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NEW EXTENDED WEYL TYPE THEOREMS

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Abstract. In this paper we introduce and study the new properties (ab), (gab), (aw) and (gaw) as a continuation of our previous article [4], where we introduced the two properties (b) and (gb).

Among other, we prove that if T is a bounded linear operator acting on a Banach space X, then T possesses property (gb) if and only if T possesses property (gab) and $\text{ind}(T - \lambda I) = 0$ for all $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$; where $\sigma_{SBF_+^-}(T)$ is the essential semi-B-Fredholm spectrum of T and $\sigma_a(T)$ is the approximate spectrum of T. We prove also that T possesses property (gaw) if and only if T possesses property (gab) and $E_a(T) = \Pi_a(T)$.

1. Introduction

Throughout this paper, X will denote an infinite-dimensional complex Banach space, $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. For $T \in \mathcal{L}(X)$, let T^*, $N(T)$, $R(T), \sigma(T)$ and $\sigma_a(T)$ denote respectively the adjoint, the null space, the range, the spectrum and the approximate point spectrum of T. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) = \dim N(T)$and $\beta(T) = \text{codim} R(T)$. Recall that an operator $T \in \mathcal{L}(X)$ is called upper semi-Fredholm if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in \mathcal{L}(X)$ is called lower semi-Fredholm if $\beta(T) < \infty$. Let $SF_+(X)$ denotes the class of all upper semi-Fredholm operators. If $T \in \mathcal{L}(X)$ is either an upper or a lower semi-Fredholm operator, then T is called a semi-Fredholm operator, and the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. Let $F(X)$ denotes the class of all Fredholm operators. Define $SF_+^-(X) = \{T \in SF_+(X) : \text{ind}(T) \leq 0\}$. The class of Weyl operators is defined by $W(X) = \{T \in F(X) : \text{ind}(T) = 0\}$. The classes of operators defined above generate the following spectra : The Weyl spectrum is defined by $\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W(X)\}$, while the Weyl essential approximate spectrum is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(X)\}$.

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Following Coburn [10], we say that Weyl's theorem holds for $T \in \mathcal{L}(X)$ if $\sigma(T) \setminus \sigma_W(T) = E^0(T)$, where $E^0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\}$. Here and elsewhere in this paper, for $A \subset \mathbb{C}$, isoA is the set of isolated points of A. According to Rakočević [14], an operator $T \in \mathcal{L}(X)$ is said to satisfy a-Weyl's theorem if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where $E_a^0(T) = \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\}$.

For $T \in \mathcal{L}(X)$ and a nonnegative integer n define $T_{[n]}$ to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular, $T_{[0]} = T$). If for some integer n the range space $R(T^n)$ is closed and $T_{[n]}$ is an upper (a lower) semi-Fredholm operator, then T is called an upper (a lower) semi-B-Fredholm operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [7]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. An operator T is said to be a B-Weyl operator [6, Definition 1.1] if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{is not a B-Weyl operator }\}$.

Recall that the ascent $a(T)$ of an operator T is defined by $a(T) = \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, and the descent $\delta(T)$ of T is defined by $\delta(T) = \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ with $\inf \emptyset = \infty$. An operator $T \in \mathcal{L}(X)$ is called Drazin invertible if it has a finite ascent and descent. The Drazin spectrum $\sigma_D(T)$ of an operator T is defined by $\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}$.

Define also the set $LD(X)$ by $LD(X) = \{T \in \mathcal{L}(X) : a(T) < \infty \text{ and } R(T^{a(T)+1}) \text{ is closed }\}$ and $\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin LD(X)\}$. Following [5], an operator $T \in \mathcal{L}(X)$ is said to be left Drazin invertible if $T \in LD(X)$. We say that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I \in LD(X)$, and that $\lambda \in \sigma_a(T)$ is a left pole of T of finite rank if λ is a left pole of T and $\alpha(T - \lambda I) < \infty$. Let $\Pi_a(T)$ denotes the set of all left poles of T and let $\Pi_a^0(T)$ denotes the set of all left poles of T of finite rank. From [5, Theorem 2.8] it follows that if $T \in \mathcal{L}(X)$ is left Drazin invertible, then T is an upper semi-B-Fredholm operator of index less than or equal to 0.

Let $\Pi(T)$ be the set of all poles of the resolvent of T and let $\Pi^0(T)$ be the set of all poles of the resolvent of T of finite rank, that is $\Pi^0(T) = \{\lambda \in \Pi(T) : \alpha(T - \lambda I) < \infty\}$. According to [12], a complex number λ is a pole of the resolvent of T if and only if $0 < \max\ (a(T - \lambda I), \delta(T - \lambda I)) < \infty$. Moreover, if this is true then $a(T - \lambda I) = \delta(T - \lambda I)$. According also to [12], the space $R((T - \lambda I)^{a(T-\lambda I)+1})$ is closed for each $\lambda \in \Pi(T)$. Hence we have always $\Pi(T) \subset \Pi_a(T)$ and $\Pi^0(T) \subset \Pi_a^0(T)$.

We say that Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$, and that a-Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$. Following [6], we say that generalized Weyl's theorem holds for $T \in \mathcal{L}(X)$ if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$, where $E(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda I)\}$ is the set of all isolated eigenvalues of T, and that generalized Browder's theorem holds for $T \in \mathcal{L}(X)$ if $\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$. It is proved in [3, Theorem 2.1] that generalized

Browder's theorem is equivalent to Browder's theorem. In [5, Theorem 3.9], it is shown that an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem, but the converse does not hold in general. Nonetheless and under the assumption $E(T) = \Pi(T)$, it is proved in [8, Theorem 2.9] that generalized Weyl's theorem is equivalent to Weyl's theorem.

Let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, $SBF_+^-(X) = \{T \in SBF_+(X) \ : \ \text{ind}(T) \leq 0\}$. The upper B-Weyl spectrum of T is defined by $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SBF_+^-(X)\}$. We say that generalized a-Weyl's theorem holds for $T \in \mathcal{L}(X)$ if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in \text{iso}\sigma_a(T) \ : \ 0 < \alpha(T - \lambda I)\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathcal{L}(X)$ obeys generalized a-Browder's theorem if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$. It is proved in [3, Theorem 2.2] that generalized a-Browder's theorem is equivalent to a-Browder's theorem, and it is known from [5, Theorem 3.11] that an operator satisfying generalized a-Weyl's theorem satisfies a-Weyl's theorem, but the converse does not hold in general and under the assumption $E_a(T) = \Pi_a(T)$ it is proved in [8, Theorem 2.10] that generalized a-Weyl's theorem is equivalent to a-Weyl's theorem.

Following [15], we say that $T \in \mathcal{L}(X)$ possesses property (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. The property (w) has been studied in [1, 15]. In [1, Theorem 2.8], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general.

We say that $T \in \mathcal{L}(X)$ possesses property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gw) has been introduced and studied in [2]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [2] that an operator possessing property (gw) possesses property (w) but the converse is not true in general. According to [4], an operator $T \in \mathcal{L}(X)$ is said to possess property (gb) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$, and is said to possess property (b) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$. It is shown [4, Theorem 2.3] that an operator possessing property (gb) possesses property (b) but the converse is not true in general.

In this paper we define and study the new properties (ab), (gab), (aw) and (gaw) in connection with Weyl type theorems [5]. We prove that an operator $T \in \mathcal{L}(X)$ possessing property (gab) possesses property (ab) but the converse is not true in general as shown by Example 2.3, nonetheless and under the assumption that $\Pi(T) = \Pi_a(T)$ we prove that the two properties are equivalent.

We show also that an operator possessing property (ab) satisfies Browder's theorem, and we show that an operator possessing property (gab) satisfies generalized Browder's theorem, but the converses of these results are not true in general.

We prove that an operator possessing property (gb) possesses property (gab) and that an operator possessing property (b) possesses property (ab), but the converses of these theorems are not true in general.

We prove also that an operator possessing property (gaw) possesses property (aw) but the converse is not in general, and we show that an operator possessing property (gaw) possesses property (gab) but the converse does not hold in general, however, under the assumption that $E_a(T) = \Pi_a(T)$ the two properties are equivalent.

In the last part, as a conclusion, we give a diagram summarizing the different relations between Weyl type theorems and properties, extending a similar diagram given in [4].

2. Properties (gab) and (ab)

For $T \in \mathcal{L}(X)$, let $\Delta(T) = \sigma(T) \setminus \sigma_W(T)$ and $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$, $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$ and $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.

Definition 2.1. A bounded linear operator $T \in \mathcal{L}(X)$ is said to possess property (ab) if $\Delta(T) = \Pi_a^0(T)$, and is said to possess property (gab) if $\Delta^g(T) = \Pi_a(T)$.

Theorem 2.2 *Let $T \in \mathcal{L}(X)$. If T possesses property (gab), then T possesses property (ab).*

Proof. Suppose that T possesses property (gab), then $\Delta^g(T) = \Pi_a(T)$. If $\lambda \in \Delta(T)$, then $\lambda \in \Delta^g(T) = \Pi_a(T)$. Hence λ is a left pole of T. Since $T - \lambda I \in SF_+(X)$, then $\alpha(T - \lambda I)$ is finite. So $\lambda \in \Pi_a^0(T)$.

Conversely, if $\lambda \in \Pi_a^0(T)$, then λ is a left pole of T, $\alpha(T-\lambda I)$ and $a(T-\lambda I)$ are finite. Since T possesses property (gab), we have $\lambda \in \Delta^g(T)$ and $\mathrm{ind}(T - \lambda I) = 0$. As $a(T - \lambda I) < \infty$, then $\delta(T - \lambda I) < \infty$. Hence $\alpha(T - \lambda I) = \beta(T - \lambda I) < \infty$. Thus $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Finally, we have $\Delta(T) = \Pi_a^0(T)$, and T possesses property (ab). ■

The converse of Theorem 2.2 does not hold in general as shown by the following example.

Example 2.3. Let R be the unilateral right shift operator defined on the Hilbert space $\ell^2(\mathbb{N})$. It is known from [13, Theorem 3.1] that $\sigma(R) = D(0, 1)$ is the closed unit disc in $\mathbb{C}$, $\sigma_a(R) = C(0, 1)$ is the unit circle of $\mathbb{C}$ and R has empty eigenvalues set. Moreover, $\sigma_W(R) = D(0, 1)$ and $\Pi_a^0(R) = \emptyset$. Define T on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = 0 \oplus R$. Then $\sigma(T) = D(0, 1)$, $N(T) = \ell^2(\mathbb{N}) \oplus \{0\}$, $\sigma_a(T) = \{0\} \cup C(0, 1)$, $\sigma_W(T) = D(0, 1)$, $\sigma_{BW}(T) = D(0, 1)$, $\Pi_a(T) = \{0\}$ and $\Pi_a^0(T) = \emptyset$. Hence $\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) = \emptyset \neq \Pi_a(T)$. Consequently, T possesses property (ab) but it does not possess property (gab).

Theorem 2.4. *Let $T \in \mathcal{L}(X)$. IfT possesses property (ab), then T satisfies Browder's theorem.*

Proof. Suppose that T possesses property (ab), then $\Delta(T) = \Pi_a^0(T)$. If $\lambda \in \Delta(T)$, then $a(T - \lambda I)$ is finite and $T - \lambda I$ is a Weyl operator. Hence $\delta(T - \lambda I) < \infty$ and $T - \lambda I$ is Drazin invertible. Since $\lambda \in \sigma(T)$, then $\lambda \in \Pi^0(T)$.

Conversely, if $\lambda \in \Pi^0(T)$, then $T-\lambda I$ is a Weyl operator and $\lambda \in \sigma(T)$, so $\Pi^0(T) \subset \Delta(T)$. Hence $\Delta(T) = \Pi^0(T)$, i.e. T satisfies Browder's theorem. ■

The following example shows that a-Browder's theorem and Browder's theorem do not imply property (ab).

Example 2.5. Let $R \in \mathcal{L}(\ell^2(\mathbb{N}))$ be the unilateral right shift and $S \in \mathcal{L}(\ell^2(\mathbb{N}))$ the operator defined by $S(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_2, \xi_3, \xi_4, \dots)$.

Consider the operator T defined on the Banach space $X = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = R \oplus S$, then $\sigma(T) = D(0,1)$ is the closed unit disc in $\mathbb{C}$, $\mathrm{iso}\sigma(T) = \emptyset$ and $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is the unit circle of $\mathbb{C}$, $\sigma_{SF_+^-}(T) = C(0,1)$ and $\sigma_W(T) = D(0,1)$. This implies that $\sigma(T) \setminus \sigma_W(T) = \emptyset$ and $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \{0\}$. Moreover, we have $\Pi_a^0(T) = \{0\}$. Hence T satisfies a-Browder's theorem and so T satisfies Browder's theorem. But T does not possess property (ab).

However, from Theorem 2.2 and Theorem 2.4 we have immediately the following result.

Corollary 2.6. *Let $T \in \mathcal{L}(X)$. Then T possesses property (ab) if and only if T satisfies Browder's theorem and $\Pi^0(T) = \Pi_a^0(T)$. In particular, If T possesses property (gab), then T satisfies generalized Browder's theorem.*

The converse of the last assertion of the preceding corollary is not true in general. Indeed, if we consider the operator T defined in Example 2.5, then $\sigma_{BW}(T) = D(0,1)$, $\sigma_{SBF_+^-}(T) = C(0,1)$ and $\Pi(T) = \emptyset$. This implies that $\sigma(T) \setminus \sigma_{BW}(T) = \emptyset$and $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{0\}$. Moreover, we have $\Pi_a(T) = \{0\}$. Hence T satisfies generalized a-Browder's theorem and T satisfies generalized Browder's theorem. But T does not possess property (gab) because $\sigma(T) \setminus \sigma_{BW}(T) \neq \Pi_a(T)$.

However, we have the following result.

Corollary 2.7. *Let $T \in \mathcal{L}(X)$. Then T possesses property (gab) if and only if T satisfies generalized Browder's theorem and $\Pi(T) = \Pi_a(T)$.*

Proof. Assume that T possesses property (gab), i.e. $\Delta^g(T) = \Pi_a(T)$. Then from Corollary 2.6, T satisfies generalized Browder's theorem, i.e. $\Delta^g(T) = \Pi(T)$, and hence $\Pi(T) = \Pi_a(T)$.

Conversely, assume that T satisfies generalized Browder's theorem and $\Pi(T) = \Pi_a(T)$, then $\Delta^g(T) = \Pi(T)$ and $\Pi(T) = \Pi_a(T)$, which implies that $\Delta^g(T) = \Pi_a(T)$ and T possesses property (gab). ■

Theorem 2.8. *Let $T \in \mathcal{L}(X)$. The following statements are equivalent:*

(i) T possesses property (gab);

(ii) T possesses property (ab) and $\Pi(T) = \Pi_a(T)$.

Proof. Assume that T possesses property (gab), then from Theorem 2.2 and Corollary 2.7, T possesses property (ab) and $\Pi(T) = \Pi_a(T)$. Conversely, assume that property (ab) holds for T and $\Pi(T) = \Pi_a(T)$. From Theorem 2.4, T satisfies Browder's theorem. As we know from [3, Theorem 2.1] that Browder's theorem is equivalent to generalized Browdre's theorem, it follows that T satisfies generalized Browder's theorem. Hence we have $\Delta^g(T) = \Pi(T)$. As by hypothesis $\Pi(T) = \Pi_a(T)$, then $\Delta^g(T) = \Pi_a(T)$ and T possesses property (gab). ■

Theorem 2.9. *Let $T \in \mathcal{L}(X)$. Then T possesses property (b) if and only if*

(i) T possesses property (ab);

(ii) $\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a(T)$.

Proof. Suppose that T possesses property (b), then by [4, Theorem 2.5], T satisfies Browder's theorem, i.e. $\Delta(T) = \Pi^0(T)$, and from [4, Corollary 2.7] we have $\Pi^0(T) = \Pi^0_a(T)$. So $\Delta(T) = \Pi^0_a(T)$ and T possesses property (ab). If $\lambda \in \Delta_a(T)$, as T possesses property (b) then $\lambda \in \Pi^0(T)$. So $\mathrm{ind}(T - \lambda I) = 0$. Conversely, assume that T possesses property (ab) and $\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a(T)$. Since T possesses property (ab), then T satisfies Browder's theorem. From [4, Theorem 2.11], we see that T possesses property (b). ■

Now we give a characterization similar to Theorem 2.9, in the case of property (gab).

Theorem 2.10. *Let $T \in \mathcal{L}(X)$. Then T possesses property (gb) if and only if*

(i) T possesses property (gab);

(ii) $\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta^g_a(T)$.

Proof. Assume that T possesses property (gb), then from [4, Corollary 2.8], T satisfies generalized Browder's theorem, i.e. $\Delta^g(T) = \Pi(T)$, and from [4, Corollary 2.9] we have $\Pi(T) = \Pi_a(T)$. Therefore $\Delta^g(T) = \Pi_a(T)$ and T possesses property (gab). If $\lambda \in \Delta^g_a(T)$, as T possesses property (gb) then $\lambda \in \Pi(T)$. Hence $T - \lambda I$ is a B-Weyl operator and so $\mathrm{ind}(T - \lambda I) = 0$. Conversely, assume that T possesses property (gab) and $\mathrm{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta^g_a(T)$. Since T possesses property (gab) then by Corollary 2.7, T satisfies generalized Browder's theorem. From [4, Theorem 2.12], T possesses property (gb). ■

The following example shows that in general properties (ab) and (gab) do not imply properties (b) and (gb) respectively.

Example 2.11. Let R and L denote the unilateral right shift operator and the unilateral left shift operator, respectively on the Hilbert space $\ell^2(\mathbb{N})$ and we consider the operator T defined by $T = L \oplus R \oplus R$. Then $\alpha(T) = 1$, $\beta(T) = 2$ and $a(T) = \infty$. This implies in particular that $0 \notin \sigma_{SF_+^-}(T)$. Since $a(T) = \infty$, it follows that T does not satisfy a-Browder's theorem and so T does not satisfy generalized a-Browder's theorem. From [4, Theorem 2.5], T does not possess property (b) and from [4, Theorem 2.3], T does not possess property (gb). On the other hand, $\sigma(T) = D(0,1)$ the closed unit disc in $\mathbb{C}$, $\Pi_a(T) = \emptyset$ and $\sigma_{BW}(T) = D(0,1)$. Hence $\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$ and T possesses property (gab). So T possesses also property (ab).

Remark 2.12. The properties (gab) and (ab) are not transmitted from an operator to its adjoint. To see this, if we consider the operator T defined in Example 2.5, then $\sigma(T^*) = D(0,1)$, $\sigma_a(T) = C(0,1) \cup \{0\}$, $\sigma_{BW}(T^*) = D(0,1)$. Moreover, we have $\Pi_a(T^*) = \emptyset$. So $\sigma(T^*) \setminus \sigma_{BW}(T^*) = \Pi_a(T^*)$. Hence T^* possesses property (gab), which implies that T^* possesses also property (ab). But $T = (T^*)^*$ does not possess properties (ab) and (gab).

3. Properties (gaw) and (aw)

Definition 3.1. A bounded linear operator is said to possess property (aw) if $\Delta(T) = E_a^0(T)$, and is said to possess property (gaw) if $\Delta^g(T) = E_a(T)$.

Lemma 3.2. *Let $T \in \mathcal{L}(X)$ be an upper semi-B-Fredholm operator. If $\alpha(T) < \infty$, then T is an upper semi-Fredholm operator.*

Proof. Since $T \in SBF_+(X)$, there exists an integer n such that $R(T^n)$ is closed and $T_{[n]}\colon R(T^n) \to R(T^n)$ is an upper semi-Fredholm operator. Since $\alpha(T) < \infty$, it follows from [16, Lemma 3.3] that $\alpha(T^n) < \infty$. As we know that $R(T^n)$ is closed, hence T^n is an upper semi-Fredholm operator. Thus T is also an upper semi-Fredholm operator. ∎

Theorem 3.3. *Let $T \in \mathcal{L}(X)$. If T possesses property (gaw), then T possesses property (aw).*

Proof. Assume that T possesses property (gaw), then $\Delta^g(T) = E_a(T)$. If $\lambda \in \Delta(T)$, then $\lambda \in \Delta^g(T) = E_a(T)$. Hence λ is an eigenvalue of T isolated in $\sigma_a(T)$. Since $T - \lambda I$ is a Weyl operator, then $\alpha(T - \lambda I)$ is finite. So $\lambda \in E_a^0(T)$.

Conversely, if $\lambda \in E_a^0(T)$, then λ is an eigenvalue of T isolated in $\sigma_a(T)$ and $\alpha(T - \lambda I) < \infty$. Since T possesses property (gaw), then $\lambda \in \Delta^g(T)$, and $T - \lambda I$ is an upper semi-B-Fredholm operator. As $\alpha(T - \lambda I)$ is finite, then from Lemma 3.2 we have $T - \lambda I$ is an upper semi-Fredholm operator. Since $\mathrm{ind}(T - \lambda I) = 0$, then $T - \lambda I$ is a Weyl operator and $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Finally, we have $\Delta(T) = E_a^0(T)$, and T possesses property (aw). ∎

The converse of Theorem 3.3 does not hold in general as shown by the following example.

Example 3.4. Let T the operator defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by
$$T(x_1, x_2, x_3, \dots) = 0 \oplus (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \dots).$$
Then $\sigma(T) = \{0\}$, $\sigma_W(T) = \{0\}$, $\sigma_{BW}(T) = \{0\}$, $E_a^0(T) = \emptyset$, and $E_a(T) = \{0\}$. Therefore $\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$ and $\sigma(T) \setminus \sigma_{BW}(T) \neq E_a(T)$. So T possesses property (aw) but T does not possess property (gaw).

Theorem 3.5. *Let $T \in \mathcal{L}(X)$. Then T possesses property (gaw) if and only if T possesses property (gab) and $E_a(T) = \Pi_a(T)$.*

Proof. Suppose that T possesses property (gaw), then $\sigma(T)\backslash\sigma_{BW}(T) = E_a(T)$. If $\lambda \in \sigma(T)\setminus\sigma_{BW}(T)$, then $\lambda \in \text{iso}\sigma_a(T)$, and $T-\lambda I$ is an upper semi-B-Fredholm operator of index less or equal than zero. Hence from [5, Theorem 2.8] we have $\lambda \in \Pi_a(T)$. This implies that $\sigma(T)\setminus\sigma_{BW}(T) \subset \Pi_a(T)$. Now if $\lambda \in \Pi_a(T)$, since $\Pi_a(T) \subset E_a(T)$ is always true, then $\lambda \in E_a(T)$, and as T possesses property (gaw) we have $\lambda \in \sigma(T)\setminus\sigma_{BW}(T)$. Hence $\sigma(T)\setminus\sigma_{BW}(T) = \Pi_a(T)$, i.e. T possesses property (gab) and $\Pi_a(T) = E_a(T)$.

Conversely, assume that T possesses property (gab) and $E_a(T) = \Pi_a(T)$. Then $\sigma(T)\setminus\sigma_{BW}(T) = \Pi_a(T)$ and $E_a(T) = \Pi_a(T)$. So $\sigma(T)\setminus\sigma_{BW}(T) = E_a(T)$ and T possesses property (gaw). ■

Similarly to Theorem 3.5 we have the following result in the case of property (aw), which we give without proof.

THEOREM 3.6. *Let $T \in \mathcal{L}(X)$. Then T possesses property (aw) if and only if T possesses property (ab) and $E_a^0(T) = \Pi_a^0(T)$.*

The following example shows that in general properties (gab) and (ab) do not imply properties (gaw) and (aw) respectively.

EXAMPLE 3.7. Let $T \in \mathcal{L}(\ell^2(\mathbb{N}))$ defined by $T(x_1, x_2, x_3, \dots) = (\frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{5}, \dots)$. Then $\sigma(T) = \{0\}$, $\sigma_{BW}(T) = \{0\}$ and $E_a(T) = \{0\}$. This implies that $\sigma(T)\setminus\sigma_{BW}(T) \neq E_a(T)$ and T does not possess property (gaw). Moreover, we have $\sigma_W(T) = \{0\}$ and $E_a^0(T) = \{0\}$. Therefore $\sigma(T)\setminus\sigma_W(T) \neq E_a^0(T)$ and T does not possess property (aw). On the other hand, $\Pi_a(T) = \emptyset$. Then $\sigma(T)\setminus\sigma_{BW}(T) = \Pi_a(T)$, i.e. T possesses property (gab), and so T possesses property (ab).

REMARK 3.8. Properties (gaw) and (aw) are not transmitted from an operator to its adjoint. Indeed, let $T \in \mathcal{L}(\ell^2(\mathbb{N}))$ defined by $T(\xi_1, \xi_2, \xi_3, \dots) = (0, \frac{\xi_1}{2}, \frac{\xi_2}{3}, \dots)$, then $T^*(\xi_1, \xi_2, \xi_3, \dots) = (\frac{\xi_2}{2}, \frac{\xi_3}{3}, \dots)$. Moreover $\sigma(T) = \{0\}$, $\sigma_{BW}(T) = \{0\}$, $\sigma_W(T) = \{0\}$, $E_a(T) = \emptyset$ and $E_a^0(T) = \emptyset$. This implies that $\sigma(T)\setminus\sigma_{BW}(T) = E_a(T)$ and $\sigma(T)\setminus\sigma_W(T) = E_a^0(T)$. Therefore T possesses properties (gaw) and (aw). On the other hand, $\sigma(T^*) = \{0\}$, $\sigma_{BW}(T^*) = \{0\}$, $\sigma_W(T^*) = \{0\}$, $E_a(T^*) = \{0\}$ and $E_a^0(T^*) = \{0\}$. Thus $\sigma(T^*)\setminus\sigma_{BW}(T^*) \neq E_a(T^*)$ and $\sigma(T^*)\setminus\sigma_W(T^*) \neq E_a^0(T^*)$. So T^* does not possesses properties (aw) and (gaw).

We conclude this section by some examples.

EXAMPLES 3.9. 1° Every unilateral right shift operator R defined on $\ell^2(\mathbb{N})$ possesses property (gaw). Indeed, $\sigma_a(R) = C(0,1)$ is the unit circle of $\mathbb{C}$, $\sigma(R) = D(0,1)$ is the closed unit disc in $\mathbb{C}$, $\sigma_{BW}(R) = D(0,1)$ and $E_a(R) = \emptyset$. Hence $\sigma(R)\setminus\sigma_{BW}(R) = E_a(R)$ and R possesses property (gaw).

2° Every unilateral left shift operator L defined on $\ell^2(\mathbb{N})$ possesses property (gaw). Indeed $\sigma_a(L) = D(0,1)$, $\sigma(L) = D(0,1)$, $\sigma_{BW}(L) = D(0,1)$ and $E_a(L) = \emptyset$. Hence $\sigma(L)\setminus\sigma_{BW}(L) = E_a(L)$ and L possesses property (gaw).

4. Conclusion

In this last part, we give a summary of the known Weyl type theorems as in [5], including the properties introduced in [15], [2], [4] and in this paper. We use the abbreviations gaW, aW, gW, W, (gw), (w), (gaw) and (aw) to signify that an operator $T \in \mathcal{L}(X)$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem, Weyl's theorem, property (gw), property (w), property (gaw) and property (aw). Similarly, the abbreviations gaB, aB, gB, B, (gb), (b), (gab) and (ab) have analogous meaning with respect to Browder's theorem or the properties introduced in [4] or the new properties introduced in this paper.

The following table summarizes the meaning of various theorems and properties.

gaW	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	gaB	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
aW	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	aB	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$
gW	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$	gB	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$
W	$\sigma(T) \setminus \sigma_W(T) = E^0(T)$	B	$\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(w)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$	(b)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(gaw)	$\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$	(gab)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$
(aw)	$\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$	(ab)	$\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$

Table 1

In the following diagram, which extends the similar diagram presented in [4], arrows signify implications between various Weyl type theorems, Browder type theorems, property (gw), property (gb), property (gab) and property (gaw). The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$\begin{tikzcd}
gW & gaB & (gaw) \arrow[r, "3.3"] \arrow[d, "3.5"] & (aw) & & & & & \\
(gw) \arrow[u, "{[2]}"] \arrow[r, "{[4]}"] \arrow[d, "{[2]}"] & (gb) \arrow[u, "{[4]}"] \arrow[r, "2.10"] \arrow[d, "{[4]}"] & (gab) \arrow[r, "2.6"] \arrow[d, "2.2"] & gB \arrow[d, "{[5]}"] & gW \arrow[l, "{[5]}"'] \arrow[d, "{[5]}"] & gaW \arrow[l, "{[5]}"'] \arrow[r, "{[5]}"] \arrow[d, "{[5]}"] & gaB \arrow[r, "{[5]}"] \arrow[d, Leftrightarrow, "{[3]}"] & gB \arrow[d, Leftrightarrow, "{[3]}"] \\
(w) \arrow[r, "{[4]}"'] \arrow[d, "{[1]}"] & (b) \arrow[r, "2.9"'] \arrow[d, "{[4]}"] & (ab) \arrow[r, "2.4"'] & B & W \arrow[l, "{[9]}"] & aW \arrow[l, "{[14]}"] \arrow[r, "{[5]}"'] & aB \arrow[r, "{[11]}"'] & B \\
W & aB & (aw) \arrow[u, "3.6"] & & & & &
\end{tikzcd}$$

Table 2

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