu, \nu) = n/\nu$ implies that the surjection $\Sigma \to [H; G]_H$ is an isomorphism. In particular, $\Sigma \cap \Theta = 0$ so that $\Pi_{G*H} = \Sigma \oplus \Theta$. In the same way, $\Theta \to [G; H]_G$ is an isomorphism. Thus β is an isomorphism and so $H_2(P) = 0$.

Now suppose that $H_2(P) = 0$. Then β is injective, which implies that the map $\Sigma/(\Sigma \cap \Theta) \to [H; G]_H$ is an isomorphism. This in turn implies that $\operatorname{lcm}(m/\mu, n/\nu) = n/\nu$ so that m/μ divides n/ν. Analogous considerations applied to the map $\Theta/(\Sigma \cap \Theta) \to [G; H]_G$ show that n/ν divides m/μ. Thus $m/\mu = n/\nu$.

The order of the subgroup $\langle x\rangle = \bar{G}$ of P is $\mu \gcd(\mu, \nu)$ by (9), (10) and Lemma 4.2. This implies that $\mu \gcd(\mu, \nu) \le m = \mu n/\nu$, so that $\gcd(\mu, \nu) \le n/\nu$. On the other hand, the fact that $n \mid b^2$ can be used to show that $n/\nu \mid \gcd(\mu, \nu)$. Thus $m/\mu = n/\nu = \gcd(\mu, \nu)$. $\square$

Corollary 4.5. *With the notation and hypotheses of Theorem* 4.4*, if $H_2(G \bowtie H)$ is trivial, then the factors G and H embed in the Peiffer product $G \bowtie H$.*

Proof. The order of $\bar{G}$ is $\mu \gcd(\mu, \nu) = \mu m/\mu = m$, so that $G \cong \bar{G}$. Similarly, $H \cong \bar{H}$. $\square$

Examples with $H_2 = 0$. Given nonzero integers a and b with $g = \gcd(a, b)$, the divisor relations (11) are satisfied if we set $m = ag$ and $n = bg$. One notes that $m/\mu = n/\nu = \gcd(\mu, nu) = g$, so the group $P = P(a, b; ag, bg) = \mathbb{Z}/ag \bowtie \mathbb{Z}/bg$ has order abg, $H_2(P) = 0$, and the factors embed in the Peiffer product.

More with $H_2 = 0$. Suppose that $a = pq$, $b = qr$, and $m = n = q^2$ where p, q, and r are pairwise relatively prime. The divisor relations (11) are satisfied and $m/\mu = n/\nu = \gcd(\mu,\nu) = q$, so $P = P(pq, qr; q^2, q^2) = \mathbb{Z}/q^2 \bowtie \mathbb{Z}/q^2$ has order q^3, $H_2(P) = 0$, and the factors embed in the Peiffer product.

Examples with $H_2 \neq 0$. Consider a fixed integer $p \geq 2$. The divisor relations (11) are satisfied for $a = p^r$, $b = p^s$, $m = p^{r+t}$, and $n = p^{r+s+t}$ if $r \geq 1$, $s \geq 0$, and $0 \leq t \leq r$. We have $m/\mu = n/\nu = p^t$ and $\gcd(\mu,\nu) = p^r$. For fixed r and s and $t = 0 \ldots r$, let G_t denote the Peiffer product

$$G_t = P(p^r, p^{r+s}; p^{r+t}, p^{r+s+t}) \cong \mathbb{Z}/p^{r+t} \bowtie \mathbb{Z}/p^{r+s+t}.$$

Taking $t = r$, Theorem 4.4 shows that $H_2(G_r) = 0$. With this, Lemma 4.2 provides that G_r has order p^{3r+s} and Corollary 4.5 shows that the element x has order p^{2r} in $G_r \cong \mathbb{Z}/p^{2r} \bowtie \mathbb{Z}/p^{2r+s}$.

For each t, we have a central extension

$$0 \to \langle x^{p^{r+t}} \rangle \to G_r \to G_t \to 1$$

and the five-term sequence for this extension shows that $H_2(G_t) \cong \langle x^{p^{r+t}} \rangle$ is cyclic of order p^{r-t}. Knowing the order of G_r and the order of x in G_r, we also conclude that G_t has order p^{2r+s+t}. Finally, we note that the factors embed in the Peiffer product $G_t \cong \mathbb{Z}/p^{r+t} \bowtie \mathbb{Z}/p^{r+s+t}$.

4.4. **A double wreath product.** Given positive integers m and n, let $G = C_m^{(n)}$ be the direct product of n copies of the multiplicative cyclic group C_m of order m and let $H = C_n^{(m)}$ be the direct product of m copies of the cyclic group C_n of order n.

$$G = \langle x_1, \ldots, x_n : x_i^m, [x_i, x_j] \rangle$$
$$H = \langle y_1, \ldots, y_m : y_i^n, [y_i, y_j] \rangle$$

Then G and H act on each other by cyclic permutation of indices.

$$x_i^{y_j} = x_{i+1} \quad \text{(subscripts mod n)} \tag{12}$$
$$y_j^{x_i} = y_{j+1} \quad \text{(subscripts mod m)}$$

The Peiffer product $G \bowtie H$ is a homomorphic image of the standard wreath products $C_m \wr C_n$ and $C_n \wr C_m$, so we think of $G \bowtie H = C_m^{(n)} \bowtie C_n^{(m)}$ as a *double wreath product* of C_m and C_n.

Lemma 4.6. *For the double wreath product* $G \bowtie H = C_m^{(n)} \bowtie C_n^{(m)}$, *the map*

$$\beta : \Pi_{G*H} \to [G;H]_G \oplus [H;G]_H$$

is an isomorphism. In addition, the natural map

$$H_2(G) \oplus H_2(H) \to H_2(G \bowtie H)$$

is surjective.

Proof. We show that the kernel of the map $S_{G*H} \to [H;G]_H$ is contained in the kernel of the map $S_{G*H} \to \Pi_{G*H}$. An analogous result holds for the map $T_{G*H} \to \Pi_{G*H}$ and from this it follows that the map β is an isomorphism.

First note that $[H;G]_H = [H;G] \cong C_n^{(m-1)}$ is generated by the elements

$$y_1^{-1}y_2,\ y_2^{-1}y_3, \ldots,\ y_{m-1}^{-1}y_m.$$

Next, Lemma 2.1 shows that S_{G*H} is generated by the elements

$$\langle x_i, y_j \rangle = x_i^{-1} y_j^{-1} x_i y_{j+1} [S, G*H].$$

The map β carries $\langle x_i, y_j \rangle$ to the element $y_j^{-1} y_{j+1} \in [H;G] = [H;G]_H$. Since H is abelian, Theorem 3.2 shows that $H \otimes_G \mathfrak{g} \cong S_{G*H}$ via the map $h \otimes (g-1) \mapsto \langle g, h \rangle$. It follows that

$$\langle x_i, y_j \rangle^n = 1$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. In addition,

$$\prod_{j=1}^{m} \langle x_i, y_j \rangle = \langle x_i, y_1 \ldots y_m \rangle = \langle x_i, y_1 y_1^{x_i} \ldots y_1^{x_i^{m-1}} \rangle = \langle x_i, y_1^{1+x_i+\cdots+x_i^{m-1}} \rangle = 1$$

for all $i = 1, \ldots, n$ since $(1 + x_i + \cdots + x_i^{m-1})(x_i - 1) = 0$ in the integral group ring $\mathbb{Z}G$. With this we see that it suffices to show that for each i and j, the elements $\langle x_i, y_j \rangle$ and $\langle x_{i+1}, y_j \rangle$ have the same image in Π_{G*H}. Working in $G*H$, notice that $x_i^{-1} y_j^{-1} x_i y_{j+1} = [x_i, y_j] y_j^{-1} y_{j+1} \in S$ and $y_j^{-1} x_i^{-1} y_j x_{i+1} = [y_j, x_i] x_i^{-1} x_{i+1} \in T$ so that $y_j^{-1} y_{j+1} x_i^{-1} x_{i+1} \in ST = \Pi$. This implies that

$$[y_{j+1}, y_j^{-1} y_{j+1} x_i^{-1} x_{i+1}] = [y_{j+1}, x_i^{-1} x_{i+1}] \in [\Pi, G*H].$$

The image of the element $\langle x_i^{-1} x_{i+1}, y_{j+1} \rangle$ under the map $S_{G*H} \to \Pi_{G*H}$ is

$$(x_i^{-1} x_{i+1})^{-1} y_{j+1}^{-1} (x_i^{-1} x_{i+1}) y_{j+1}^{(x_i^{-1} x_{i+1})} [\Pi, G*H] = [x_i^{-1} x_{i+1}, y_{j+1}][\Pi, G*H] = 1$$

so that $\langle x_i^{-1} x_{i+1}, y_{j+1} \rangle$ lies in the kernel of $S_{G*H} \to \Pi_{G*H}$. Now, using the relations of Lemma 2.1 for S_{G*H}, we have

$$\begin{aligned} \langle x_i^{-1} x_{i+1}, y_{j+1} \rangle &= \langle x_{i+1}, y_{j+1}^{x_i^{-1}} \rangle \langle x_i^{-1}, y_{j+1} \rangle \\ &= \langle x_{i+1}, y_j \rangle \langle x_i^{-1}, y_j^{x_i} \rangle \\ &= \langle x_{i+1}, y_j \rangle \langle x_i, y_j \rangle^{-1}. \end{aligned}$$

This shows that the elements $\langle x_i, y_j \rangle$ and $\langle x_{i+1}, y_j \rangle$ have the same image in Π_{G*H} and completes the proof that β is an isomorphism.

To prove the second assertion of the Lemma, note that the groups G_H and H_G are both cyclic, so that $H_2(G_H) = H_2(H_G) = 0$. Referring to Figure 1.1, the map α_2 is the zero map. Since β is injective, it follows that $H_2(G \bowtie H) \to \Pi_{G*H}$ is the zero map and so $H_2(G*H) = H_2(G) \oplus H_2(H) \to H_2(G \bowtie H)$ is surjective. $\square$

We will not attempt to determine the kernel of the map $H_2(G) \oplus H_2(H) \to H_2(G \bowtie H)$ here, but we can use Lemma 4.6 to compute the order of the double wreath product $C_m^{(n)} \bowtie C_n^{(m)}$.

Proposition 4.7. *The order of the double wreath product* $C_m^{(n)} \bowtie C_n^{(m)}$ *of* C_m *and* C_n *is* $mn \gcd(m,n)$.

Proof. Letting $G = C_m^{(n)}$ and $H = C_n^{(m)}$ we use Proposition 1.1 to compute as follows:

$$\begin{aligned}(G \bowtie H)/\bar{G} &\cong H_G \cong C_n \\ \bar{G}/(\bar{G} \cap \bar{H}) &\cong (G \bowtie H)/\bar{H} \cong C_m \\ \bar{G} \cap \bar{H} &= [\bar{G}, \bar{H}] \\ &= sgp\{x_i^{-1} x_{i+1} : i = 1, \ldots, n\} \\ &= sgp\{y_j^{-1} y_{j+1} : j = 1, \ldots, m\}.\end{aligned}$$

Now $[\bar{G}, \bar{H}] = [G \bowtie H, G \bowtie H]$ is central in $G \bowtie H$ and the five-term sequence for the central extension

$$0 \to [\bar{G}, \bar{H}] \to G \bowtie H \to G_H \times H_G \to 1$$

has the form

$$H_2(G \bowtie H) \to H_2(G_H \times H_G) \to [\bar{G}, \bar{H}] \to H_1(G \bowtie H) \stackrel{\cong}{\to} G_H \times H_G \to 0.$$

The composite maps $G \to G \bowtie H \to G_H \times H_G$ and $H \to G \bowtie H \to G_H \times H_G$ factor through the cyclic groups G_H and H_G and so induce the trivial map on second homology. And since $H_2(G) \oplus H_2(H) \to H_2(G \bowtie H)$ is surjective, it follows that the map $H_2(G \bowtie H) \to H_2(G_H \times H_G)$ is the trivial map. This implies that $[\bar{G}, \bar{H}] \cong H_2(G_H \times H_G) \cong C_m \otimes C_n$. Thus the order of $[\bar{G}, \bar{H}]$ is $\gcd(m, n)$ and so the order of $G \bowtie H$ is $mn \gcd(m, n)$. $\square$

A parting shot. Let p be a prime number. We close by calling attention to the groups $P(p, p; p^2, p^2) = \mathbb{Z}/p^2 \bowtie \mathbb{Z}/p^2$ of §4.3 and $C_p^{(p)} \bowtie C_p^{(p)}$ of §4.4, both of which have order p^3. These are nonisomorphic nonabelian groups whose center and derived subgroups coincide and have order p. Such groups are called extra-special p-groups. According to [12, pages 140–141], every nonabelian group of order p^3 is isomorphic to either $\mathbb{Z}/p^2 \bowtie \mathbb{Z}/p^2$ or to $C_p^{(p)} \bowtie C_p^{(p)}$. Further, the extra-special p-groups “play an important role in some of the deeper parts of finite group theory” and every extra-special p-group can be exhibited as a “central product” of nonabelian groups of order p^3. It is therefore satisfying to see how these groups can be constructed from cyclic groups using the Peiffer product construction.

References

[1] Y. G. Baik and S. J. Pride, *Generators of the second homotopy module of presentations arising from group constructions*, preprint, University of Glasgow, 1992.

[2] Y. G. Baik, J. Harlander, and S. J. Pride, *The geometry of group extensions*, J. Group Theory **1** (1998), 395–416, MR 1 660 412.

[3] W. A. Bogley and M. A. Gutierrez, *Mayer-Vietoris sequences in homotopy of 2-complexes and in homology of groups*, J. Pure Appl. Algebra **77** (1992), 39–65, MR 93e:20069, Zbl 761.55016.

[4] W. A. Bogley and S. J. Pride, *Calculating generators of π_2*, Two-dimensional Homotopy and Combinatorial Group Theory, (C. Hog-Angeloni, W. Metzler, and A. J. Sieradski, eds.), London Math. Soc. Lecture Notes, no. 197, Cambridge University Press, Cambridge, 1993, pp. 157–188, MR 95g:57006, Zbl 811.57005.

[5] R. Brown, *Coproducts of crossed P–modules: applications to second homotopy groups and to the homology of groups*, Topology **23** (1984), 337–345, MR 86e:18017, Zbl 565.55017.

[6] M. N. Dyer, *Asphericity of 2–complexes and the homology of groups*, preprint, University of Oregon, 1991.

[7] N. D. Gilbert, *Identities between sets of relations*, J. Pure Appl. Algebra **83** (1993), 263–276, MR 94e:55023, Zbl 778.57003.

[8] N. D. Gilbert and P. J. Higgins, *The non-abelian tensor product of groups and related constructions*, Glasgow Math. J. **31** (1989), 17–29, MR 90e:20026, Zbl 681.20034.

[9] M. A. Gutierrez and J. G. Ratcliffe, *On the second homotopy group*, Quart. J. Math. Oxford (2) **32** (1981), 45–55, MR 82g:57003, Zbl 457.57002.

[10] N. S. Hekster, *On the structure of n–isoclinism classes of groups*, J. Pure Appl. Algebra **40** (1986), 63–85, MR 87k:20063, Zbl 593.20040.

[11] T. MacHenry, *The tensor product and the second nilpotent product of groups*, Math. Z. **73** (1960), 134–145, MR 22 #11027, Zbl 121.27002.

[12] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, no. 80, Springer-Verlag, New York–Berlin, 1982, MR 84k:20001, Zbl 483.20001.

[13] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, no. 38, Cambridge University Press, Cambridge, 1994, MR 95f:18001, Zbl 797.18001.

[14] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. **42** (1941), 409–428, MR 2,323c, Zbl 027.26404.

Department of Mathematics, Kidder 368, Oregon State University, Corvallis, OR 97331-4605, USA

bogley@math.orst.edu http://ucs.orst.edu/~bogleyw/

Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, Scotland

N.D.Gilbert@ma.hw.ac.uk http://www.ma.hw.ac.uk/~nick/

This paper is available via http://nyjm.albany.edu:8000/j/2000/6-3.html.