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A GENERALIZED CLASS OF STARLIKE FUNCTIONS ASSOCIATED WITH THE WRIGHT HYPERGEOMETRIC FUNCTION

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Abstract. In terms of Wright's generalized hypergeometric function we define some classes of analytic functions. The class generalize well known classes of starlike functions. Necessary and sufficient coefficient bounds are given for functions in this class. Further distortion bounds, extreme points and results on partial sums are investigated.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting functions f which are univalent in U. Also we denote by $\mathcal{T}$, the class of analytic functions with negative coefficients introduced by Silverman [14] consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0,\ n = 2, 3, \dots ;\ z \in U). \tag{2}$$

For functions $f \in \mathcal{A}$ given by (1) and $g(z) \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad z \in U,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U.$$

Definition 1. Let $k \geq 0,\ 0 \leq \gamma < 1$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}(k, \gamma)$ if it satisfies the condition

$$\operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} + k\frac{z^2 f''(z)}{f(z)} \right\} > \gamma, \quad z \in U.$$

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The class $\mathcal{S}(k,\gamma)$ was studied in [8] (see also [6, 9, 11]). In particular the class

$$\mathcal{S}(\gamma) := \mathcal{S}(0,\gamma)$$

is the well-known class of starlike functions.

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_p, A_p$ and $\beta_1, B_1 \ldots,\ \beta_q, B_q$ $(p,q \in N = 1,2,3,\ldots)$ such that

$$1 + \sum_{n=1}^{q} B_n - \sum_{n=1}^{p} A_n \geq 0, \tag{3}$$

the Wright's generalized hypergeometric function [20]

$$ {}_p\Psi_q[(\alpha_1, A_1), \ldots, (\alpha_p, A_p); (\beta_1, B_1), \ldots, (\beta_q, B_q); z] = {}_p\Psi_q[(\alpha_n, A_n)_{1,p}; (\beta_n, B_n)_{1,q}; z]$$

is defined by

$$ {}_p\Psi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] = \sum_{n=0}^{\infty} \{\textstyle\prod_{t=0}^{p} \Gamma\left(\alpha_t + nA_t\right)\}\{\textstyle\prod_{t=0}^{q} \Gamma\left(\beta_t + nB_t\right)\}^{-1} \frac{z^n}{n!},$$

$z \in U$. If $p \leq q+1$, $A_t = 1 (t = 1, \ldots, p)$ and $B_t = 1 (t = 1, \ldots, q)$, we have the relationship:

$$\Theta\, {}_p\Psi_q[(\alpha_n, 1)_{1,p}; (\beta_n, 1)_{1,q}; z] = {}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q; z), \quad z \in U, \tag{4}$$

where ${}_pF_q(\alpha_1, \ldots, \alpha_p;\ \beta_1, \ldots, \beta_q; z)$ is the generalized hypergeometric function and

$$\Theta = \left(\textstyle\prod_{t=0}^{p} \Gamma(\alpha_t)\right)^{-1} \left(\textstyle\prod_{t=0}^{q} \Gamma(\beta_t)\right). \tag{5}$$

In [2] Dziok and Raina defined the linear operator by using Wright's generalized hypergeometric function. Let

$$ {}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] = \Theta z\ {}_p\Psi_q[(\alpha_t, A_t)_{1,p} (\beta_t, B_t)_{1,q}; z], \quad z \in U,$$

and

$$\mathcal{W} = \mathcal{W}[(\alpha_n, A_n)_{1,p}; (\beta_n, B_n)_{1,q}] : A \to A$$

be a linear operator defined by

$$\mathcal{W}f(z) := z\ {}_p\phi_q[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}; z] * f(z), \quad z \in U.$$

We observe that, for f of the form (1), we have

$$\mathcal{W}f(z) = z + \sum_{n=2}^{\infty} \sigma_n\ a_n z^n, \quad z \in U, \tag{6}$$

where

$$\sigma_n = \frac{\Theta\ \Gamma(\alpha_1 + A_1(n-1)) \ldots \Gamma(\alpha_p + A_p(n-1))}{(n-1)!\Gamma(\beta_1 + B_1(n-1)) \ldots \Gamma(\beta_q + B_q(n-1))}\ ,$$

and Θ is given by (5). In view of the relationship (4) the linear operator (6) includes the Dziok-Srivastava operator [3] (see [4]).

Corresponding to the family $\mathcal{S}(\gamma,k)$ we define the class $\mathcal{W}_q^p(k,\gamma)$ of function f of the form (1) such that

$$\operatorname{Re}\left\{\frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)}+k\frac{z^2(\mathcal{W}f(z))''}{\mathcal{W}f(z)}\right\}>\gamma,\quad z\in U. \tag{7}$$

We also let

$$\mathcal{T}\mathcal{W}_q^p(k,\gamma)=\mathcal{T}\cap\mathcal{W}_q^p(k,\gamma).$$

Further we define some subclasses of the class $\mathcal{W}_q^p(k,\gamma)$ as given below.

Suppose that: If $A_t=1$ $(t=1,\dots,p)$ and $B_t=1(t=1,\dots,q)$, $p=2$ and $q=1$ we have

(i) $\alpha_1=\delta+1\,(\delta>-1)$, $\alpha_2=1$, $\beta_1=1$, then

$$\mathcal{W}(\delta+1,1;1)f(z)\equiv D^\delta f(z):=\frac{z}{(1-z)^{\delta+1}}*f(z)$$

is called Ruscheweyh derivative of order δ $(\delta>-1)$(see [12]). A function $f\in\mathcal{A}$ is in $\mathcal{RS}(\gamma,k)$ if

$$\operatorname{Re}\left\{\frac{z(D^\delta f(z))'}{D^\delta f(z)}+k\frac{z^2(D^\delta f(z))''}{D^\delta f(z)}\right\}>\gamma,\quad z\in U, \tag{8}$$

(ii) $\alpha_1=a\,(a>0)$, $\alpha_2=1$, $\beta_1=c\,(c>0)$,

$$\mathcal{W}(a,1;c)f(z)\equiv\mathcal{L}(a,c)f(z):=\left(\sum_{n=0}^{\infty}\tfrac{(a)_n}{(c)_n}z^{n+1}\right)*f(z)$$

called Carlson and Shaffer operator [1]. A function $f\in\mathcal{A}$ is in $\mathcal{LS}(\gamma,k)$ if

$$\operatorname{Re}\left\{\frac{z(\mathcal{L}(a,c)f(z))'}{\mathcal{L}(a,c)f(z)}+k\frac{z^2(\mathcal{L}(a,c)f(z))''}{\mathcal{L}(a,c)f(z)}\right\}>\gamma,\quad z\in U, \tag{9}$$

(iii) $\alpha_1=2$ $\alpha_2=1$, $\beta_1=2-\eta$,

$$\mathcal{W}(2,1;2-\eta)f(z)\equiv\Omega^\eta f(z)=\Gamma(2-\eta)z^\eta D_z^\eta f(z)$$

where $(\eta\in\mathcal{R};\eta\neq 2,3,4,\dots)$ the operator $\Omega^\eta f(z)$ was introduced by Owa and Srivastava [19]. A function $f\in\mathcal{A}$ is in $\mathcal{GS}(\gamma,k)$ if

$$\operatorname{Re}\left\{\frac{z(\Omega^\eta f(z))'}{\Omega^\eta f(z)}+k\frac{z^2(\Omega^\eta f(z))''}{\Omega^\eta f(z)}\right\}>\gamma,\quad z\in U. \tag{10}$$

In this paper we obtain a sufficient coefficient condition for functions f given by (1) to be in the class $W_q^p(k,\gamma)$ and we show that it is also necessary condition for functions to belong to the class. Distortion results and extreme points for functions in $\mathcal{T}\mathcal{W}_q^p(k,\gamma)$ are obtained. Finally, we investigate partial sums for the class $\mathcal{T}\mathcal{W}_q^p(k,\gamma)$.

2. Coefficients inequalities and distortions theorem

First we obtain a sufficient condition for functions in $\mathcal{W}_q^p(k,\gamma)$.

THEOREM 1. *Let $0 \leq \gamma < 1$ and $k \geq 0$. Suppose also that $f(z) \in \mathcal{A}$ is given by (1). If*

$$\sum_{n=2}^{\infty} \left[kn^2 + n - kn - \gamma\right] \sigma_n \left|a_n\right| \leq 1-\gamma, \tag{11}$$

then $f \in \mathcal{W}_q^p(k,\gamma)$.

Proof. If we put

$$P(z) = \frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)} + k\frac{z^2(\mathcal{W}f(z))''}{\mathcal{W}f(z)}, \quad z \in U,$$

then it is sufficient to prove that

$$|P(z)-1| < 1-\gamma, \quad z \in U.$$

Indeed if $f(z) \equiv z$ $(z \in U)$, then we have $P(z) \equiv 1$ $(z \in U)$. This implies that the desired in equality (11). If $f(z) \neq z$ $(|z| = r < 1)$, then there exist a coefficient $\sigma_n a_n \neq 0$ for some $n \geq 2$. It follows that

$$\sum_{n=2}^{\infty} \sigma_n \left|a_n\right| r^n > 0.$$

Further note that the sequence $b_n = kn^2 + n - kn - \gamma$ is increasing. Therefore

$$\sum_{n=2}^{\infty} \left[kn^2 + n - kn - \gamma\right] \sigma_n |a_n| r^n > (1-\gamma) \sum_{n=2}^{\infty} \sigma_n |a_n| r^n,$$

which implies that $\sum\limits_{n=2}^{\infty} \sigma_n |a_n| < 1$.

Thus by coefficient inequality (11), we obtain

$$\begin{aligned}
|P(z)-1| &= \left| \frac{\sum\limits_{n=2}^{\infty}(n-1)(nk+1)\sigma_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\sigma_n a_n z^{n-1}} \right| < \frac{\sum\limits_{n=2}^{\infty}(n-1)(nk+1)\sigma_n |a_n|}{1-\sum\limits_{n=2}^{\infty}\sigma_n |a_n|} \\
&= \frac{\sum\limits_{n=2}^{\infty}(n-1)(nk+1)\sigma_n |a_n|}{1-\sum\limits_{n=2}^{\infty}\sigma_n |a_n|} \\
&< \frac{\sum\limits_{n=2}^{\infty}\left[kn^2+n-kn-\gamma\right]\sigma_n |a_n| - (1-\gamma)\sum\limits_{n=2}^{\infty}\sigma_n |a_n|}{1-\sum\limits_{n=2}^{\infty}\sigma_n |a_n|} \\
&\leq \frac{(1-\gamma)-(1-\gamma)\sum\limits_{n=2}^{\infty}\sigma_n |a_n|}{1-\sum\limits_{n=2}^{\infty}\sigma_n |a_n|} \leq 1-\gamma, \quad z \in U.
\end{aligned}$$

Hence we obtain

$$\operatorname{Re}\left\{\frac{z(\mathcal{W}f(z))'}{\mathcal{W}f(z)}+k\frac{z^2(\mathcal{W}f(z))''}{\mathcal{W}f(z)}\right\}=\operatorname{Re}(P(z))>1-(1-\gamma)=\gamma \quad (z\in U).$$

That is $f\in\mathcal{W}_q^p(k,\gamma)$. This completes the proof. ■

In the next theorem, we show that the condition (11) is also necessary for functions from the class $\mathcal{TW}_q^p(k,\gamma)$.

Theorem 2. *Let f be given by* (2). *Then the function f belongs to the class $\mathcal{TW}_q^p(k,\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}\left[kn^2+n-kn-\gamma\right]\sigma_n a_n\leq 1-\gamma. \tag{12}$$

Proof. In view of Theorem 1 we need only to show that $f\in\mathcal{TW}_q^p(k,\gamma)$ satisfies the coefficient inequality (11). If $f\in\mathcal{TW}_q^p(k,\gamma)$ then the function

$$P(z)=\frac{z(\mathcal{W}f(z))'+kz^2(\mathcal{W}f(z))''}{\mathcal{W}f(z)} \quad (z\in U)$$

satisfies

$$\operatorname{Re}\{P(z)\}>\gamma \quad (z\in U).$$

This implies that

$$\mathcal{W}f(z)=z-\sum_{n=2}^{\infty}\sigma_n a_n z^n\neq 0 \quad (z\in U\setminus\{0\}).$$

Noting that $\frac{\mathcal{W}f(r)}{r}$ is the real continuous function in the open interval $(0,1)$ with $f(0)=1$, we have

$$1-\sum_{n=2}^{\infty}\sigma_n a_n r^{n-1}>0 \quad (0<r<1). \tag{13}$$

Now

$$\gamma<P(r)=\frac{1-\sum\limits_{n=2}^{\infty}n\sigma_n a_n r^{n-1}-k\sum\limits_{n=2}^{\infty}\left(n^2-n\right)\sigma_n a_n r^{n-1}}{1-\sum\limits_{n=2}^{\infty}\sigma_n a_n r^{n-1}}$$

and consequently by (13) we obtain

$$\sum_{n=2}^{\infty}\left[kn^2+n-kn-\gamma\right]\sigma_n a_n r^{n-1}<1-\gamma.$$

Letting $r\to 1$, we get (12). This proves the converse part. ■

Corollary 1. *If a function f of the form* (2) *belongs to the class $\mathcal{TW}_q^p(k,\gamma)$, then*

$$a_n\leq\frac{1-\gamma}{[kn^2+n-kn-\gamma]\,\sigma_n}, \quad n=2,3,\ldots.$$

The equality holds for the functions

$$h_n(z) = z - \frac{1-\gamma}{[kn^2+n-kn-\gamma]\,\sigma_n} z^n, \quad z \in U,\ n = 2, 3, \dots . \tag{14}$$

Next we obtain the distortion bounds for functions belonging to the class $\mathcal{TW}_q^p(k,\gamma)$.

THEOREM 3. *Let f be in the class $\mathcal{TW}_q^p(k,\gamma)$, $|z| = r < 1$. If the sequence*

$$\left\{\left[kn^2+n-kn-\gamma\right]\sigma_n\right\}_{n=2}^{\infty}$$

is nondecreasing, then

$$r - \frac{1-\gamma}{(k\gamma+2)\sigma_2} r^2 \le |f(z)| \le r + \frac{1-\gamma}{(2k-\gamma+2)\sigma_2} r^2. \tag{15}$$

If the sequence $\left\{\frac{kn^2+n-kn-\gamma}{n}\sigma_n\right\}_{n=2}^{\infty}$ *is nondecreasing, then*

$$1 - \frac{2(1-\gamma)}{(2k-\gamma+2)\sigma_2} r \le |f'(z)| \le 1 + \frac{2(1-\gamma)}{(2k-\gamma+2)\sigma_2} r. \tag{16}$$

The result is sharp. The extremal function is the function h_2 of the form (14).

Proof. Since $f \in \mathcal{TW}_q^p(k,\gamma)$, we apply Theorem 2 to obtain

$$(2k-\gamma+2)\sigma_2 \sum_{n=2}^{\infty} |a_n| \le \sum_{n=2}^{\infty} \left[kn^2+n-kn-\gamma\right]\sigma_n |a_n| \le 1-\gamma.$$

Thus

$$|f(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} |a_n| \le r + \frac{1-\gamma}{(2k-\gamma+2)\sigma_2} r^2.$$

Also we have

$$|f(z)| \ge |z| - |z|^2 \sum_{n=2}^{\infty} |a_n| \ge r - \frac{1-\gamma}{(2k-\gamma+2)\sigma_2} r^2$$

and (15) follows. In similar manner for f', the inequalities

$$|f'(z)| \le 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \le 1 + |z| \sum_{n=2}^{\infty} na_n$$

and

$$\sum_{n=2}^{\infty} n|a_n| \le \frac{2(1-\gamma)}{(2k-\gamma+2)\sigma_2}$$

lead to (16). This completes the proof. ∎

COROLLARY 2. *Let f be in the class $\mathcal{TW}_q^p(k,\gamma)$, $|z| = r < 1$. If*

$$p > q,\ \alpha_{q+1} \ge 1,\ \alpha_j \ge \beta_j \text{ and } A_j \ge B_j \ \ (j = 2, \dots, q), \tag{17}$$

then the assertions (15), (16) *holds true.*

Proof. From (17) we have that the sequences

$$\left\{\left[kn^2+n-kn-\gamma\right]\sigma_n\right\}_{n=2}^{\infty} \quad \text{and} \quad \left\{\frac{kn^2+n-kn-\gamma}{n}\sigma_n\right\}_{n=2}^{\infty}$$

are nondecreasing. Thus, by Theorem 3, we have Corollary 2. ∎

Theorem 4. *Let $h_1(z) = z$ and h_n be defined by* (14). *Then $f \in \mathcal{TW}_q^p(k,\gamma)$ if and only if f can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n h_n(z), \quad \mu_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \mu_n = 1. \tag{18}$$

Proof. If a function f is of the form (18), then

$$\begin{aligned} f(z) &= \sum_{n=2}^{\infty} \left[kn^2 + n - kn - \gamma\right] \sigma_n \frac{1-\gamma}{\left[kn^2 + n - kn - \gamma\right] \sigma_n} \mu_n \\ &= \sum_{n=2}^{\infty} \mu_n (1-\gamma) = (1-\mu_1)(1-\gamma) \leq 1-\gamma. \end{aligned}$$

Hence $f \in \mathcal{TW}_q^p(k,\gamma)$.

Conversely, if f is in the class $\mathcal{TW}_q^p(k,\gamma)$, then we may set $\mu_n = \frac{\left[kn^2+n-kn-\gamma\right]\sigma_n}{1-\gamma}$, $n \geq 2$ and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$. Then the function f is of the form (18) and this completes the proof. ■

3. Partial sums

For a function $f \in A$ given by (1) Silverman [15] and Silvia [18] investigated the partial sums f_1 and f_m defined by

$$f_1(z) = z; \text{ and } f_m(z) = z + \sum_{n=2}^{m} a_n z^n, \; m = 2, 3 \ldots. \tag{19}$$

We consider in this section partial sums of functions in the class $\mathcal{TW}_q^p(k,\gamma)$ and obtain sharp lower bounds for the ratios of real part of f to f_m and f' to f'_m.

Theorem 5. *Let a function f of the form* (2) *belong to the class $\mathcal{TW}_q^p(k,\gamma)$ and assume* (17). *Then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\} \geq 1 - \frac{1}{d_{m+1}}, \quad z \in U, \; m \in N \tag{20}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\} \geq \frac{d_{m+1}}{1+d_{m+1}}, \quad z \in U, \; m \in N, \tag{21}$$

where

$$d_n := \frac{kn^2 + n - kn - \gamma}{1-\gamma} \sigma_n. \tag{22}$$

Proof. By (17) it is not difficult to verify that

$$d_{n+1} > d_n > 1, \quad n = 2, 3 \ldots. \tag{23}$$

Thus by Theorem 2 we have

$$\sum_{n=2}^{m} a_n + d_{m+1} \sum_{n=m+1}^{\infty} a_n \leq \sum_{n=2}^{\infty} d_n a_n \leq 1. \tag{24}$$

Setting

$$g(z) = d_{m+1}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{d_{m+1}}\right)\right\} = 1 + \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{m} a_n z^{n-1}}, \tag{25}$$

it suffices to show that $\operatorname{Re} g(z) \geq 0, \quad z \in U$. Applying (24), we find that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{n}|a_n| - d_{m+1}\sum\limits_{n=m+1}^{\infty}|a_n|} \leq 1, \;\; z \in U,$$

which readily yields the assertion (20) of Theorem 5. In order to see that

$$f(z) = z + \frac{z^{m+1}}{d_{m+1}}, \quad z \in U, \tag{26}$$

gives the sharp result, we observe that for $z = re^{i\pi/m}$ we have

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{d_{m+1}} \overset{z\to 1^-}{\longrightarrow} 1 - \frac{1}{d_{m+1}}.$$

Similarly, if we take

$$\begin{aligned} h(z) &= (1+d_{m+1})\left\{\frac{f_m(z)}{f(z)} - \frac{d_{m+1}}{1+d_{m+1}}\right\} \\ &= 1 - \frac{(1+d_{n+1})\sum\limits_{n=m+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} a_n z^{n-1}}, \quad z \in U, \end{aligned}$$

and making use of (24), we can deduce that

$$\left|\frac{h(z)-1}{h(z)+1}\right| \leq \frac{(1+d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|}{2-2\sum\limits_{n=2}^{m}|a_n| - (1-d_{m+1})\sum\limits_{n=m+1}^{\infty}|a_n|} \leq 1, \;\; z \in U,$$

which leads us immediately to the assertion (21) of Theorem 5. The bound in (21) is sharp for each $m \in N$ with the extremal function f given by (26), and the proof is complete. ■

THEOREM 6. *Let a function f of the form* (1) *belong to the class $\mathcal{TW}_q^p(k,\gamma)$ and assume* (17). *Then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq 1 - \frac{m+1}{d_{m+1}} \tag{27}$$

and

$$\operatorname{Re}\left\{\frac{f_m'(z)}{f'(z)}\right\} \geq \frac{d_{m+1}}{m+1+d_{m+1}}, \tag{28}$$

where d_m *is defined by* (22).

Proof. By setting

$$g(z) = d_{m+1}\left\{\frac{f'(z)}{f_m'(z)} - \left(1 - \frac{m+1}{d_{m+1}}\right)\right\}, \quad z \in U,$$

and

$$h(z) = [(m+1) + d_{m+1}]\left\{\frac{f_m'(z)}{f'(z)} - \frac{d_{m+1}}{m+1+d_{m+1}}\right\}, \quad z \in U,$$

the proof is analogous to that of Theorem 5, and we omit the details. ∎

4. Integral transform and integral means

First we prove that the class $\mathcal{TW}_q^p(k,\gamma)$ is closed under some integral transforms.

For $f \in \mathcal{A}$ we define the Komatu operator [10]

$$V_{\delta,c}(f)(z) = \frac{(c+1)^\delta}{\Gamma(\delta)} \int_0^1 t^{c-1}\left(\log\frac{1}{t}\right)^{\delta-1} f(tz)dt, \;\; c > -1,\, \delta \geq 0.$$

In particular, for $\delta = 1$ we obtain well-known the Bernardi operator.

A simple calculation gives

$$V_{\delta,c}(f)(z) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n z^n.$$

First we show that the class $\mathcal{TW}_q^p(k,\gamma)$ is closed under $V_{\delta,c}(f)(z)$.

THEOREM 7. *Let* $f(z) \in \mathcal{TW}_q^p(k,\gamma)$. *Then* $V_{\delta,c}(f)(z) \in \mathcal{TW}_q^p(k,\gamma)$.

Proof. We need to prove that

$$\sum_{n=2}^{\infty}\frac{\Phi(\lambda,\gamma,k,n)}{(1-\gamma)}\left(\tfrac{c+1}{c+n}\right)^\delta a_n \leq 1, \tag{29}$$

where

$$\Phi(\gamma,k,n) = [kn^2 + n - kn - \gamma]\sigma_n. \tag{30}$$

On the other hand by Theorem 2, $f \in \mathcal{TW}_q^p(k,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\Phi(\lambda,\gamma,k,n)}{(1-\gamma)} a_n \leq 1,$$

where $\Phi(\gamma,k,n)$ is defined in (30). Since $\frac{c+1}{c+n} < 1$, then (29) holds and the proof is complete. ∎

THEOREM 8. *Let* $f \in \mathcal{TW}_q^p(k,\gamma)$. *Then* $V_{\delta,c}(f)(z)$ *is starlike of order* ξ, $0 \le \xi < 1$ *in* $|z| < R_1$, *where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^{\delta} \frac{(1-\xi)\Phi(\gamma,k,n)}{(n-\xi)(1-\gamma)} \right]^{\frac{1}{n-1}} \qquad (n \ge 2),$$

where $\Phi(\gamma,k,n)$ *is defined in* (30).

Proof. It is sufficient to prove

$$\left| \frac{z(V_{\delta,c}(f)(z))'}{V_{\delta,c}(f)(z)} - 1 \right| < 1-\xi, \quad |z| < R_1. \tag{31}$$

For the left hand side of (30) we have,

$$\left| \frac{z(V_{\delta,c}(f)(z))'}{V_{\delta,c}(f)(z)} - 1 \right| = \left| \frac{\sum\limits_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n} \right)^{\delta} a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^{\delta} a_n z^{n-1}} \right| \le \frac{\sum\limits_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n} \right)^{\delta} a_n |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^{\delta} a_n |z|^{n-1}}.$$

The last expression is less than $1-\xi$ if

$$|z|^{n-1} < \left(\frac{c+n}{c+1} \right)^{\delta} \frac{(1-\xi)\Phi(\gamma,k,n)}{(n-\xi)(1-\gamma)}.$$

Therefore, the proof is complete. ■

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following.

THEOREM 9. *Let* $f \in \mathcal{TW}_q^p(k,\gamma)$. *Then* $V_{\delta,c}(f)(z)$ *is convex of order* $0 \le \xi < 1$ *in* $|z| < R_2$ *where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^{\delta} \frac{(1-\xi)\Phi(\gamma,k,n)}{n(n-\xi)(1-\gamma)} \right]^{\frac{1}{n-1}} \qquad (n \ge 2),$$

where $\Phi(\lambda,\gamma,k,2)$ *is defined in* (30).

In [14], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [16] and settled in [17], that

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\eta} d\theta \le \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^{\eta} d\theta,$$

for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [17], he also proved his conjecture for the subclasses $T^*(\gamma)$ and $C(\gamma)$ of T.

Now, we prove the Silverman's conjecture for the functions in the family $\mathcal{TW}_q^p(k,\gamma)$.

Lemma 1. [7] *If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta \, d\theta \le \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta \, d\theta. \tag{32}$$

Applying Lemma 1, Theorem 2 and Theorem 4 we prove the following result.

Theorem 10. *Suppose $f \in \mathcal{TW}_q^p(k,\gamma)$, $\eta > 0$, $0 \le \lambda \le 1$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{1-\gamma}{\Phi(\gamma,k,2)} z^2,$$

where $\Phi(\gamma,k,2)$ is defined in (30). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \le \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{33}$$

Proof. By (2), it is equivalent to prove that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{(1-\gamma)}{\Phi(\lambda,\gamma,k,2)} z\right|^\eta d\theta.$$

By Lemma 1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\Phi(\lambda,\gamma,k,2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\gamma}{\Phi(\lambda,\gamma,k,2)} w(z), \tag{34}$$

and using (11), we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \tfrac{\Phi(\gamma,k,n)}{1-\gamma} a_n z^{n-1}\right| \le |z| \sum_{n=2}^{\infty} \frac{\Phi(\gamma,k,n)}{1-\gamma} a_n \le |z|.$$

This completes the proof. ∎

5. Neighbourhood results

In this section we discuss neighbourhood results of the class $\mathcal{TW}_q^p(k,\gamma)$. Following [5, 13], we define the δ-neighbourhood of function $f(z) \in T$ by

$$N_\delta(f) := \left\{ h \in T : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|a_n - d_n| \le \delta \right\}. \tag{35}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ h \in T : g(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \le \delta \right\}. \tag{36}$$

THEOREM 11. *If*

$$\delta := \frac{2(1-\gamma)}{\Phi(\lambda, \gamma, k, 2)}, \tag{37}$$

where $\Phi(\lambda, \gamma, k, 2)$ *is defined in* (30), *then* $\mathcal{TW}_q^p(k, \gamma) \subset N_\delta(e)$.

Proof. For $f \in \mathcal{TW}_q^p(k, \gamma)$, Theorem 2 immediately yields $\Phi(\lambda, \gamma, k, 2) \sum_{n=2}^{\infty} a_n \le 1 - \gamma$, so that

$$\sum_{n=2}^{\infty} a_n \le \frac{(1-\gamma)}{\Phi(\lambda, \gamma, k, 2)}. \tag{38}$$

On the other hand, from (11) and (38) that

$$\begin{aligned}\sigma_2(1+\lambda) \sum_{n=2}^{\infty} n a_n &\le (1-\gamma) - (1+\lambda)(k-\gamma)\sigma_2 \sum_{n=2}^{\infty} a_n \\ &\le (1-\gamma) - (1+\lambda)(k-\gamma)\sigma_2 \frac{(1-\gamma)}{\Phi(\lambda, \gamma, k, 2)} \le \frac{2(1-\gamma)}{[2+k-\gamma]},\end{aligned}$$

that is

$$\sum_{n=2}^{\infty} n a_n \le \frac{2(1-\gamma)}{\Phi(\lambda, \gamma, k, 2)} := \delta \tag{39}$$

which, in view of the definition (36) proves Theorem 11. ∎

REMARK. We observe that, if we specialize the parameters of the class $\mathcal{TW}_q^p(k, \gamma)$, we obtain the analogous results for the classes $\mathcal{RS}(\gamma, k)$, $\mathcal{LS}(\gamma, k)$ and $\mathcal{GS}(\gamma, k)$ defined in this paper.

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