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SUBSPACE AND ADDITION THEOREMS FOR EXTENSION AND COHOMOLOGICAL DIMENSIONS. A PROBLEM OF KUZMINOV

V. V. Fedorchuk

Abstract. Let K be either a CW or a metric simplicial complex. We find sufficient conditions for the subspace inequality

$$A \subset X, \quad K \in \mathrm{AE}(X) \Rightarrow K \in \mathrm{AE}(A).$$

For the Lebesgue dimension ($K = S^n$) our result is a slight generalization of Engelking's theorem for a strongly hereditarily normal space X. As a corollary we get the inequality

$$A \subset X \Rightarrow \dim_G A \leq \dim_G B.$$

for a certain class of paracompact spaces X and an arbitrary abelian group G.

As for the addition theorems

$$K \in \mathrm{AE}(A), \;\; L \in \mathrm{AE}(B) \Rightarrow K * L \in \mathrm{AE}(A \cup B),$$
$$\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1,$$

we extend Dydak's theorems for metrizable spaces (G is a ring with unity) to some classes of paracompact spaces.

Introduction

One of the main results of theory of Lebesgue dimension is the inequality

(1) $\quad \dim(A \cup B) \leq \dim A + \dim B + 1.$

This inequality is called *Urysohn–Menger formula.* Inequality (1) was established for separable metric spaces by P.S. Urysohn in [40] (announcement [39]) and was extended to hereditarily normal spaces by Ju.M. Smirnov [37].

When cohomological dimension theory was extended to paracompact spaces by means of sheaves (look at [22], [23], [28]), W.I. Kuzminov posed the following question ([28], Problem 10).

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Let X be a hereditarily paracompact space and let $X = A \cup B$. Is it true that the inequality

(2) $\dim_G X \le \dim_G A + \dim_G B + 1$

holds for an arbitrary abelian group?

L.R. Rubin [39] was the first who gave a partial answer to this question. Namely, he proved that

(3) $\dim_{\mathbb{Z}}(A \cup B) \le \dim_{\mathbb{Z}} A + \dim_{\mathbb{Z}} B + 1$

for subspaces of metrizable spaces. Thereafter J. Dydak and J.J. Walsh [19] proved the inequality (2) for $G = \mathbb{Z}_p$ or G is a subring of the ring of rationals $\mathbb{Q}$, and $\dim_G A$, $\dim_G B \ge 2$. A.N. Dranishnikov and D. Repovš [13] proved the inequality (2) for $G = \mathbb{Z}_p$ or G is subring of $\mathbb{Q}$, $\dim(A \cup B)$ is finite and $\dim_G A = 1$ or $\dim_G B = 1$. At last J. Dydak [16] proved that inequality (2) holds for any ring G with unity. All these results were obtained for metrizable spaces.

A.Ch. Chigogidze [2] using the method of inverse systems and applying Rubin's theorem proved

Theorem A. *If X is a perfectly normal space and $X = A \cup B$, then*
$\dim^0_{\mathbb{Z}} X \le \dim^0_{\mathbb{Z}} A + \dim^0_{\mathbb{Z}} B + 1$. ∎

Here $\dim^0_G X \le n$ means that $K(G, n) \in \mathrm{AE}(X)$. Applying Ghigogidze's technique and using mentioned above Dydak's theorem, one can prove

Theorem B. *If X is a perfectly normal space and $X = A \cup B$, then*
$\dim^0_G X \le \dim^0_G A + \dim^0_G B + 1$
for an arbitrary countable ring with unity G. ∎

It should be mentioned that generally Kuzminov's question has a negative answer. A.N. Dranishnikov, D. Repovš and E.V. Ščepin [14] constructed subsets A and B of $\mathbb{R}^4$ such that

$\dim_{\mathbb{Q}/\mathbb{Z}}(A \cup B) > \dim_{\mathbb{Q}/\mathbb{Z}} A + \dim_{\mathbb{Q}/\mathbb{Z}} B + 1.$

Thereafter J. Dydak [16] proved that if $G \otimes G = 0$, then for any $m \ge 2$ there is a subset A of S^{2m+1} such that $\dim_G A \le m - 1$ and $\dim_G(S^{2m+1} \setminus A) \le m$.

The purpose of this paper is to find general conditions under which Kuzminov's hypothesis is true. Let us start with the following theorem that was proved by P.J. Huber [15], Y. Kodama [27] and I.A. Shvedov [36] (look at [28]).

Theorem C. *If X is a paracompact space, then*
$\dim_G X = \dim^0_G X$. ∎

Combining Theorems B and C we get

Theorem D. *If X is a paracompact perfectly normal space and $X = A \cup B$, then*

$\dim_G X \leq \dim_G A + \dim_G B + 1$

for an arbitrary countable ring with unity G. ■

Our main result is the following generalization of Dydak's theorem ([16], Theorem 1.4, the first part).

Theorem 5.2. *Suppose A, B are subsets of a hereditarily paracompact p_1-space X and G is a ring with unity. Then*

$\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1$

in the following cases:

1) *G is countable;*

2) *X is perfectly normal.*

A space X is said to be a *p_1-space* if X admits a perfect mapping onto a first countable space. Every first countable hereditarily paracompact space is a hereditarily paracompact p_1-space.

We prove also Subspace Theorem 2.20. On the one hand, it generalizes Pasynkov–Zambahidze's [43] and Engelking's [21] theorems for Lebesgue dimension. On the other hand, Theorem 2.20 is a generalization of Dydak's subspace theorem for extension dimension [16]. Corollary 2.22 is a subspace theorem for cohomological dimension.

Another auxiliary result is Theorem 3.4 which is a generalization of Urysohn–Menger addition theorem for Lebesgue dimension. Theorem 3.4 is also a generalization of Dydak's addition theorem for extension dimension of metrizable spaces [16].

Yet another auxiliary result is the following extension of Dranishnikov's theorem [8] to paracompact spaces.

Theorem 4.8. *Let X be a paracompact p_1-space and let $K \in \mathrm{AE}(X)$ be a* CW-*complex such that either:*

1) *K is countable;*

2) *X is perfectly normal.*

Then

$\dim_{H_m(K,\mathbb{Z})} X \leq m$ *for all $m > 0$.*

1. Preliminaries

All spaces are assumed to be normal T_1, all mappings are continuous. Any additional information concerning general topology and dimension theory one can find in [20], [21], [9], [11], [12], [18]. Recall some known notions and facts.

1.1. Definition. Let X and Y be spaces and let $Z \subset X$. The property that all partial mappings $f : Z \to Y$ extend over X will be denoted by $Y \in \mathrm{AE}(X, Z)$. If every mapping $f : Z \to Y$ extends over an open set $U_f \supset Z$, then we write

$Y \in \mathrm{ANE}(X, Z)$. If every partial mapping $f : C \to Y$ on X, C is a closed subset of X, can be extended over X (respectively over an open set $U_f \supset C$), then Y is called an *absolute (neighbourhood) extensor* of X (notation: $Y \in \mathrm{A(N)E}(X)$).

1.2. Proposition. *If $Y \in \mathrm{A(N)E}(X)$ and $F \subset X$ is closed, then $Y \in \mathrm{A(N)E}(F)$.* ∎

Proposition 1.2 yields

1.3. Proposition. *Let $Y \in \mathrm{ANE}(X)$ and let F be a closed subspace of X such that $Y \in \mathrm{AE}(F)$ and $Y \in \mathrm{AE}(C)$ for every closed subset C of X containing in $X \setminus F$. Then $Y \in \mathrm{AE}(X)$.* ∎

1.4. Corollary. *If $X = X_1 \cup \cdots \cup X_n$, where X_i are closed in X, and $Y \in \mathrm{AE}(X_i)$, $i = 1, \ldots, n$, and $Y \in \mathrm{ANE}(X)$, then $Y \in \mathrm{AE}(X)$.* ∎

1.5. Proposition. *If a space $X = \bigoplus_{\alpha \in A} X_\alpha$ is the union of a discrete family of its subspaces X_α such that $Y \in A(N)E(X_\alpha)$ for any $\alpha \in A$, then $Y \in A(N)E(X)$.* ∎

1.6. Proposition ([7], (2.2)). *Let $\{F_\alpha : \alpha \in A\}$ be a locally finite collection of closed sets of a countably paracompact space X. Then there exists a locally finite collection $\{G_\alpha : \alpha \in A\}$ of open sets, with $F_\alpha \subset G_\alpha$, in the following cases:*

1) *A is countable;*

2) *X is collectionwise normal.* ∎

1.7. Proposition. *Let F be a closed subspace of a countably paracompact space X and let $u = \{U_\alpha : \alpha \in A\}$ be a locally finite cover of F by functionally open in F sets. Then u can be extended to a locally finite functionally open cover $v = \{V_\alpha : \alpha \in A\}$ of X in the following cases:*

1) *A is countable;*

2) *X is collectionwise normal.*

Proof. Since the family $\overline{u} = \{\overline{U}_\alpha : \alpha \in A\}$ is locally finite, we can apply Proposition 1.6 to $F_\alpha = \overline{U}_\alpha$. There exists a locally finite collection $\{G_\alpha : \alpha \in A\}$ of open sets in X, with $U_\alpha \subset G_\alpha$. Without loss of generality we can assume that $U_\alpha = G_\alpha \cap F$. Now we can find functionally open sets H_α such that

$$U_\alpha = F \cap H_\alpha \subset H_\alpha \subset G_\alpha.$$

Let $H = \bigcup\{H_\alpha : \alpha \in A\}$ and let $C = X \setminus H$. Then $F \cap C = \varnothing$. Hence there exists a functionally open set W such that

$$C \subset W \subset X \setminus F.$$

Fix some $\alpha_0 \in A$ and set

$$V_{\alpha_0} = H_{\alpha_0} \cup W,\ V_\alpha = H_\alpha \text{ for } \alpha \neq \alpha_0.$$ ∎

Proposition 1.7 implies

1.8. Proposition. *Let F be a closed subspace of a countably paracompact space X. Then every locally finite partition of unity $\{\varphi_\alpha : \alpha \in A\}$ on F can be extended to a locally finite partition of unity on X in the following cases:*

1) *A is countable;*

2) *X is collectionwise normal.* ∎

1.9. Remark. Point 2) for a paracompact space X was proved by J. Dydak ([17], Theorem 8.4).

We say that X is a *p_1-space* if X admits a perfect mapping onto a first countable space.

1.10. Theorem ([17], Theorem 11.1). *Suppose F is a closed subset of a p_1-space X and m is an infinite cardinal number. Then the following conditions are equivalent:*

1) $K \in \mathrm{ANE}(X, F)$ *for every* CW*-complex, with* $\mathrm{card}(K^0) \le m$*;*

2) *every locally finite partition of unity $\{\varphi_\alpha : \alpha \in A\}$ (with* $\mathrm{card}(A) \le m$*) on F extends to a locally finite partition of unity on X.* ∎

Proposition 1.8 and Theorem 1.10 yield

1.11. Theorem. *Let X be a normal space and let K be a CW-complex. Then* $K \in \mathrm{ANE}(X)$ *in the following cases:*

1) *K is compact;*

2) *K^0 is countable and X is a countably paracompact p_1-space;*

3) *X is a collectionwise normal countably paracompact p_1-space.* ∎

1.12. Theorem. *Let X be s space and let K be a metric simplicial (contractible) complex. Then* $K \in \mathrm{ANE}(X)$ ($K \in \mathrm{AE}(X)$) *in the following cases:*

1) *K is countable and topologically complete;*

2) *X is perfectly normal and K is countable;*

3) *X is perfectly normal and collectionwise normal.* ∎

Remark on proof. O. Hanner [24], E. Michael [29], C.H. Dowker [5] proved assertions 1), 2), 3) respectively for a metrizable space K which is an ANE for metrizable spaces. On the other hand, every metric simplicial (contractible) complex is an $\mathrm{ANE}(\mathcal{M})(\mathrm{AE}(\mathcal{M}))$. ∎

1.13. Proposition [7]. *Let A be a closed set of a normal space X, let Y be a space, and let $f : (X \times 0) \cup (A \times I) \to Y$ be a mapping. If f extends over $X \times \{0\} \cup U$, where U is open in $X \times I$ and $A \times I \subset U$, then f extends over $X \times I$.* ∎

1.14. Definition. We say that a pair (X, Y) of spaces satisfies *homotopy extension theorem* (notation: $(X, Y) \in \mathrm{het}$) if the following theorem is fulfilled.

Homotopy Extension Theorem. *Let A be a closed subset of X. Then any mapping $f : (X \times 0) \cup (A \times I) \to Y$ extends over $X \times I$.*

Proposition 1.13 yields

1.15. Proposition. *Let X be a normal space such that $Y \in \mathrm{ANE}(X \times I)$, then $(X, Y) \in \mathrm{het}$.* ∎

1.16. Proposition [4]. *The product $X \times Y$ of a countably paracompact space X and a compact metric space Y is countably paracompact.* ∎

1.17. Proposition [7]. *The product $X \times Y$ of a countably paracompact and collectionwise normal space X and a compact metric space Y is collectionwise normal.* ∎

1.18. Proposition ([4], [26], [32]). *Every perfectly normal space is countably paracompact.* ∎

1.19. Theorem. *Let X be a space and let K be a* CW*-complex. Then $(X, K) \in \mathrm{het}$ in the following cases:*

1) *X is normal, K is compact;*

2) *X is countably paracompact p_1-space, K^0 is countable;*

3) *X is countably paracompact collectionwise normal p_1-space.*

If L is a metric simplicial complex, then $(X, L) \in \mathrm{het}$ in the following cases:

4) *X is countably paracompact, L is countable and topologically complete;*

5) *X is perfectly normal, L is countable;*

6) *X is collectionwise normal and perfectly normal.*

Proof. Point 1) was considered by M. Starbird [39] and K. Morita [34]. Point 2) is a consequence of Theorem 1.11. 2) and Propositions 1.13, 1.15, 1.16. Point 3) is a consequence of Theorem 1.11. 3) and Propositions 1.13, 1.15, 1.17.

Points 4), 5), 6) were considered by C.H. Dowker [7] in more general situation: L is a metrizable ANR. ∎

Recall that a space X *dominates* a space Y (notation: $Y \leq_h X$) if there exist mappings $f : Y \to X$ and $g : X \to Y$ such that $g \circ f \simeq \mathrm{id}_Y$.

1.20. Proposition. *If X dominates $Y, (Z, Y) \in \mathrm{het}$, and $X \in \mathrm{A(N)E}(Z)$, then $Y \in \mathrm{A(N)E}(Z)$.* ∎

Proposition 1.20 yields

1.21. Proposition. *Let X and Y be homotopy equivalent spaces and let $(Z, X) \in \mathrm{het}$, $(Z, Y) \in \mathrm{het}$. Then $X \in \mathrm{A(N)E}(Z)$ if and only if $Y \in \mathrm{A(N)E}(Z)$.* ∎

1.22. Proposition [42]. *Every (countable)* CW*-complex is homotopy equivalent to a (locally finite countable) metric simplicial complex.* ∎

1.23. Definition. Let K and L be spaces and let $\mathcal{P}$ be a class of spaces. We say that K $\mathrm{AE}_{\mathcal{P}}$-*dominates* L (notation: $L \leq_{\mathcal{P}} K$) if for every $X \in \mathcal{P}$ we have

$K \in \mathrm{AE}(X) \Longrightarrow L \in \mathrm{AE}(X)$.

If $L \leq_{\mathcal{P}} K$ and $K \leq_{\mathcal{P}} L$ we write $K =_{\mathcal{P}} L$.

From Proposition 1.18 we get

1.24. Proposition. *If K dominates L and $(X, L) \in$ het for every $X \in \mathcal{P}$, then $L \leq_{\mathcal{P}} K$.* ∎

Proposition 1.20 yields

1.25. Proposition. *Let K and L be homotopy equivalent spaces and let $(X, K) \in$ het, $(X, L) \in$ het for all $X \in \mathcal{P}$. Then $K =_{\mathcal{P}} L$.* ∎

1.26. Definition. We say that a class $\mathcal{K}$ of CW-complexes is equivalent to a class of $\mathcal{L}$ of metric simplicial complexes with respect to a class $\mathcal{P}$ of spaces (notation: $\mathcal{K} \sim_{\mathcal{P}} \mathcal{L}$) if for every $K \in \mathcal{K}$ there exists $L \in \mathcal{L}$ such that $K =_{\mathcal{P}} L$ and for every $L \in \mathcal{L}$ there exists $K \in \mathcal{K}$ such that $K =_{\mathcal{P}} L$.

1.27. Theorem. *Let $\mathcal{K}$ be a class of* CW*-complexes, $\mathcal{L}$ be a class of metric simplicial complexes, and let $\mathcal{P}$ be a class of spaces. Then $\mathcal{K} \sim_{\mathcal{P}} \mathcal{L}$ in the following cases:*

1) *$\mathcal{K}$ and $\mathcal{L}$ consist of all finite complexes, $\mathcal{P}$ is the class of all normal spaces;*

2) *$\mathcal{K}$ consists of all countable complexes, $\mathcal{L}$ consists of all topologically complete countable complexes, and $\mathcal{P}$ consists of all countably paracompact p_1-spaces.*

3) *$\mathcal{K}$* and *$\mathcal{L}$ consist of all countable complexes, and $\mathcal{P}$ consists of all perfectly normal p_1-space.*

4) *$\mathcal{P}$ consists of all perfectly normal and collectionwise normal p_1- space.*

Proof. It is well known that every finite CW-complex is an ENR. So point 1) is a consequence of Theorem 1.19. 1).

Let K be a countable CW-complex. By Proposition 1.22 there exists a locally finite countable metric simplicial complex L such that $K \simeq L$. Theorem 1.19 implies that $(X, K) \in$ het for every $X \in \mathcal{P}$. According to Proposition 1.15 and Theorem 1.12 we have $(X, L) \in$ het for every $X \in \mathcal{P}$. So $K =_{\mathcal{P}} L$, since Proposition 1.25.

Now let L be a topologically complete countable metric simplicial complex. Put $K = L_{\mathrm{CW}}$. It is well known that $L \simeq L_{\mathrm{CW}}$. So repeating the previous argument we get $L =_{\mathcal{P}} K$.

If L is a countable metric simplicial complex and X is a perfectly normal p_1-space, then $(X, L) \in$ het by Theorem 1.19. 5). Let $K = L_{\mathrm{CW}}$. Then $(X, K \in$ het according to Theorem 1.19. 2), because of Proposition 1.18. Applying Proposition 1.25 we get $K =_{\mathcal{P}} L$.

If K is a countable CW-complex, then it was proved above that $K =_{\mathcal{P}} L$ for every countable metric simplicial complex L such that $L \simeq K$.

To prove assertion 4) we use the previous argument changing Theorems 1.19. 5) and 1.19. 2) for Theorems 1.19. 6) and 1.19. 3) respectively. ∎

1.28. Lemma. *Let A be a subspace of a space X and $u = \{U_\lambda : \lambda \in \Lambda\}$ be a cover of A. Assume that either:*

1) *X is hereditarily normal and u is finite;*

2) *X is hereditarily normal and countably paracompact, and u is countable;*

3) *X is hereditarily paracompact and u is locally finite in A.*

Then there exist open in X sets $V_\lambda, \lambda \in \Lambda$, such that

(i) $A \subset V = \bigcup\{V_\lambda : \lambda \in \Lambda\}$;

(ii) $V_\lambda \cap A \subset U_\lambda$ *for any* $\lambda \in \Lambda$;

(iii) $\bigcap\limits_{\lambda\in\Lambda_0} V_\lambda \neq \varnothing \Rightarrow \bigcap\limits_{\lambda\in\Lambda_0} U_\lambda \neq \varnothing$ *for any finite* $\Lambda_0 \subset \Lambda$;

(iv) *the family $v = \{V_\lambda : \lambda \in \Lambda\}$ is locally finite in V.* ■

If K is a simplicial complex, $z \in K$ and $a \in K^{(0)}$ is a vertex, then $\mu_a(z)$ is a barycentric coordinate of z. So we get a *barycentric mapping* $\mu_a : K \to [0;1] = I = I_a$.

1.29. Proposition. *Let K be a metric simplicial complex. Then the diagonal product*

$$\mu_K = \mathop{\triangle}\limits_{a\in K^{(0)}} \mu_a \colon K \to I^{K^{(0)}}$$

is a topological embedding. ■

From Proposition 1.29 we get

1.30. Proposition. *If K is a metric simplicial complex and X is a space, then $f : X \to K$ is continuous if and only if $\mu_a \circ f$ is continuous for each vertex a of K.* ■

If $a \in K^{(0)}$ is a vertex of a simplicial complex K, then $\mathrm{St}(a)$ denotes the union of all open simplices $s \subset K$ such that a is a vertex of s, i.e.

(1.1) $\mathrm{St}(a) = \{z \in K : \mu_a(z) > 0\}$.

1.31. Proposition [1]. *Let $f, g : X \to K$ be mappings to a metric simplicial complex K. If for any $x \in X$ there exists $a \in K^{(0)}$ such that $f(x), g(x) \in \mathrm{St}(a)$, then $f \simeq g$.* ■

1.32. Proposition. *Let X be a space and let K be either a metric simplicial complex or a* CW*-complex such that:*

1) *either X hereditarily normal and K is finite;*

2) *or X hereditarily normal and hereditarily countably paracompact, and K is countable;*

3) *or X is hereditarily paracompact.*

Then for any $A \subset X$ and any mapping $f : A \to K$ there exist a neighbourhood OA and a mapping $g : OA \to K$ such that $g|_A$ is homotopic to f.

Proof. Let K be a metric simplicial complex. Consider more complicated case of a hereditarily paracompact space X and an arbitrary complex K. Let

(1.2) $\quad U_a = f^{-1}(\mathrm{St}(a)),\ a \in K^{(0)}.$

Then $u = \{U_a : a \in K^{(0)}\}$ is a point-finite cover of A. Since A is paracompact, there exists a locally finite cover $u_1 = \{U_a^1 : a \in K^{(0)}\}$ of A such that

(1.3) $\quad U_a^1 \subset U_a,\ a \in K^{(0)}.$

According to Lemma 1.28 there exist open in X sets $V_a,\ a \in K^{(0)}$, satisfying conditions (i)–(iv), where we replace $\lambda \in \Lambda$ by $a \in K^{(0)}$. Since V is paracompact, there is a locally finite partition $\{\varphi_a\}$ of unity for V combinatorially subordinated to the cover $v = \{V_a\}$, i.e.

(1.4) $\quad \overline{\varphi_a^{-1}((0;1])}^V \subset V_a,\ a \in K^{(0)}.$

Conditions (1.3), (1.4), and (ii) imply that

(1.5) $\quad A \cap \varphi_a^{-1}((0;1]) \subset U_a,\ a \in K^{(0)}.$

Let

(1.6) $\quad \varphi = \triangle_{a\in K^{(0)}}\ \varphi_a : V \to I^{K^{(0)}}.$

Conditions (ii), (iii), (1.1)–(1.4) and (1.6) yield

(1.7) $\quad \varphi(V) \subset \mu_K(K).$

Putting

$$g = \mu_K^{-1} \circ \varphi : V \to K,$$

we get the required mapping g. Indeed, condition (1.5) implies that

$$g(A \cap \varphi_a^{-1}((0;1])) \subset \mathrm{St}(a) = f(U_a).$$

It remains to apply Proposition 1.31.

Now let K be a CW-complex. Point 1) coincides with Point 1) for simplicial complexes. Consider points 2) and 3) simultaneously. Let $A \subset X$ and let $f : A \to K$ be a mapping. By Proposition 1.22 there exist a (locally finite countable) metric simplicial complex L and mappings $f_1 : L \to K$ and $f_2 : K \to L$ such that

(1.8) $\quad f_1 \circ f_2 \simeq \mathrm{id}_K,\ f_2 \circ f_1 \simeq \mathrm{id}_L.$

According to Proposition 1.32 for simplicial complexes there exist a neighbourhood OA and a mapping $h : OA \to L$ such that

(1.9) $\quad h|_A \simeq f_2 \circ f.$

Put $g = f_1 \circ h$. Then $g|_A = f_1 \circ h|_A \simeq (1.9) \simeq f_1 \circ f_2 \circ f|_A \simeq (1.8) \simeq \mathrm{id}_K \circ f|_A \simeq f|_A.$ ■

2. Subspace theorem

Recall that an open set $U \subset X$ is said to be *regular open* in X if $U = \mathrm{Int}(\overline{U})$.

2.1. Lemma. *Let U_1, U_2 be open disjoint subsets of X and B be a closed subset of X such that*

(2.1) $\quad \overline{U}_1 \cap \overline{U}_2 \subset B \subset \overline{U}_1.$

If U_2 is regular open in $Y = X \setminus B$, then U_2 is regular open in X.

Proof. Since $U_1 \cap U_2 = \varnothing$, we have $U_1 \cap \overline{U}_2^X = \varnothing$. Hence $\overline{U}_1^X \cap \mathrm{Int}(\overline{U}_2^X) = \varnothing$. From (2.1) we get $B \cap \mathrm{Int}(\overline{U}_2^X) = \varnothing$ and, consequently, $\mathrm{Int}(\overline{U}_2^X) \subset Y$. Thus $\mathrm{Int}(\overline{U}_2^X)$ is a regular open in Y set containing a regular open in Y set U_2 as a dense subset. So $\mathrm{Int}(\overline{U}_2^X) = U_2$. ■

2.2. Proposition. *Let X be a space, let K be a* CW*-complex, and let L be a metric simplicial complex such that either:*

1) *X is hereditarily normal and $K(L)$ is finite;*

2) *X is a hereditarily normal, hereditarily countably paracompact first countable space, and K is countable;*

3) *X is hereditarily paracompact first countable space;*

4) *X is hereditarily normal and hereditarily countably paracompact, and L is countable and topologically complete;*

5) *X is perfectly normal and L is countable;*

6) *X is perfectly normal and paracompact and L is arbitrary.*

Then the following conditions are equivalent:

(i) $K \in \mathrm{AE}(A)$ *for any* $A \subset X$;

(ii) $K \in \mathrm{AE}(U)$ *for any regular open* $U \subset X$.

Proof. It suffices to check that (ii) $\Rightarrow$ (i). Let F be a closed subset of A and let $f : F \to K$ be a mapping. By Proposition 1.32 there exist an open set $U \subset X$ and a mapping $g : U \to K$ such that

(2.2) $F \subset U$;

(2.3) $g|_F \simeq f$.

Let $B = (X \setminus U) \cap \overline{F}^X$ and $Y = X \setminus B$. Then the sets $C = U \cap \overline{F}^X$ and $D = Y \setminus U$ are closed in Y and disjoint. Since Y is normal, there exist regular open in Y sets V and W such that

(2.4) $C \subset V,\ D \subset W$;

(2.5) $V \cap W = \varnothing$.

Inclusions $F \subset U$ and $F \subset \overline{F}^X$ imply that $\overline{F}^X \subset \overline{C}^X$. Hence

(2.6) $B \subset \overline{C}^X \subset \overline{V}^X$.

From (2.5) and (2.6) we get

(2.7) $\overline{V}^X \cap \overline{W}^X \subset B \subset \overline{V}^X$.

So we may apply Lemma 2.1 for $U_1 = V$ and $U_2 = W$. According to this lemma, W is regular open in X.

In view of (2.4) there exists an open set G such that

(2.8) $D \subset G \subset \overline{G}^Y \subset W$.

Hence

(2.9) $\Gamma \equiv W \setminus G \subset X \setminus G \subset X \setminus D \subset U$.

Since W is regular open in X, the mapping $g_1 = g|_\Gamma : \Gamma \to K$ can be extended to a mapping $g_2 : W \to K$ according to condition (ii). Now we can define a mapping $g_3 : Y \to K$ in the following way:

$$g_3(y) = \begin{cases} g(y) & \text{if } y \in Y \setminus \overline{G}^Y; \\ g_2(y) & \text{if } y \in W. \end{cases}$$

The mapping g_3 is defined on Y because $Y \setminus \overline{G}^Y \subset$ (by (2.8)) $\subset W$. It is continuous since $Y \setminus \overline{G}^Y$ and W are open.

Now let $h = g_3|_A$. By definition $h|_F = g|_F$. Hence f extends over A by Theorem 1.19. ■

2.3. Definition. Let $A \subset X$ be an arbitrary subset. A space Y is called an *absolute extensor* of a space A *with respect* to X (notation: $Y \in AE(A, X)$) provided $Y \in \mathrm{AE}(F)$ for every set F such that, $F \subset A$, F closed in X.

2.4. Remark. The class $\mathrm{AE}(X, A)$ from Definition 1.1 does not coincide with the class $\mathrm{AE}(A, X)$.

2.5. Lemma. *Let X be a space, let K be either a* CW-*complex or a metric simplicial complex.*

If X can be represented as the union of a sequence $X_1, X_2, \dots$ of subspaces such that $K \in \mathrm{AE}(X_i, X)$ and the union $\bigcup_{i \le n} X_i \equiv Y_n$ is closed for $n = 1, 2, \dots,$ then $K \in \mathrm{AE}(X)$ in the following cases:

1) *X is normal and K is finite;*

2) *X is a countably paracompact p_1-space and K is a countable* CW-*complex;*

3) *X is a collectionwise normal countably paracompact p_1-space and K is a* CW-*complex;*

4) *X is normal and K is a countable and topologically complete simplicial complex;*

5) *X is perfectly normal and K is a countable simplicial complex;*

6) *X is perfectly normal and collectionwise normal and K is a simplicial complex.*

Proof. Consider a mapping $f : F \to K$, when F is closed in X. We have

(2.10) $\quad F_n = F \cap Y_n$ is closed in X.

We shall construct inductively open sets $U_n \subset X$ and mappings $f_n : [U_n] \to K$, $n \ge 1$, such that

$(2.11)_n \quad Y_n \subset U_n$;

$(2.12)_n \quad [U_{n-1}] \subset U_n$;

$(2.13)_n \quad f_n|_{F_n} = f|_{F_n}$;

$(2.14)_n \quad f_n|_{[U_{n-1}]} = f_{n-1}$.

Here we assume that $U_0 = \varnothing$. Let $n = 1$. According to (2.10) and to $K \in \mathrm{AE}(Y_1)$ there exists a mapping $f' : Y_1 \to K$ such that

(2.15) $\quad f'|_{F_1} = f|_{F_1}.$

By Theorems 1.11 and 1.12, there exist a neighbourhood OY_1 and a mapping $f^1 : OY_1 \to K$ such that

(2.16) $\quad f^1|_{Y_1} = f'|_{Y_1}.$

Take an open set U_1 such that

(2.17) $\quad F_1 \subset U_1 \subset [U_1] \subset OF_1$

and set

(2.18) $\quad f_1 = f'|_{[U_1]}.$

Conditions (2.17) and ((2.15), (2.16), (2.18)) respectively imply $(2.11)_1$, and $(2.13)_1$. As for conditions $(2.12)_1$ and $(2.14)_1$, they are fulfilled trivially.

Assume that the sets U_i and the mappings f_i satisfying $(2.11)_i$–$(2.14)_i$ are defined for all $i < n \geq 2$. From $(2.11)_{n-1}$ it follows that

(2.19) $\quad Z_n = Y_n \setminus U_{n-1} \subset X_n.$

Since Z_n is closed in X and $K \in \mathrm{AE}(X_n, X)$, we have

(2.20) $\quad K \in \mathrm{AE}(Z_n).$

Let

(2.21) $\quad A_n = (Z_n \cap [U_{n-1}]) \cup (F_n \setminus U_{n-1}).$

The set A_n is closed and contained in Z_n by virtue of (2.10) and (2.19). Let $f'_{n-1} : A_n \to K$ be a mapping defining by

(2.22) $$f'_{n-1}(x) = \begin{cases} f(x) & \text{if } x \in F_n \setminus U_{n-1}, \\ f_{n-1}(x) & \text{if } x \in Z_n \cap [U_{n-1}]. \end{cases}$$

According to (2.20) there exists a mapping $f^1_{n-1} : Z_n \to K$ such that

(2.23) $\quad f^1_{n-1}|_{A_n} = f'_{n-1}.$

Set

(2.24) $\quad B_n = [U_{n-1}] \cup Z_n.$

Define a mapping $f'_n : B_n \to K$ in the following way:

(2.25) $$f'_n(x) = \begin{cases} f_{n-1}(x) & \text{if } x \in [U_{n-1}], \\ f^1_{n-1}(x) & \text{if } x \in Z_n. \end{cases}$$

By Theorems 1.11 and 1.12 there exist a neighbourhood OB_n and a mapping $f^1_n : OB_n \to K$ such that

(2.26) $\quad f^1_n|_{B_n} = f'_n.$

Take an open set U_n such that

(2.27) $\quad B_n \subset U_n \subset [U_n] \subset OB_n$

and define a mapping $f_n : [U_n] \to K$ by

(2.28) $\quad f_n = f^1_n|_{[U_n]}.$

Conditions (2.24) and (2.27) imply $(2.12)_n$. Conditions (2.10), (2.19), (2.24), (2.27), and $(2.12)_n$ yield $(2.11)_n$. From (2.25), (2.26), and (2.28) we get $(2.14)_n$. Finally, conditions (2.22), (2.23), (2.25), and $(2.13)_{n-1}$ imply $(2.13)_n$.

Hence the construction of the sets U_n and the mappings f_n satisfying $(2.11)_n$–$(2.14)_n$ for $n = 1, 2, \dots$ is completed. Putting

$$\overline{f}(x) = f_n(x) \text{ if } x \in [U_n],$$

we get a mapping $\overline{f} : X \to K$ which extends the mapping $f : F \to K$. ■

2.6. Corollary. *Let X and K satisfy conditions of Lemma 2.5. If X can be represented as the union of a sequence $X_1, X_2, \dots$ of subspaces such that $K \in \mathrm{AE}(X_i)$ and the union $\bigcup_{i \le n} X_i = Y_n$ is closed for $n = 1, 2 \dots$, then $K \in \mathrm{AE}(X)$.* ■

Lemma 2.5 also yields

2.7. Countable sum theorem. *Let X and K satisfy conditions of Lemma 2.5. If X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces such that $K \in \mathrm{AE}(F_n)$ for $n = 1, 2, \dots$, then $K \in \mathrm{AE}(X)$.* ■

Theorem 2.7 is also an immediate consequence of Theorems 1.11, 1.12, and the following Dranishnikov's theorem.

2.8. Theorem [10]. *If a normal space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces, then $K \in \mathrm{AE}(X)$ provided $K \in \mathrm{AE}(F_n)$ for all n and $K \in \mathrm{ANE}(X)$.* ■

Proposition 1.5 and Theorem 2.7 yield

2.9. σ-discrete sum theorem. *Let X and K satisfy conditions of Lemma 2.5, and let $\varphi_n = \{F^n_\alpha : \alpha \in A_n\}$, $n \in \mathbb{N}$, be discrete families of closed subsets of X such that $K \in AE\big(F^n_\alpha\big)$ and*

$$X = \textstyle\bigcup_{n,\alpha} F^n_\alpha.$$

Then $K \in AE(X)$. ■

2.10. Lemma ([21], Lemma 2.3.3). *Let $u = \{U_\alpha : \alpha \in A\}$ be a point-finite open cover of a space X. For $i = 1, 2, \dots$ denote by X_i the set of all points of the space X which belong to exactly i members of the cover u, by $\mathcal{T}_i$ the family of all subsets of A that have exactly i elements, and let*

$$X_T = X_i \cap \big(\textstyle\bigcap\{U_\alpha : \alpha \in T\}\big)$$

for each $T \in \mathcal{T}_i$. Then

(2.29) $X = \bigcup\{X_i : i = 1, 2, \dots\}$;

(2.30) $X_i \cap X_j = \varnothing$ *whenever* $i \ne j$;

(2.31) $Y_n = \bigcup\{X_i : i \le n\}$ *is closed for* $i = 1, 2, \dots$;

(3.32) $X_i = \bigcup\{X_T : T \in \mathcal{T}_i\}$ *and X_T are open in X_i and pairwise disjoint.* ■

2.11. Point-finite sum theorem. *Let X and K satisfy conditions of Lemma 2.5. Let X can be represented as the union of a family $\{F_\alpha : \alpha \in A\}$ of closed subspaces such that $K \in \mathrm{AE}(F_\alpha)$ for $\alpha \in A$, and there exists a point-finite open cover $u = \{U_\alpha : \alpha \in A\}$ of the space X such that $F_\alpha \subset U_\alpha$ for $\alpha \in A$. Then $K \in \mathrm{AE}(X)$.*

Proof. Consider the decomposition of the space X described in Lemma 2.10. Because of Lemma 2.5 it suffices to show that

(2.33) $\quad K \in \mathrm{AE}(X_i, X),\ i = 1, 2, \ldots.$

Let F be a closed subspace of the space X containing in X_i. It follows from (2.32) that for every $T \in \mathcal{T}_i$ the set $F \cap X_T$ is closed in X. Since

$$F \cap X_T \subset \bigcup\{F_\alpha : \alpha \in T\},$$

by Proposition 1.2 and Theorem 2.7 we have $K \in \mathrm{AE}(F \cap X_T)$ for every $T \in \mathcal{T}_i$. From (2.32) we get

$$F = \bigoplus_{T \in \mathcal{T}_i} (F \cap X_T).$$

Consequently, $K \in \mathrm{AE}(F)$ by Proposition 1.5. ∎

2.12. Theorem. *Let X and K satisfy conditions of Lemma 2.5 and let X be weakly paracompact. If X can be represented as the union of a family $u = \{U_\alpha : \alpha \in A\}$ of open subspaces such that $K \in \mathrm{AE}(\overline{U}_\alpha)$ for $\alpha \in A$, then $K \in \mathrm{AE}(X)$.*

Proof. The space X being weakly paracompact, one can assume that the cover u is point-finite and thus has a closed shrinking $\{F_\alpha : \alpha \in A\}$. To complete the proof it suffices to apply Proposition 1.2 and Theorem 2.11. ∎

Recall that two subsets A and B of a space X are *separated* if $A \cap \overline{B} = \varnothing = \overline{A} \cap B$.

2.13. Remark. Theorems 2.11 and 2.12 generalize well known results for Legesgue dimension. Theorem 2.12 ($K = S^n$) for paracompact X was established by C.H. Dowker [6] and K. Nagamy [35]. Theorem 2.11 and 2.12 (both for $K = S^n$) were obtained by A.V. Zarelua [44].

2.14. Definition [21]. A space X is called *strongly hereditarily normal* if for every pair A, B of separated subset of X there exist open sets $U, V \subset X$ such that

$$A \subset U,\ B \subset V,\ U \cap V = \varnothing,$$

and both U and V can be represented as the union of point finite families of open F_σ-sets.

2.15. Proposition. *A space X is strongly hereditarily normal if and only if X is hereditarily normal and every regular open set $U \subset X$ can be represented as the union of a point-finite family of open F_σ-sets.* ∎

2.16. Definition [43]. A disjoint covering u of a space X is said to be *scaled* if u can be represented as the union of families $u_i,\ i = 0, 1, \ldots,$ such that:

(i) $\quad u_0 = \varnothing$;

(ii) $\quad$ for $i \geq 1$, the family u_i is discrete in $Z_i = X \setminus \bigcup_{j<i}\left(\bigcup u_j\right)$ and consists of closed subsets of Z_i.

It is clear that every Z_i is open in X and every $Y_i = \bigcup_{j<i}\left(\bigcup u_j\right)$ is closed in X.

2.17. Definition [43]. A hereditarily normal space X is called *totally scaled* if every open set $U \subset X$ has a scaled covering consisting of F_σ-sets of X.

Formally classes SHN (of strongly hereditarily normal spaces) and TS (of totally scaled spaces) are incomparable. We define a new class which contains both of them.

2.18. Definition. A hereditarily normal space X is called *regular scaled* (notation: $X \in \mathrm{RS}$) if every regular open set $U \subset X$ has a scaled covering consisting of F_σ-sets of X.

2.19. Proposition. *If* RS *is the class of all regular scaled spaces, then*

$$\mathrm{SHN} \cup \mathrm{TS} \subset \mathrm{RS}. \blacksquare$$

2.20. Subspace Theorem. *Let X be a regular scaled space, and let K be either a* CW-*complex or a metric simplicial complex. If X and K satisfy conditions of Lemma 2.5, then $K \in \mathrm{AE}(X) \Rightarrow K \in \mathrm{AE}(A)$ for any $A \subset X$.*

Proof. According to Proposition 2.2 it suffices to check that $K \in \mathrm{AE}(U)$ for every regular open set $U \subset X$. Since X is regular scaled, there exists a scaled covering $u = \bigcup_{i=0}^{\infty} u_i$ of U consisting of F_σ-sets of X. Set

$$Y_i = \bigcup_{j \le i} \big(\bigcup u_j\big); \tag{2.34}$$

$$X_i = Y_i \setminus Y_{i-1} = \bigcup u_i. \tag{2.35}$$

In view of Corollary 2.6 it remains to show that $K \in \mathrm{AE}(X_i)$ for every i. By definition of a scaled covering, the equality (2.35) implies that

$$X_i = \bigcup \{B_\gamma : \gamma \in \Gamma\},$$

where the family $\{B_\gamma : \gamma \in \Gamma\}$ is discrete in X_i and consists of F_σ-sets of X. Since Proposition 1.2 and Theorem 2.7, $K \in \mathrm{AE}(B_\gamma)$ for every $\gamma \in \Gamma$. Applying Proposition 1.5, we get $K \in \mathrm{AE}(X_i)$. $\blacksquare$

2.21. Corollary. *If either X is a perfectly normal first countable space and K is a countable* CW-*complex, or X is perfectly normal and K is a countable metric simplicial complex then $K \in \mathrm{AE}(X) \Rightarrow K \in \mathrm{AE}(A)$ for any $A \subset X$.* $\blacksquare$

From Theorems C and 2.20 we get

2.22. Corollary. *Let X be regular scaled paracompact p_1-space and $A \subset X$. Then*

$$\dim_G A \le \dim_G X$$

for an arbitrary abelian group G. $\blacksquare$

2.23. Remark. Theorem 2.20, for regular scaled X and $K = S^n$, is contained in ([43], Theorem 6). For strongly hereditarily normal X and $K = S^n$, Theorem 2.20 was proved in ([21], Theorem 3.1.19). Subspace theorem, for metrizable X and arbitrary CW-complex K, was proved in ([16], Proposition 1.1). Corollary 2.22 is also a generalization of just mentioned Dydak's theorem.

3. Addition theorem for simplicial complexes

3.1. Definition [16] (see also [12]). Given two simplicial complexes K and L, their *simplicial join* $K * L$ is formed by declaring $\sigma = \langle a_0, \dots, a_k,\ b_0, \dots, b_l\rangle$ to be its simplex if and only if and only if $\sigma_1 = \langle a_0, \dots, a_k\rangle$ is a simplex in K and $\sigma_2 = \langle b_0, \dots, b_l\rangle$ is a simplex in L. So $\sigma = \sigma_1 * \sigma_2$ is a topological join of simplices σ_1 and σ_2. Both K and L are naturally embedded in $K * L$. The simplicial join is related to the abstract join. As in the case of the abstract join, there are canonical projections

$\pi : K * L \to [0; 1];$

$\pi_K : K * L \setminus L \to K;$

$\pi_L : K * L \setminus K \to L.$

Namely, any point $x \in K * L$ can be expressed as $t \cdot y + (1 - t) \cdot z$, where $t \in [0, 1]$, $y \in K$, and $z \in L$. Here t is unique, and we put $\pi(x) = t$, y is unique if $x \notin L$, and we put $\pi_K(x) = y$. Similarly, z is unique if $x \notin K$, and we put $\pi_L(x) = z$.

3.2. Lemma ([16], [12]). *Suppose K, L are metric simplicial complexes and X is a normal space. Then there is a one-to-one correspondence between mappings $f : X \to K * L$ and 5-tuples*

$(U, V, g : U \to K,\ h : V \to L,\ \alpha : X \to [0; 1])$

satisfying the following conditions:

(3.1) $\quad X = U \cup V$, *U and V are functionally open in X;*

(3.2) $\quad \alpha^{-1}[0; 1) = U$ *and* $\alpha^{-1}(0; 1] = V$.

Namely, given $f : X \to K * L$, define

$U = f^{-1}(K * L \setminus L),\ V = f^{-1}(K * L \setminus K);$

$g = \pi_K \circ f,\ h = \pi_L \circ f,\ \alpha = \pi \circ f.$

Conversely, given (U, V, g, h, α), one defines f as follows:

$$f(x) = \begin{cases} g(x) & \textit{if } x \in U \setminus V, \\ (1 - \alpha(x)) \cdot g(x) + \alpha(x) \cdot h(x) & \textit{if } x \in U \cap V, \\ h(x) & \textit{if } x \in V \setminus U. \end{cases}$$

The mapping f defined as above will be denoted by $g *_\alpha h$.

Proof. The only thing we need to verify that f is continuous. In accordance with Proposition 1.29 we need to show that $\mu_a \circ f$ is continuous for all vertices a of $K * L$. Since $(K * L)^{(0)} = K^{(0)} \cup L^{(0)}$, without loss of generality we may assume that $a \in K^{(0)}$. Since $\mu_\alpha(h(x)) = 0$ for $x \in U$, we have

(3.3) $\quad \mu_a \circ f(x) = (1 - \alpha(x)) \cdot \mu_a \circ g(x)$ for all $x \in U$

and

(3.4) $\quad \mu_a \circ f(x) = 0$ for all $x \in U \setminus V$.

Clearly, $\mu_a \circ f|_U$ is continuous. Let $x_0 \in U \setminus V$ and $\epsilon > 0$. We have to find a neighbourhood Ox_0 such that

$\mu_a \circ f(x) \le \epsilon$ for all $x \in Ox_0$.

Set

(3.5) $\quad Ox_0 = \alpha^{-1}(1-\epsilon; 1]$.

If $x \in Ox_0 \setminus V$, then $\mu_a \circ f(x) = 0$ according to (3.4). If $x \in Ox_0 \cap U$, then

$\mu_0 \circ f(x) =$ (by (3.3)) $= (1-\alpha(x)) \cdot \mu_a \circ g(x) \le$ (in view of (3.5)) $\le \epsilon \cdot \mu_a \circ g(x) \le \epsilon$. ■

3.3. Lemma. *Suppose $f : A \to [0;1]$ is continuous, A is a closed subset of a space X, and U and V are open in X sets such that*

$U \cup V = X,\ U \cap A = f^{-1}[0;1),\ V \cap A = f^{-1}(0;1]$.

Then there is an extension $f_1 : X \to [0;1]$ of f such that

$f_1^{-1}[0;1) \subset U,\ f_1^{-1}(0;1] \subset V$. ■

3.4. Urysohn–Menger theorem. *Let a space X and metric simplicial complexes K and L be such that either:*

1) *X is hereditarily normal and K and L are finite;*

2) *X is hereditarily normal and hereditarily countably paracompact, and K and L are countable and topologically complete;*

3) *X is perfectly normal, K and L are countable;*

4) *X is perfectly normal and paracompact.*

Let $X = A \cup B$ and $K \in \mathrm{AE}(A)$, $L \in \mathrm{AE}(B)$. Then $K * L \in \mathrm{AE}(X)$.

Proof. Suppose C is a closed subset of X, and $f : C \to K * L$ is a mapping. In view of Theorem 1.12 we may assume that C is functionally closed in X. By Lemma 3.2, f defines two closed, disjoint subsets $C_K = f^{-1}(K)$, $C_L = f^{-1}(L)$ of C and maps

$f_K : C \setminus C_L \to K,\ f_L : C \setminus L,\ \alpha : C \to [0;1]$

such that

(3.6) $\quad \alpha^{-1}(0) = C_K,\ \alpha^{-1}(1) = C_L$;

(3.7) $\quad f(x) = (1-\alpha(x)) \cdot f_K(x) + \alpha(x) \cdot f_L(x)$.

Since C_L is a G_δ-set, $K \in AE(A \setminus C_L)$ according to Theorem 2.7. Consequently, f_K extends to a mapping

$f_K : C \cup A \setminus C_L \to K$,

Applying Proposition 1.32, we can find an open set U_A and a mapping $g_K : U_A \to K$ such that

(3.8) $\quad C \cup A \setminus C_L \subset U_A \subset X \setminus C_L;\ \ g_K|_{C \setminus C_L} \simeq f_K$.

Since $C \setminus C_L$ is closed in U_A, by Theorem 1.19 there exists a mapping $f'_K : U_A \to K$ such that

(3.9) $\quad f'_K|_{C \setminus C_L} = f_K$.

Similarly, there exist an open set U_B and a mapping $f'_L : U_B \to L$ such that

(3.10) $C \cup A \setminus C_K \subset U_B \subset X \setminus C_K$;

(3.11) $f'_L|_{C \setminus C_K} = f_L$.

Since $U_A \cup U_B = X$, conditions (3.8) and (3.10) yield

(3.12) $U_A \cap C = C \setminus C_L,\ U_B \cap C \setminus C_K$.

Now, according to (3.6) and (3.12), we can apply Lemma 3.3 to the set C, the function α and the sets U_A and U_B. We get a function $\beta : X \to [0;1]$ such that $\beta|C = \alpha$ and

(3.13) $\beta^{-1}[0;1) \subset U_A,\ \beta^{-1}(0;1] \subset U_B$.

Setting

$U^1_A = \beta^{-1}[0;1),\ U^1_B = \beta^{-1}(0;1]$,

$f^1_K = f'_K|_{U^1_A},\ f^1_L = f'_L|_{U^1_B}$,

and applying Lemma 3.2, we get the mapping

$f^1_K *_\beta f^1_L : X \to K * L$

which extends f, because of (3.9) and (3.11). ∎

3.5. Remark. Since $\dim X \le n$ means that $S^n \in AE(X)$ for any normal space X and $S^n * S^m = S^{n+m+1}$, Theorem 3.4 is a generalization of the Urysohn–Menger theorem for hereditarily normal spaces. Theorem 3.4 also is a generalization of Dydak's theorem ([16], Theorem 1.2) for metrizable X and arbitrary K and L.

Point 4) of Theorem 3.4 is also an immediate corollary of Theorem 1.19.6) and the following Dydak's theorem.

3.6. Theorem ([18], Theorem 4.3). *Suppose X is a hereditarily paracompact space and K and L are* CW*-complexes. If K is an absolute extensor of $A \subset X$ up to homotopy and L is an absolute extensor of $B \subset X$ up to homotopy, then the joint $K * L$ is an absolute extensor of $A \cup B$ up to homotopy.*

Recall, that K is an *absolute extensor up to homotopy* of X [18] if every mapping $f : A \to K$, A closed in X, extends over X up to homotopy.

4. Dranishnikov's theorem for paracompact spaces

The purpose of this section is to generalize a theorem of A.N. Dranishnikov [15] stating that if a CW-complex K is an absolute extensor of a compactum X, then $\dim_{H_m(K,\mathbb{Z})} X \le m$ for all $m > 0$. This theorem was extended to metrizable spaces X by J.Dydak [16]. We follow his argument.

4.1. Proposition. *Let $f : X' \to X$ be an open mapping of a weakly paracompact space X' onto a space X such that* card *$f^{-1}(x) \le n$ for all $x \in X$ and some $n < \infty$, and let K be a metric simplicial complex such that $K \in$* AE*(X). Then $K \in$* AE*(X') in the following cases:*

1) *K is countable and topologically complete;*

2) *X' is perfectly normal and K is countable;*

3) *X' is perfectly normal and paracompact.*

Proof. Induction on n. For $n = 1$ the mapping f is a homeomorphism. Suppose Proposition 4.1 holds for $n \leq m$ and consider $n = m + 1$. The set

$$A = \{x' \in X' : \mathrm{card} f^{-1} f(x') \leq m\}$$

is closed in X' and $K \in \mathrm{AE}(A)$ by inductive assumption. Further $K \in \mathrm{ANE}(X')$ according to Theorem 1.12. By virtue of Proposition 1.2 it suffices to prove that $K \in \mathrm{AE}(C)$ for every closed subset C of X' containing in $X' \setminus A$. Given $x \in X' \setminus A$, there is an open neighbourhood U_x of x in X' such that $f|_{\overline{U}_x} : \overline{U}_x \to f(\overline{U}_x)$ is a homeomorphism. So $K \in \mathrm{AE}(C)$ by Theorem 2.12. ■

4.2. Definition [3]. Given a space X and $m > 0$, the m-th symmetric product $SP^m(X)$ of X is the space of orbits of the action of the symmetric group S_m on X^m. Points of $SP^m(X)$ will be written in the form $\sum_{i=1}^m x_i$. The set $SP^m(X)$ is equipped with the quotient topology given by the natural mapping $\pi : X^m \to SP^m(X)$.

If X is a Hausdorff space, then $\pi : X^m \to SP^m(X)$ is both open and closed, since S_m is a compact group. So if X is metrizable, then $SP^m(X)$ is metrizable, too. If X has a base point a, then $SP^m(X)$ has $\sum_{i=1}^m a$ as its base point; this base point will be denoted by a, too. There is a natural inclusion $i_m^n = i : SP^n(X) \to SP^m(X)$ for all $n < m$. It is given by formula

$$i\big(\sum_{i=1}^n x_i\big) = \sum_{i=1}^n x_i + (m-n)a.$$

In this way, points of the form $\sum_{i=1}^n x_i,\ n < m$, can be considered as belonging to $SP^m(X)$.

The direct limit of the sequence

$$X = SP^1(X) \to SP^2(X) \to \cdots \to SP^m(X) \to \cdots$$

is denoted by $SP^\infty(X)$.

4.3. Proposition [31]. *If every pair $\big(SP^{m+1}(X),\ SP^m(X)\big)$ satisfies homotopy extension property, then the infinite symmetric product $SP^\infty(X)$ is homotopy equivalent to the telescope*

$$\bigcup_{m=1}^\infty SP^m(X) \times [m-1, m]. ■$$

4.4. Corollary. *If X is metrizable, then $SP^\infty(X)$ is homotopy equivalent to the telescope*

$$\bigcup_{m=1}^\infty SP^m(X) \times [m-1, m]. ■$$

The next statement was formulated in ([16], Lemma 3.2) for metrizable X and Z. But the metrizability condition is excessive. Repeating the original proof with more details we get

4.5. Lemma. *Suppose* (X, x_0), (Z, a) *are pointed spaces and* $f : (X, x_0) \to (SP^k(Z), a)$ *is a mapping,* $k \geq 1$. *Let*

$$\begin{array}{ccc} (X', x_0') & \longrightarrow & (Z^k, a) \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ (X, x_0) & \xrightarrow{f} & (SP^k(Z), a) \end{array}$$

be the pull-back diagram. Then, the function

$$f^* : (X, x_0) \to (SP^{k!}(X'), x_0')$$

defined by

$$f^*(x) = \sum_{y \in (\pi')^{-1}(x)} y$$

is continuous.

Proof. Every element $\sigma \in S_k$ defines a mapping $\sigma_Z : Z^k \to Z^k$ by

$$\sigma_Z(z_1, \ldots, z_k) = (z_{\sigma(1)}, \ldots, z_{\sigma(k)}).$$

Let $\alpha : Z^k \to SP^{k!}(Z^k)$ be the mapping defining as follows:

$$\alpha(z_1, \ldots, z_k) = \sum_{\sigma \in S_k} (z_{\sigma(1)}, \ldots, z_{\sigma(k)}) = \sum_{\sigma \in S_k} \sigma_Z(z_1, \ldots, z_k).$$

There is a mapping $g : SP^k(Z) \to SP^{k!}(Z^k)$ such that $\alpha = g \circ \pi$.
Indeed, it suffices to put

$$g(z_1 + \cdots + z_k) = \sum_{\sigma \in S_k} (z_{\sigma(1)}, \ldots, z_{\sigma(k)}).$$

Let

$$i : X \times SP^{k!}(Z^k) \to SP^{k!}(X \times Z^k)$$

be the mapping defined by

$$i(x, z_1 + \cdots + z_k) = (x, z_1) + \cdots + (x, z_k).$$

It remains to check that $f^*(x) = i(x, gf(x))$. ■

The following lemma was formulated in ([16], Lemma 3.3) for metrizable X and arbitrary K. Using Proposition 4.1 and Lemma 4.5 we find more strict restrictions.

4.6. Lemma. *Suppose* X *is a weakly paracompact space and* $K \in \mathrm{AE}(X)$ *is a pointed metric simplicial complex such that:*

1) *either* K *is countable and topologically complete;*

2) X *is perfectly normal and* K *is countable;*

3) X *is perfectly normal and paracompact.*

Given a closed subset A *of* X *and a mapping* $g : A \to SP^k(K)$*, there is an extension* $g' : X \to SP^{k \cdot k!}(K)$ *of* g.

Proof. Add a discrete base point x_0 to A and map it to the base point $a \in SP^k(K)$. Let $C(K)$ be the metric cone over K. Then $C(K)$ is a contractible metric simplicial complex. Since SP^k is a functor from the homotopy category (see [3]), $SP^k(C(K))$ is homotopic to a contractible metric simplicial complex. According to Theorems 1.12, 1.19 and Proposition 1.20 there is an extension $G : X \to SP^k(C(K))$ of the mapping g. Let $f : X' \to X$ be the pull-back of the projection $C(K)^k \to SP^k(C(K))$ under G. Thus,

$$X' = \{(x, x_1, \dots, x_k) \in X \times C(K)^k : G(x) = x_1 + \dots + x_k\}$$

and $f(x, x_1, \dots, x_k) = x$. The space X' contains as closed subset, the space A' obtained as a pull-back of $K^k \to SP^k(K)$ under g. Since X' is perfect preimage of X, X' is weakly paracompact and paracompact if X is paracompact. On the other side, if X is perfectly normal, then $X \times C(K)^k$ is perfectly normal as a product of a perfectly normal space X with a metrizable space $C(K)^k$ [33]. So X' is perfectly normal being a subset of $X \times C(K)^k$. Thus, we can apply Proposition 4.1. It follows that $K^k \in AE(X')$. Hence we can extend the natural projection $A' \to K^k$ over X' and compose it with $K^k \to SP^k(K)$. Since SP^m is a functor, the resulting mapping induces

$$SP^{k!}(X') \to SP^{k!}(SP^k(K) \to SP^{k.k!}(K),$$

which, when composed with the mapping $f^* : X \to SP^{k!}(X')$ from Lemma 4.5, is an extension of $g : A \to SP^k(K)$. ■

4.7. Proposition. *Let X be a weakly paracompact p_1-space and let $K \in$ AE(X) be a* CW-*complex such that:*

1) *either K is countable;*

2) *or X is perfectly normal and paracompact.*

Then

$$K(H_m(K, \mathbb{Z}), m) \in \mathrm{AE}(X)$$

for all $m > 0$.

Proof. It suffices to show that $SP^\infty(K) \in \mathrm{AE}(X)$ as $SP^\infty(K)$ homotopy dominates $K(H_m(K; \mathbb{Z}), m)$ for each $m > 0$ (see [3]). Let L be a metric simplicial complex (locally finite countable if K is countable) which is homotopy equivalent to K. Since Theorem 1.27, it is sufficient to check that $SP^\infty(L) \in \mathrm{AE}(X)$. According to Corollary 4.4 we can replace $SP^\infty(L)$ by the telescope

$$\bigcup_{m=1}^{\infty} SP^m(L) \times [m-1, m].$$

After this we use Lemma 4.6. ■

From Proposition 4.7 and Theorem C we get

4.8. Theorem. *Let X be a paracompact p_1-space and let $K \in$ AE(X) be a* CW-*complex such that either:*

1) *K is countable;*

2) *X is perfectly normal.*

Then

$\dim_{H_m(K,\mathbb{Z})} X \le m$ *for all $m > 0$.* ■

4.9. Remark. From Theorem 1.27 it follows that the assertion of Theorem 4.8 holds for a metric simplicial complex K.

5. Dydak's theorem for paracompact spaces

J.Dydak proved the following theorem.

5.1. Theorem ([16], Theorem 1.4, the first part). *Suppose A, B are subsets of a metrizable space. Then*

$$\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1$$

for any ring G with unity. ■

We shall extend this theorem to classes of spaces which are larger that the class of all metrizable spaces. A space X is called a *hereditarily paracompact p_1-space* if each its subspace is a paracompact p_1-space.

5.2. Theorem. *Suppose A, B are subsets of a hereditarily paracompact p_1-space X and G is a ring with unity. Then*

$$\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1$$

in the following cases:

1) *G is countable;*

2) *X is perfectly normal.* ■

To prove Theorem 5.2 we need an additional information. The next assertion is was proved by J. Dydak for metrizable spaces ([16], Theorem 1.3, the first part). We give more general result.

5.3. Theorem. *Let A and B be subspaces of a hereditarily paracompact p_1-space X. Then*

$$\dim_{G\otimes H}(A \cup B) \leq \dim_G A + \dim_H B + 1 \tag{5.1}$$

in the following cases:

1) *G and H are countable*;

2) *X is perfectly normal.*

To prove Theorem 5.3 we need an additional information. J. Milnor defined [30] the join $X_1 * X_2$ of topological spaces X_1 and X_2 in the following way. A point of the join is a formal linear combination $t_1x_1 + t_2x_2$, where $x_i \in X_i$, $t_i \geq 0$, $t_1 + t_2 = 1$, and the element x_i may be chosen arbitrarily or omitted whenever the corresponding t_i vanishes. A subbase for the open sets is given by the sets of the following two types:

1) $\{t_1x_1 + t_2x_2 : \alpha_i < t_i < \beta_i\}$;

2) $\{t_1x_1 + t_2x_2 : t_i \neq 0$ and $x_i \in U$, where U is an arbitrary open subset of $X_i\}$.

This topology has the property that coordinate functions

$t_i : X_1 * X_2 \to [0;1]$ and $x_i : t_i^{-1}(0,1] \to X_i$

are continuous. It is easy to see that the topology of the metric simplicial join defined at the beginning of § 3 coincides with the Milnor's topology.

So we can apply a partial version of Milnor's result.

5.4. Proposition ([30], Lemma 2.1). *Let K and L be metric simplicial complexes. Then the reduced singular homology groups of the join $K * L$ with coefficients in a principal ideal domain D are given by*

(5.2) $\tilde{H}_{r+1}(K*L) = \Sigma_{i+j=r}\tilde{H}_i(K)\otimes\tilde{H}_j(L)+\Sigma_{i+j=r-1}\mathrm{Tor}\big(\tilde{H}_i(K),\tilde{H}_j(L)\big)$. ∎

Proof of Theorem 5.3. Suppose $G, H \neq 0$ are abelian groups and $\dim_G A = m \geq \dim_H B = n$. If $m = n = 0$, then Theorem 5.3 reduces to the Urysohn–Menger Theorem. Assume $m > 0$. By Theorems C and 1.27 there exist metric simplicial complexes L_1 and L_2 such that

(5.3) $L_1 \in AE(A),\ L_2 \in AE(B)$;

(5.4) $L_1 = K(G,m),\ L_2 = K(H,n)$.

From (5.2) and Theorem 3.4 we get

(5.5) $L_1 * L_2 \in AE(A \cup B)$.

Because of (5.4), equality (5.2) implies that

(5.6) $H_{m+n+1}(L_1 * L_2,\ \mathbb{Z}) = G \otimes H$.

Remark 4.9 and equality (5.6) yield

$\dim_{G\otimes H}(A \cup B) \leq m+n+1$. ∎

Proof of Theorem 5.2. Since G is a ring with unity 1, G is a retract $G \otimes G$. Indeed, the homomorphism $m : G \otimes G$ induced by the multiplication $G \times G \to G$ is a left inverse of $\mathrm{id} \otimes 1 : G \to G \otimes G$. Hence there exists an exact sequence

(5.7) $0 \to G \to G \otimes G \to G \to 0$

such that

(5.8) $G \to G \otimes G \to G = \mathrm{id}_G$.

The short exact sequence (5.7) generates the Bockstein exact sequence

(5.9) $\cdots \to H^p(Y,F;G) \to H^p(Y,F;G\otimes G) \to H^p(Y,F;G) \to H^{p+1}(Y,F;G) \to \cdots$

for any paracompact space Y and its closed subset F. In view of (5.8) we have

(5.10) $H^p(Y,F;G) \to H^p(Y,F;G\otimes G) \to H^p(Y,F;G) = \mathrm{id}$.

Let $Y = A \cup B$, $\dim_G A = m$, $\dim_G B = n$, and $p \geq m+n+2$. Then Theorem 5.3 implies that

(5.11) $H^p(Y,F;G\otimes G) = 0$

for any F. From (5.10) and (5.11) it follows that

(5.12) $H^p(Y,F;G) = 0$ for any $p \geq m+n+2$.

Hence $\dim_G(Y) = \dim_G(A \cup B) \leq m+n+1$. ∎

5.5. Corollary. *Suppose A, B are subsets of a first countable hereditarily paracompact space X and G is a ring with unity. Then*

$$\dim_G(A \cup B) \leq \dim_G A + \dim_G B + 1$$

in the following cases:

1) *G is countable*;

2) *X is perfectly normal.* ∎

Theorem 5.2 gives the best possible (at this time) answer to Kuzminov's question, if G is a ring with unity.

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(received 25.02.2009)

Mech.-Math. Faculty, Moscow State University, Moscow 119992, Russia.

E-mail: `vvfedorchuk@gmail.com`