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# SOME RESULTS ON COMPLEX ALGEBRAS OF SUBALGEBRAS[1]

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**Abstract.** In this paper we investigate a property of the algebras of complexes (or power algebras or globals) which is a natural generalization of the notion of having all subgroups to be quasinormal in group theory. We say that an algebra $\mathcal{A}$ has the *complex algebras of subalgebras* if the set of all non-empty subuniverses of this algebra forms a subuniverse of the algebra of complexes of $\mathcal{A}$. For example, all conservative and all entropic algebras have this property. Among other things, we prove that the class of finite algebras which have the complex algebra of subalgebras is not closed under finite direct products and it is not globally determined.

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## 1. Power constructions

The natural generalization of the multiplication of cosets in group theory is the following "lifting" of an arbitrary $n$-ary operation $f$ on a set $A$ to the $n$-ary operation $f^+$ on the power set $\mathcal{P}(A)$ of all non-empty subsets of $A$:

**Definition 1.** *Let $A$ be a non-empty set, $\mathcal{P}(A)$ the set of all non-empty subsets of $A$, and $f : A^n \to A$. We define $f^+ : \mathcal{P}(A)^n \to \mathcal{P}(A)$ in the following way:*

$$f^+(X_1, \dots, X_n) = \{f(x_1, \dots, x_n) \mid x_1 \in X_1, \dots, x_n \in X_n\}.$$

*If $\mathcal{A} = \langle A, \{f \mid f \in \mathcal{F}\}\rangle$ is an algebra, the* **complex algebra** *(or* **power algebra***, or* **global***) $\mathcal{P}(\mathcal{A})$ is defined as:*

$$\mathcal{P}(\mathcal{A}) = \langle \mathcal{P}(A), \{f^+ \mid f \in \mathcal{F}\}\rangle.$$

Beside group theory and semigroup theory, power operations are implicitly used in some other fields. For instance, the set of ideals of a distributive lattice $L$ again forms a lattice, and meets and joins in the new lattice are precisely the power operations of meets and joins in $L$. In the formal language theory the product of two languages is simply the power operation of concatenation of words.

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Note that there are some other types of power constructions in universal algebra. In some approaches, it is convenient to include the empty set to the universe of the power algebra. Also, the set-theoretic operations of the union, intersection and complementation could be added to the fundamental operation of the complex algebra. This is the case in the more general power construction, i.e. the powering of relational structures, which was introduced by Jónsson and Tarski in [14]. This construction has proved to be very useful in various areas of algebraic logic and theory of non-classical logics. Also, there are several different ways to lift a relation from a base set to its power set. A general definition of $n$-ary power relation was given by Whitney ([22]). This type of construction is widely used in theoretical computer science, in the context of powerdomains. Some attempts to give a general view on power structures and present common ideas used by different authors were made in [2] and [4].

## 2. Complex algebras of subalgebras

We investigate a property of complex (power) algebras which is a natural generalization of the notion of having all the subgroups of a group to be quasinormal. Namely, for a subgroup $H$ of a group $G$ we say that it is **quasinormal** in $G$ if $H$ commutes with all the subgroups $K$ of $G$ i.e. $HK = KH$. Of course, normal subgroups are always quasinormal, but converse is not true. For example, if $3 \leq p$ is a prime, then any cyclic group $C_{p^n}$ extended by any cyclic group $C_{p^m}$ has all subgroups to be quasinormal. For an overview of some old, recent and new results on quasinormal groups see [21].

It is easy to see that if $G$ is a group, then the set $Sub(G)$ of all subgoups of $G$ forms a subgroup of the power group $\mathcal{P}(G)$ iff all subgroups of $G$ are quasinormal. In the case of a universal algebra $\mathcal{A}$, the set $Sub(\mathcal{A})$ of all non-empty subuniverses of $\mathcal{A}$ is a subset of the universe of the complex algebra $\mathcal{P}(\mathcal{A})$, but it is not always a subuniverse of $\mathcal{P}(\mathcal{A})$.

**Definition 2.** *Let $\mathcal{A}$ be an algebra. We say that $\mathcal{A}$* **has the complex algebra of subalgebras** *(or briefly, has* **CaSa***) if the set $Sub(\mathcal{A})$ of all non-empty subuniverses of $\mathcal{A}$ forms a subuniverse of the complex algebra $\mathcal{P}(\mathcal{A})$; the corresponding subalgebra $CSub(\mathcal{A})$(with the universe $Sub(\mathcal{A})$ we call the* **complex algebra of subalgebras** *of $\mathcal{A}$. For a variety of algebras $V$ we say that it* **has the complex algebra of subalgebras** *if any $\mathcal{A} \in V$ has the complex algebra of subalgebras.*

For example, the variety of left-zero (or right-zero) semigroups has CaSa. More generally, any conservative algebra has CaSa (for an algebra $\mathcal{A}$ we say that it is **conservative** if for any $n$-ary fundamental operation $f$, and for any $a_1, a_2, \ldots, a_n \in A$ we have $f(a_1, a_2, \ldots, a_n) \in \{a_1, a_2, \ldots, a_n\}$). A very natural and widely studied variety of algebras having CaSa is the variety of idempotent entropic algebras (the so called **modes**).

**Definition 3.** *An algebra $\mathcal{A}$ is called* **entropic** *if it satisfies for every $n$-ary* *fundamental operation $f$, and every $m$-ary fundamental operation $g$ the identity*

$$g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))\approx$$

$$f(g(x_{11},\dots,x_{1m}),\dots,g(x_{n1},\dots,x_{nm})).$$

In other words, an algebra $\mathcal{A}$ is entropic if all operations of $\mathcal{A}$ commute with each other. Note that a groupoid $G$ is entropic if it satisfies the identity $xy\cdot uv\approx xu\cdot yv$, called also the **mediality**. It is easy to verify that *any entropic algebra is CaSa*. The complex algebras of subalgebras were introduced and studied in the context of idempotent entropic algebras in [18] (see also [19], [20]).

There are algebras which have the complex algebra of subalgebras, but which are neither conservative nor entropic. Some of them satisfy the so called **generalized entropic property**, introduced and studied in [1].

**Definition 4.** *We say that an algebra $\mathcal{A}$ (respectively, a variety $V$) satisfies the* **generalized entropic property** *if for every $n$-ary operation $f$ and $m$-ary operation $g$ of $\mathcal{A}$ (of $V$), there exist $m$-ary terms $t_1,\dots,t_n$ such that the identity*

$$g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))\approx$$

$$f(t_1(x_{11},\dots,x_{1m}),\dots,t_n(x_{n1},\dots,x_{nm}))$$

*holds in $\mathcal{A}$ (in $V$).*

For example, a groupoid satisfies the generalized entropic property, if there are binary terms $t$ and $s$ such that the identity $xy\cdot uv=t(x,u)s(y,v)$ holds. The entropic law is a special case of the generalized entropic property, where the terms $t_1,\dots,t_n$ are equal to $g$. It is easy to verify that:

**Proposition 1.** *Every algebra satisfying the generalized entropic property has the complex algebra of subalgebras.*

In the case of varieties, more can be proved.

**Theorem 1.** *([10]) In a variety $V$ of groupoids, every groupoid in $V$ has the complex algebra of subalgebras iff $V$ satisfies the identity*

$$xy\cdot uv=t(x,u)s(y,v),$$

*for some terms $t(x,u)$ and $s(y,v)$.*

A generalization of this result is proved in [1]. Namely:

**Theorem 2.** *([1]) For a variety $V$ of algebras, every algebra $\mathcal{A}\in V$ has the complex algebra of subalgebras iff the variety $V$ satisfies the generalized entropic property.*

Generally, the generalized entropic property and the entropic law are not equivalent, but there are a lot of varieties where every algebra satisfying the generalized entropic property is also entropic (see [1]).

On the other hand, there are algebras which have the complex algebra of subalgebras, and do not satisfy the generalized entropic property. In fact, using such an example from [1], we can conclude that the property CaSa is not equationally definable, i.e.

**Proposition 2.** *Let $K$ be the class of all groupoids which have the complex algebra of subalgebras. Then, there is no set of identities $\Sigma$ such that $K = Mod(\Sigma)$.*

*Proof.* Let $G$ be the following groupoid:

| $\cdot$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $c$ | $c$ |
| $b$ | $c$ | $b$ | $c$ |
| $c$ | $a$ | $b$ | $c$ |

It can be verified that $G$ has the complex algebra of subalgebras. On the other hand, in [1] it is proved that the free algebra $\mathbf{F}_V(x, y, z)$ of the variety $V$ generated by $G$ has subgroupoids whose complex multiplication is not a subgroupoid of $\mathbf{F}_V(x, y, z)$. $\square$

**Theorem 3.** *Let $\mathcal{F}$ be any type of algebras. Then the class of all algebras of the type $\mathcal{F}$ which have the complex algebras of subalgebras is closed under formation of subalgebras and taking homomorphic images.*

*Proof.* Let an algebra $\mathcal{A}$ of type $\mathcal{F}$ have the complex algebra of subalgebras and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are subalgebras of $\mathcal{B}$, then for any $f \in \mathcal{F}_n$ we have

$$(f^{\mathcal{B}})^{+}(C_1, \ldots, C_n) \subseteq B.$$

As $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are also subalgebras of $\mathcal{A}$, and $\mathcal{A}$ has CaSa, we have

$$(f^{\mathcal{A}})^{+}(C_1, \ldots, C_n) \in \mathrm{Sub}(\mathcal{A}),$$

and consequently

$$(f^{\mathcal{A}})^{+}(C_1, \ldots, C_n) = (f^{\mathcal{B}})^{+}(C_1, \ldots, C_n) \in \mathrm{Sub}(\mathcal{B}).$$

So, the algebra $\mathcal{B}$ has CaSa.

Let $\varphi : \mathcal{A} \to \mathcal{B}$ be an epimorphism and $\mathcal{C}_1, \ldots, \mathcal{C}_n$ subalgebras of $\mathcal{B}$. We know that $\varphi^{-1}(\mathcal{C}_i) = \mathcal{D}_i$, $i \in \{1, \ldots, n\}$, are subalgebras of $\mathcal{A}$. Let us prove that for any $f \in \mathcal{F}_n$,

$$(f^{\mathcal{B}})^{+}(C_1, \ldots, C_n) \in \mathrm{Sub}(\mathcal{B}).$$

Let $b_1, \ldots, b_m \in (f^{\mathcal{B}})^{+}(C_1, \ldots, C_n)$. We have to prove that for any $g \in \mathcal{F}_m$,

$$g^{\mathcal{B}}(b_1, \ldots, b_m) \in (f^{\mathcal{B}})^{+}(C_1, \ldots, C_n).$$

As $b_1 \in (f^{\mathcal{B}})^+(C_1, \dots, C_n)$, there are elements $c_{1j} \in C_j$, $j \in \{1, \dots, n\}$, such that $b_1 = f^{\mathcal{B}}(c_{11}, \dots, c_{1n})$. Similarly,

$$b_2 = f^{\mathcal{B}}(c_{21}, \dots, c_{2n}), \dots, b_m = f^{\mathcal{B}}(c_{m1}, \dots, c_{mn}),$$

for some elements $c_{ij} \in C_j$, $i \in \{2, \dots, m\}$, $j \in \{1, \dots, n\}$. As the mapping $\varphi : A \to B$ is "onto", there are elements $d_{ij} \in D_j$ such that $\varphi(d_{ij}) = c_{ij}$, for $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$.

Now we have

$$f^{\mathcal{A}}(d_{11}, \dots, d_{1n}) \in (f^{\mathcal{A}})^+(D_1, \dots, D_n),$$

$$f^{\mathcal{A}}(d_{21}, \dots, d_{2n}) \in (f^{\mathcal{A}})^+(D_1, \dots, D_n),$$

$$\dots$$

$$f^{\mathcal{A}}(d_{m1}, \dots, d_{mn}) \in (f^{\mathcal{A}})^+(D_1, \dots, D_n).$$

As $\mathcal{A}$ has CaSa, then for any $f \in \mathcal{F}_n$ we have

$$(f^{\mathcal{A}})^+(D_1, \dots, D_n) \in \mathrm{Sub}(\mathcal{A}).$$

Therefore, there are elements $d_1 \in D_1, \dots, d_n \in D_n$ such that

$$g^{\mathcal{A}}(f^{\mathcal{A}}(d_{11}, \dots, d_{1n}), \dots, f^{\mathcal{A}}(d_{m1}, \dots, d_{mn})) = f^{\mathcal{A}}(d_1, \dots, d_n).$$

Then we have

$$\begin{array}{l} g^{\mathcal{B}}(b_1, \dots, b_m) = g^{\mathcal{B}}(f^{\mathcal{B}}(c_{11}, \dots, c_{1n}), \dots, f^{\mathcal{B}}(c_{m1}, \dots, c_{mn})) = \\ g^{\mathcal{B}}(f^{\mathcal{B}}(\varphi(d_{11}), \dots, \varphi(d_{1n})), \dots, f^{\mathcal{B}}(\varphi(d_{m1}), \dots, \varphi(d_{mn}))) = \\ \varphi(g^{\mathcal{A}}(f^{\mathcal{A}}(d_{11}, \dots, d_{1n}), \dots, f^{\mathcal{A}}(d_{m1}, \dots, d_{mn}))) = \\ \varphi(f^{\mathcal{A}}(d_1, \dots, d_n)) = f^{\mathcal{B}}(\varphi(d_1), \dots, \varphi(d_n)). \end{array}$$

As $\varphi(d_i) = c_i \in C_i$, for $i \in \{1, \dots, n\}$, we have

$$g^{\mathcal{B}}(b_1, \dots, b_m) = f^{\mathcal{B}}(c_1, \dots, c_n) \in (f^{\mathcal{B}})^+(C_1, \dots, C_n).$$

So, $\mathcal{B}$ has the complex algebra of subalgebras. □

**Corollary 1.** *The class $K^*$ of all groupoids which have the complex algebra of subalgebras is not closed under direct products.*

*Proof.* Follows from Proposition 2 and Theorem 3. □

Even the class of *finite* groupoids from $K^*$ is not closed under *finite* direct products. In order to prove this, we will use a special class of groupoids which are connected with some kind of directed graphs.

**Definition 5.** *A* **tournament** *is a complete directed graph $T = (V, E)$ (i.e. a digraph in which every pair of different vertices is connected by exactly one directed edge). If $x, y \in V$ and there is an edge from $x$ to $y$, we will write $x \to y$. The corresponding groupoid $G_T = \langle V, \cdot \rangle$ is defined in the following way: $x \cdot x = x$ and for $x \neq y$*

$$x \to y \quad \textit{iff} \quad x \cdot y = y \cdot x = x.$$

*The groupoids obtained in this way we call* **groupoids of tournaments** *(or simply* **tournaments***).*

Note that the variety generated with all groupoids of tournaments is a locally finite non-finitely based variety, which has been explored by many authors (see for example [7]). Of course, every groupoid of tournament has the complex algebra of subalgebras. We can prove the following:

**Lemma 1.** *Let $T$ be a tournament.*

*(a) The groupoid of a tournament $T$ is entropic iff $T$ is transitive.*

*(b) Groupoid $G_T$ satisfies a generalized entropic property iff it is entropic.*

**Lemma 2.** *There are two finite groupoids of tournaments $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1 \times \mathcal{T}_2$ has no complex algebra of subalgebras.*

*Proof.* Let $\mathcal{T}_1$ be the groupoid of the tournament $T_1$ with vertices $\{x, y\}$ such that $x \to y$, and $\mathcal{T}_2$ the groupoid of the tournament $T_2$ with vertices $\{a, b, c, d\}$ in which

$$b \to a, a \to c, c \to d, a \to d, c \to b, b \to d.$$

Let $X = \{\langle x, c\rangle, \langle y, b\rangle\}$ and $Y = \{\langle y, d\rangle, \langle y, a\rangle\}$. Then $X \in \mathrm{Sub}(\mathcal{T}_1 \times \mathcal{T}_2)$ and $Y \in \mathrm{Sub}(\mathcal{T}_1 \times \mathcal{T}_2)$, but $XY$ is not a subuniverse of $(\mathcal{T}_1 \times \mathcal{T}_2)$, because $XY = \{\langle x, c\rangle, \langle x, a\rangle, \langle y, b\rangle\}$ and

$$\langle x, a\rangle \cdot \langle y, b\rangle = \langle x, b\rangle \notin XY.$$

So, $\mathcal{T}_1 \times \mathcal{T}_2$ has no complex algebra of subalgebras.

**Theorem 4.** *Let $\mathcal{F}$ be some type of algebras, such that there are at least one functional symbol $f \in \mathcal{F}$ of arity at least two. Then the class of all finite algebras of the type $\mathcal{F}$ which have the complex algebra of subalgebras is not closed under finite direct products.*

*Proof.* Let $\mathcal{T}_1 = \langle \{x, y\}, f_1\rangle$ and $\mathcal{T}_2 = \langle \{x, y\}, f_2\rangle$ be the groupoids of tournaments from Lemma 2. Denote by $f_0$ the functional symbol from $\mathcal{F}$ which has arity at least 2. Let us construct two algebras $\mathcal{A}$ and $\mathcal{B}$ of the type $\mathcal{F}$ such that they have the complex algebra of subalgebras, but $\mathcal{A} \times \mathcal{B}$ does not. Algebra $\mathcal{A}$ has the universe $A = \{x, y\}$, and fundamental operations defined in the following way:

$$f_0{}^{\mathcal{A}}(x_1, x_2, \ldots, x_n) = f_1(x_1, x_2),$$

and all the other fundamental operations are the first projection, i.e. for $g \in \mathcal{F}_m$, $g \neq f_0$,

$$g^{\mathcal{A}}(x_1, x_2, \ldots, x_m) = x_1.$$

Algebra $\mathcal{B}$ has the universe $B = \{a, b, c, d\}$, and the fundamental operations defined by

$$f_0{}^{\mathcal{B}}(x_1, x_2, \ldots, x_n) = f_2(x_1, x_2),$$

and for $g \in \mathcal{F}_m$, $g \neq f_0$,

$$g^{\mathcal{B}}(x_1, x_2, \ldots, x_m) = x_1.$$

It is easy to see that both $\mathcal{A}$ and $\mathcal{B}$ have the complex algebra of subalgebras, because they are conservative algebras. In order to prove that the algebra $\mathcal{A}\times\mathcal{B}$ has no complex algebra of subalgebras, we have to find $X_1, \ldots, X_m \in \mathrm{Sub}(\mathcal{A}\times\mathcal{B})$ such that for some $g \in \mathcal{F}_m$ it holds

$$(g^{\mathcal{A}\times\mathcal{B}})^+(X_1, \ldots X_m) \notin \mathrm{Sub}(\mathcal{A}\times\mathcal{B}).$$

Let $X = \{\langle x, c\rangle, \langle y, b\rangle\}$ and $Y = \{\langle y, d\rangle, \langle y, a\rangle\}$. It is easy to see that $X, Y \in \mathrm{Sub}(\mathcal{A}\times\mathcal{B})$, because

$$\begin{gathered} f_0{}^{\mathcal{A}\times\mathcal{B}}(\langle x_1, a_1\rangle, \langle x_2, a_2\rangle, \ldots, \langle x_n, a_n\rangle) = \\ \langle f_0{}^{\mathcal{A}}(x_1, x_2, \ldots, x_n), f_0{}^{\mathcal{B}}(a_1, a_2, \ldots, a_n)) = \\ \langle f_1(x_1, x_2), f_2(a_1, a_2)\rangle = \langle x_1, a_1\rangle \cdot \langle x_2, a_2\rangle, \end{gathered}$$

where $\cdot$ is the binary operation of the groupoid $\mathcal{T}_1 \times \mathcal{T}_2$. As we have seen in the proof of Lemma 2, $X \in \mathrm{Sub}(\mathcal{T}_1 \times \mathcal{T}_2)$, so $\langle x_1, a_1\rangle \cdot \langle x_2, a_2\rangle \in X$, and $X$ is closed under the operation $f_0{}^{\mathcal{A}\times\mathcal{B}}$. As $X$ is also closed under the other fundamental operations of the algebra $\mathcal{A}\times\mathcal{B}$, it follows that $X \in \mathrm{Sub}(\mathcal{A}\times\mathcal{B})$. Analogously, $Y \in \mathrm{Sub}(\mathcal{A}\times\mathcal{B})$.

Let us prove that $(f_0{}^{\mathcal{A}\times\mathcal{B}})^+(X, Y, Y, \ldots, Y) \notin \mathrm{Sub}(\mathcal{A}\times\mathcal{B})$:

$$\begin{gathered} (f_0{}^{\mathcal{A}\times\mathcal{B}})^+(X, Y, Y, \ldots, Y) = \\ = \bigcup\{f_0{}^{\mathcal{A}\times\mathcal{B}}(\langle x_1, a_1\rangle, \langle x_2, a_2\rangle, \ldots, \langle x_n, a_n\rangle) \mid \langle x_1, a_1\rangle \in X, \langle x_2, a_2\rangle \in Y, \ldots, \\ \ldots \langle x_n, a_n\rangle \in Y\} = \\ = \bigcup\{\langle f_0{}^{\mathcal{A}}(x_1, x_2, \ldots, x_n), f_0{}^{\mathcal{B}}(a_1, a_2, \ldots, a_n)\rangle \mid \langle x_1, a_1\rangle \in X, \langle x_2, a_2\rangle \in Y, \ldots, \\ \ldots \langle x_n, a_n\rangle \in Y\} = \\ \bigcup\{\langle f_1(x_1, x_2), f_2(a_1, a_2)\rangle \mid \langle x_1, a_1\rangle \in X, \langle x_2, a_2\rangle \in Y\} = \{\langle x, c\rangle, \langle x, a\rangle, \langle y, b\rangle\}. \end{gathered}$$

Let us prove that $Z = \{\langle x, c\rangle, \langle x, a\rangle, \langle y, b\rangle\} \notin \mathrm{Sub}(\mathcal{A}\times\mathcal{B})$:

$$\begin{gathered} f_0{}^{\mathcal{A}\times\mathcal{B}}(\langle x, a\rangle, \langle y, b\rangle, \langle y, b\rangle, \ldots, \langle y, b\rangle) = \langle f_0{}^{\mathcal{A}}(x, y, y, \ldots, y), f_0{}^{\mathcal{B}}(a, b, b, \ldots, b)\rangle = \\ = \langle f_1(x, y), f_2(a, b)\rangle = \langle x, b\rangle \notin Z. \end{gathered}$$

Consequently, the algebra $\mathcal{A}\times\mathcal{B}$ does not have the complex algebra of subalgebras. $\square$

## 3. The class of finite CaSa algebras is not globally determined

If $K$ is a class of algebras, and $\mathcal{A}$ and $\mathcal{B}$ are isomorphic algebras from $K$, then $\mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B})$ are obviously isomorphic. It is natural to ask whether the converse is true, i.e. is it true that for any $\mathcal{A}, \mathcal{B}$ from $K$ it holds:

$$\mathcal{P}(\mathcal{A}) \cong \mathcal{P}(\mathcal{B}) \Rightarrow \mathcal{A} \cong \mathcal{B} \ ?$$

If the class $K$ has this property, we say that $K$ is a **globally determined class**. The first result in this direction comes from 1967, when T. Tamura and J. Shafer proved that the class of groups is globally determined. It follows immediately from this result that rings are globally determined. Later, Tamura proved that some other important classes of semigroups, such as completely simple semigroups, completely 0-simple semigroups, left (right) zero semigroups and rectangular bands, are also globally determined. Nevertheless, the class of all semigroups is not globally determined, which was proved by E. M. Mogiljanskaja. In particular, involutive semigroups are not globally determined ([8]). An important result was obtained by Y. Kobayashi ([15]). He proved that semilattices are globally determined, which implies the same result for lattices and Boolean algebras. It is proved in [3] that the class of groupoids of tournaments is globally determined. Unary algebras have been also investigated in this context. A. Drapal ([9]) proved the following theorem:

**Theorem 5.** *([9]) Finite (partial) monounary algebras are globally determined. while the class of all monounary algebras is not globally determined.*

As monounary algebras have (trivially) complex algebras of subalgebras, we immediately obtain the following result:

**Corollary 2.** *The class of all algebras which has CaSa is not globally determined.*

To prove a stronger result on *finite* algebras having complex algebras of subalgebras we will use a result and construction from [16]. For a group $G$, by a $G$**-algebra** we mean a permutation representation $\langle A, G\rangle$ considered as a unary algebra.

**Proposition 3.** *Let $G$ be a group. Then any $G$-algebra $\langle A, G\rangle$ satisfies the generalized entropic property.*

*Proof.* The unary algebra $\langle A, G\rangle$ satisfies the generalized entropic property if for any two permutations $f, g \in G$ there exists a term $t$ such that the identity $g(f(x)) \approx f(t(x))$ holds in $\langle A, G\rangle$. As $G$ is a group, for the term $t$ we can take $t(x) = f^{-1}(g(f(x)))$. $\square$

**Proposition 4.** *Let $G$ be a group. Then any $G$-algebra has the complex algebra of subalgebras.*

*Proof.* It follows from Proposition 3 and Theorem 2. □

In the sequel we suppose that $G$ and $A$ are finite. The set of fixed-points of $g \in G$ or a subgroup $H$ of $G$ in the representation $\langle A, G\rangle$ we denote by $\mathrm{Fix}_A(g)$ and $\mathrm{Fix}_A(H)$, respectively. The **permutation character** of the representation $\langle A, G\rangle$ is the function $\chi : G \to \mathbf{N}$ such that $\chi(g) = |\mathrm{Fix}_A(g)|$. The proof of the following result from group theory can be found, for example, in [13]:

**Proposition 5.** *Let $G$ be any non-cyclic finite group. Then $G$ has non-isomorphic representations with the same permutation character.*

**Proposition 6.** *([16]) Let $G$ be a finite group, $\langle A, G\rangle$ and $\langle B, G\rangle$ two permutation representation of $G$. If $\langle A, G\rangle$ and $\langle B, G\rangle$ have the same permutation character, then they have isomorphic complex algebras.*

**Theorem 6.** *The class of all finite algebras having complex algebras of subalgebras is not globally determined.*

*Proof.* It follows from Propositions 4, 5 and 6. □

## References

[1] Adaricheva, K., Pilitowska, A., Stanovsky, D., On complex algebras of subalgebras. arXiv:math/0610832v1 [math.RA] 2006.

[2] Bošnjak, I., Madarász, R., On power structures. Algebra and Discrete Mathematics 2 (2003), 14-35.

[3] Bošnjak, I., On complex algebras. Ph.D. thesis, University of Novi Sad 2002. (In Serbian)

[4] Brink, C., Power structures and their applications, preprint. Department of Mathematics, University of Cape Town (1992), pp. 152.

[5] Brink, C., Power structures. Algebra Univers. 30 (1993), 177-216.

[6] Burris, S., Sankappanavar, H. P., A course in universal algebra. New York: Springer-Verlag 1981.

[7] Crvenković S., Dolinka I., Marković, P., A survey of algebra of tournaments. Novi Sad J. of Math. Vol. 29 No. 2 (1999), 95-130, Proc. VIII Int. Conf. "Algebra and Logic" (Novi Sad, 1998).

[8] Crvenković, S., Dolinka, I., Vinčić, M., Involution semigroups are not globally determined. Semigroup Forum 62 (2001), 477-481.

[9] Drapal, A., Globals of unary algebras. Czech. Math. J. 35 (1985), 52-58.

[10] Evans, T., Properties of algebras almost equivalent to identities. J. London Math. Soc. 35 (1962), 53-59.

[11] Gould, M., Iskra, J. A., Tsinakis C., Globally determined lattices and semilattices. Algebra Univers. 19 (1984), 137-141.

[12] Gould, M., Iskra, J. A., Globally determined classes of semigroups. Semigroup Forum 28 (1984), 1-11.

[13] Isaacs, I. M., Character Theory of Finite Groups. Academic Press 1976.

[14] Jónsson, B., Tarski, A., Boolean algebras with operators I, II. Amer. J. Math. 73 (1951), 891-939; 74 (1952), 127-167.

[15] Kobayashi, Y., Semilattices are globally determined. Semigroup Forum Vol. 29 (1984), 217-222.

[16] Lukács, E., Globals of G-algebras. Houston J. Math. 13 (1987), 241-244.

[17] Mogiljanskaja E. M., Non-isomorphic semigroups with isomorphic semigroups of subsets. Semigroup Forum Vol. 6 (1973), 330-333.

[18] Romanowska, A., Smith, J. D. H., Modal Theory - An Algebraic Approach to Order, Geometry, and Convexity. Berlin: Heldermann Verlag 1985.

[19] Romanowska, A., Smith, J. D. H., Subalgebra systems of idempotent entropic algebras. J. Algebra 120 (1989), 247-262.

[20] Romanowska, A., Smith, J. D. H., On the structure of the subalgebra systems of idempotent entropic algebras. J. Algebra 120 (1989), 262-283.

[21] Stonehewer, S. E., Old, Recent and New Results on Quasinormal Subgroups. Irish Math. Soc. Bulletin 56 (2005), 125-133.

[22] Whitney, S., Théories linéaries. Ph.D. thesis, Université Laval, Québec 1977.

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