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ON THE CONVERGENCE OF A MULTICOMPONENT THREE LEVEL ALTERNATING DIRECTION DIFFERENCE SCHEME

Boško S. Jovanović

Abstract. A multicomponent alternating direction finite difference scheme for solving the wave equation with several variables is considered. Its stability and convergence are investigated in the case when the solution of the initial–boundary value problem belongs to a Sobolev space.

We consider the first initial–boundary value problem (IBVP) for the wave equation

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \Delta u + f\,, \qquad (x,\,t)\in Q = \Omega\times(0,\,T) = (0,\,1)^n\times(0,\,T)\,,\\
&u(x,\,0) = u_0(x)\,, \qquad \frac{\partial u(x,\,0)}{\partial t} = u_1(x)\,, \qquad x\in\Omega\,,\\
&u(x,\,t) = 0, \qquad x\in\Gamma=\partial\Omega\,, \quad t\in(0,\,T)\,.
\end{aligned} \tag{1}$$

We assume that the generalized solution of IBVP (1) belongs to the Sobolev space $W_2^s(Q)$, $s\ge 2$ [4]. In this case there exists a trace $u|_{t=t'}\in W_2^{s-1/2}(\Omega)\subset L_2(\Omega)$. We also assume that the solution u can be oddly extended in space variables outside the domain Ω, preserving the Sobolev class.

Let $\overline{\omega}$ be a uniform mesh in $\overline{\Omega}$ with the step size h. Let us set $\omega=\overline{\omega}\cap\Omega$, $\gamma=\overline{\omega}\setminus\omega$ and $\omega_i=\omega\cup\{x=(x_1,\,\dots,\,x_n)\in\gamma\,|\,x_i=0\}$. Let $\overline{\theta}$ be a uniform mesh on $[-\tau/2,\,T]$ with the step size τ and $\theta=\overline{\theta}\cap(0,\,T)$. Finally, let $\overline{Q}_{h\tau}=\overline{\omega}\times\overline{\theta}$. For a function v defined on the mesh $\overline{Q}_{h\tau}$ we introduce the finite–difference operators v_{x_i}, $v_{\bar{x}_i}$, v_t and $v_{\bar{t}}$ in a usual manner [5]. Let us denote $v=v(x,\,t)$, $\hat{v}=v(x,\,t+\tau)$ and $\check{v}=v(x,\,t-\tau)$.

Let H_h be the set of discrete functions defined on the mesh $\overline{\omega}$, which vanish on γ. Let us denote

$$\Lambda_i\,v=\begin{cases}-v_{x_i\bar{x}_i}\,, & x\in\omega\\ 0\,, & x\in\gamma\end{cases} \qquad \text{and} \qquad \Lambda\,v=\sum_{i=1}^n\Lambda_i\,v\,.$$

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The unit operator on H_h will be denoted by I.

We introduce the following discrete inner product

$$(v,\, w)_\omega = h^n \sum_{x\in\omega} v(x)\, w(x)$$

and norms

$$\|v\|_\omega = (v,\, v)_\omega^{1/2} = \Big(h^n \sum_{x\in\omega} v^2(x)\Big)^{1/2} \qquad \text{and} \qquad \|v\|_{\omega_i} = \Big(h^n \sum_{x\in\omega_i} v^2(x)\Big)^{1/2} .$$

For a linear, selfadjoint and positive operator L on H_h with $\|v\|_L$ we denote so called "energy" norm

$$\|v\|_L = (L\, v,\, v)_\omega^{1/2} .$$

In particular

$$\|v\|_{\Lambda_i} = (\Lambda_i\, v,\, v)_\omega^{1/2} = \|v_{x_i}\|_{\omega_i} .$$

With T_i and T_t we denote the Steklov averaging operators in space variables x_i and time variable t (see [2])

$$T_i f(x,\, t) = \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} f(x_1,\, \dots\, ,\, x_i',\, \dots\, ,\, x_n,\, t)\, dx_i' \;,$$
$$T_t f(x,\, t) = \frac{1}{\tau}\int_{t-\tau/2}^{t+\tau/2} f(x_1,\, \dots\, ,\, x_n,\, t')\, dt' \;.$$

Finally, C will stand for a positive generic constant, independent of h and τ.

We approximate IBVP (1) with the following alternating direction finite–difference scheme (FDS) [1]

$$\begin{aligned} &(I+\sigma\tau^2\, \Lambda_i)\, v_{t\bar t}^i + \sum_{j=1}^n \Lambda_j v^j = \widetilde{f} \equiv T_1\cdots T_n T_t f\,, \qquad t\in\theta\,, \\ &v^i\big|_{t=\mp\tau/2} = T_1\cdots T_n\,(u_0 \mp 0.5\,\tau\, u_1)\,, \qquad i=1,\,2,\,\dots\,,\,n \end{aligned} \tag{2}$$

where σ is a free weight parameter. FDS (2) represents a system of n unknown mesh functions v^i. They can be determined paralelly, contrary to the other variant of the alternating direction method, such as the factorized scheme

$$(I+\sigma\tau^2\, \Lambda_1)\cdots(I+\sigma\tau^2\, \Lambda_n)\, v_{t\bar t} + \Lambda v = f\,.$$

The errors defined as $z^i = T_1\cdots T_n\, u - v^i$ satisfy the FDS

$$\begin{aligned} &(I+\sigma\tau^2\, \Lambda_i)\, z_{t\bar t}^i + \sum_{j=1}^n \Lambda_j z^j = \varphi^i\,, \qquad t\in\theta\,, \\ &z_t^i\big|_{t=-\tau/2} = \eta\,, \qquad 0.5\,(z^i+\hat z^i)\big|_{t=-\tau/2} = \xi\,, \qquad i=1,\,2,\,\dots\,,\,n \end{aligned} \tag{3}$$

where

$$\begin{aligned}
\varphi^i &= T_1 \cdots T_n \left[\left(u_{t\bar{t}} - T_t \frac{\partial^2 u}{\partial t^2} \right) + \sum_{j=1}^{n} \left(T_t \frac{\partial^2 u}{\partial x_j^2} - u_{x_j \bar{x}_j} \right) - \sigma \tau \, u_{x_i \bar{x}_i t \bar{t}} \right], \\
\eta &= T_1 \cdots T_n \left(T_t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right) \Big|_{t=0}, \\
\xi &= 0.5 \, T_1 \cdots T_n \left(u|_{t=-\tau/2} - 2 \; u|_{t=0} + u|_{t=\tau/2} \right).
\end{aligned}$$

To prove the stability and the convergence of the FDS (2) we represent equation (3) in matrix form

$$\begin{aligned}
(\mathbf{I} + \sigma \tau^2 \Lambda) \, \mathbf{z}_{t\bar{t}} + \mathbf{E} \, \Lambda \mathbf{z} = \mathbf{f}, \qquad t \in \theta, \\
\mathbf{z}_t|_{t=-\tau/2} = \mathbf{b}, \qquad 0.5 \, (\mathbf{z} + \hat{\mathbf{z}})|_{t=-\tau/2} = \mathbf{d},
\end{aligned} \tag{4}$$

where $\mathbf{z} = (z^1, \dots, z^n)^T$, $\mathbf{f} = (\varphi^1, \dots, \varphi^n)^T$, $\mathbf{b} = (\eta, \dots, \eta)^T$, $\mathbf{d} = (\xi, \dots, \xi)^T$, $\mathbf{I} = \mathrm{diag}\,(I, \dots, I)$, $\Lambda = \mathrm{diag}\,(\mathit{\Lambda}_1, \dots, \mathit{\Lambda}_n)$ and

$$\mathbf{E} = \begin{pmatrix} I & I & \dots & I \\ I & I & \dots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \dots & I \end{pmatrix}.$$

Let us also define the inner product and norm of vector–functions

$$(\mathbf{z}, \mathbf{w}) = \sum_{i=1}^{n} (z^i, w^i)_\omega, \qquad \|\mathbf{z}\| = (\mathbf{z}, \mathbf{z})^{1/2}.$$

Applying Λ to (4) we obtain a FDS in canonical form (see [5])

$$\mathbf{C} \, \mathbf{z}_{t\bar{t}} + \mathbf{A} \, \mathbf{z} = \mathbf{g}, \tag{5}$$

where $\mathbf{A} = \Lambda \, \mathbf{E} \, \Lambda = \mathbf{A}^* \geq \mathbf{0}$, $\mathbf{C} = \Lambda + \sigma \tau^2 \Lambda^2 = \mathbf{C}^* > \mathbf{0}$ and $\mathbf{g} = \Lambda \mathbf{f}$. According to Samarski's stability theory [5] the FDS (5) is stable when

$$\mathbf{C} - 0.25 \, \tau^2 \, \mathbf{A} > \mathbf{0}.$$

For $\sigma \geq n/4$ we have

$$\begin{aligned}
\big((\mathbf{C} - 0.25 \, \tau^2 \, \mathbf{A}) \, \mathbf{z}, \mathbf{z} \big) &= (\Lambda \, \mathbf{z}, \mathbf{z}) + \sigma \tau^2 (\Lambda \, \mathbf{z}, \Lambda \, \mathbf{z}) - 0.25 \, \tau^2 (\mathbf{E} \, \Lambda \, \mathbf{z}, \Lambda \, \mathbf{z}) \\
&= \sum_{i=1}^{n} (\mathit{\Lambda}_i \, z^i, z^i)_\omega + \sigma \tau^2 \sum_{i=1}^{n} (\mathit{\Lambda}_i \, z^i, \mathit{\Lambda}_i \, z^i)_\omega - 0.25 \, \tau^2 \Big\| \sum_{i=1}^{n} \mathit{\Lambda}_i \, z^i \Big\|_\omega^2 \\
= \sum_{i=1}^{n} \|z^i\|_{\mathit{\Lambda}_i}^2 + (\sigma - n/4) \, \tau^2 \sum_{i=1}^{n} \|\mathit{\Lambda}_i z^i\|_\omega^2 &+ 0.25 \, \tau^2 \sum_{i=2}^{n} \sum_{j=1}^{i-1} \|\mathit{\Lambda}_i \, z^i - \mathit{\Lambda}_j \, z^j\|_\omega^2 \\
&\geq \sum_{i=1}^{n} \|z^i\|_{\mathit{\Lambda}_i}^2 = \|\mathbf{z}\|_\Lambda^2,
\end{aligned}$$

which means that

$$\mathbf{C} - 0.25\,\tau^2\,\mathbf{A} \geq \Lambda > \mathbf{0}\,,$$

and, consequently, FDS (5) is stable.

Using energy method, multiplying (5) by $\hat{\mathbf{z}} - \check{\mathbf{z}}$, we obtain a priori estimate

$$\max_{t\in\theta} N(\mathbf{z}) \leq N(\mathbf{z})|_{t=-\tau/2} + \tau \sum_{t\in\theta} \|\mathbf{g}\|_{(\mathbf{C}-0.25\,\tau^2\,\mathbf{A})^{-1}}\,, \tag{6}$$

where

$$N^2(\mathbf{z}) = \big\|\mathbf{z}_t\big\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \Big\|\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\Big\|^2_{\mathbf{A}}\,.$$

Other standard a priori estimates (see [5]) do not hold because operators $\mathbf{A}$ and $\mathbf{C}$ do not comute.

Further

$$N^2(\mathbf{z}) = \big\|\mathbf{z}_t\big\|^2_{\mathbf{C}-0.25\,\tau^2\,\mathbf{A}} + \Big\|\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\Big\|^2_{\mathbf{A}} \geq \big\|\mathbf{z}_t\big\|^2_{\Lambda} + \Big\|\Lambda\,\frac{\mathbf{z}+\hat{\mathbf{z}}}{2}\Big\|^2_{\mathbf{E}}$$
$$= \sum_{i=1}^{n} \big\|z^i_t\big\|^2_{\Lambda_i} + \Big\|\sum_{i=1}^{n} \Lambda_i\,\frac{z^i+\hat{z}^i}{2}\Big\|^2_{\omega} \equiv \big\|\mathbf{z}\big\|^2_2\,,$$
$$N^2(\mathbf{z})\big|_{t=-\tau/2} \leq \|\mathbf{b}\|^2_{\mathbf{C}} + \|\mathbf{d}\|^2_{\mathbf{A}} = \sum_{i=1}^{n} \big\|\eta\big\|^2_{\Lambda_i+\sigma\,\tau^2\,\Lambda_i^2} + \Big\|\sum_{i=1}^{n} \Lambda_i\,\xi\Big\|^2_{\omega}\,,$$
$$\big\|\mathbf{g}\big\|_{(\mathbf{C}-0.25\,\tau^2\,\mathbf{A})^{-1}} \leq \big\|\mathbf{g}\big\|_{\Lambda^{-1}} = \big\|\mathbf{f}\big\|_{\Lambda} = \Big(\sum_{i=1}^{n} \big\|\varphi^i\big\|^2_{\Lambda_i}\Big)^{1/2}\,.$$

From here and (6), for $\tau \sim h$ (i.e. $C_1\,h \leq \tau \leq C_2\,h$), we obtain

$$\max_{t\in\theta} \|\mathbf{z}\|_2 \leq \sum_{i=1}^{n} \Big(\big\|\eta_{x_i}\big\|_{\omega_i} + \big\|\xi_{x_i\bar{x}_i}\big\|_{\omega} + \tau \sum_{t\in\theta} \big\|\varphi^i_{x_i}\big\|_{\omega_i}\Big)\,. \tag{7}$$

To prove the convergence of FDS (2) we must estimate the terms $\varphi^i_{x_i}$, η_{x_i} and $\xi_{x_i\bar{x}_i}$. That can be done using the Bramble–Hilbert lemma, in the same way as in [2]. For $\tau \sim h$ in such a way we obtain the following convergence rate estimate

$$\max_{t\in\theta} \|\mathbf{z}\|_2 \leq C\,h^{s-3}\,\|u\|_{W_2^s(Q)}\,, \qquad 3 \leq s \leq 5\,. \tag{8}$$

Remark. In some cases the assumption $\tau \sim h$ can be omited. For example, for $s = 5$ terms $\varphi^i_{x_i}$, η_{x_i} and $\xi_{x_i\bar{x}_i}$ can be represented in integral form, wherefrom directly follows

$$\max_{t\in\theta} \|\mathbf{z}\|_2 \leq C\,(h^2+\tau^2)\,\|u\|_{W_2^5(Q)}\,.$$

Another group of convergence rate estimates can be obtained in the following way. Applying $\Lambda_i\,(I+\sigma\tau\Lambda_i)^{-1}$ to (3), after summation on i we obtain

$$z_{t\bar{t}} + Az = \psi\,, \qquad t \in \theta\,,$$
$$z_t\big|_{t=-\tau/2} = \eta\,, \qquad 0.5\,(z+\hat{z})\big|_{t=-\tau/2} = \xi\,, \tag{9}$$

where

$$z = \Lambda^{-1}\sum_{i=1}^{n} \Lambda_i\, z^i\,, \qquad A=\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \Lambda_i\,(I+\sigma\tau\,\Lambda_i)^{-1}\,, \qquad \psi=\Lambda^{-1}\sum_{i=1}^{n} A_i\,\varphi^i\,.$$

For $\sigma \ge n/[4\,(1-\alpha)]$, $0<\alpha<1$, we have $0<\alpha\, I \le I-0.25\,\tau^2 A \le I$, so the FDS (9) is absolutelly stable.

The operators A and Λ satisfy the relations $I \le (I-0.25\,\tau^2 A)^{-1} \le \alpha^{-1} I$ and $A \le \Lambda$. In the case when $\tau \sim h$ we also have $\beta\,\Lambda \le A$, $0<\beta<1$. Using these relations and the energy method [5] we obtain the a priori estimate

$$\begin{aligned}\max_{t\in\theta} \|z\|_1 &\equiv \max_{t\in\theta}\Big(\big\|z_t\big\|_\omega^2 + \Big\|\frac{z+\hat z}{2}\Big\|_\Lambda^2\Big)^{1/2}\\ &\le C\Big(\|\eta\|_\omega + \sum_{i=1}^{n}\big\|\xi_{x_i}\big\|_{\omega_i} + \tau\sum_{t\in\theta}\sum_{i=1}^{n}\|\varphi^i\|_\omega\Big).\end{aligned} \tag{10}$$

Similarly, applying operator A^{k-1} $(k=2,\,3,\,\ldots)$ to (9) and repeating the same procedure, we obtain

$$\begin{aligned}\max_{t\in\theta} \|z\|_k &\equiv \max_{t\in\theta}\Big(\big\|z_t\big\|_{\Lambda^{k-1}}^2 + \Big\|\frac{z+\hat z}{2}\Big\|_{\Lambda^k}^2\Big)^{1/2}\\ &\le C\Big(\|\eta\|_{\Lambda^{k-1}} + \big\|\xi\big\|_{\Lambda^k} + \tau\sum_{t\in\theta}\sum_{i=1}^{n}\|\varphi^i\|_{\Lambda^{k-1}}\Big).\end{aligned} \tag{11}$$

In such a way, the problem of deriving the convergence rate estimate for the FDS (9), or (2), is now reduced to estimation of the right hand side terms in (10) and (11). Using the Bramble–Hilbert lemma, in the same manner as in the previous case, from (10–11) we obtain

$$\max_{t\in\theta}\|z\|_k \le C\,h^{s-k-1}\,\|u\|_{W_2^s(Q)}\,, \qquad k+1\le s\le k+3\,; \qquad k=1,\,2,\,\ldots \tag{12}$$

Analogous results for the parabolic case are obtained in [3].

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University of Belgrade, Faculty of Mathematics, 11 001 Belgrade, POB 550, Yugoslavia