

AN EFFECTIVE CRITERION FOR THE EXISTENCE OF A MASS PARTITION

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Abstract. Let μ be a proper Borel probability measure on the sphere S^2 in R^3 . It was conjectured that for every triple of rational numbers (q_1, q_2, q_3) with the property $q_1 + q_2 + q_3 = \frac{1}{2}$, there exist three planes in R^3 intersecting along the common line through the origin such that the six angular sectors on the sphere determined by those planes have respectively $q_1, q_2, q_3, q_1, q_2, q_3$ amount of the measure μ . In this paper we give an exact and explicitly realized algorithm which, for every triple (q_1, q_2, q_3) of the form $q_2 = q_3$, establishes whether there exists a configuration of three planes splitting the measure in the required proportion.

1. Introduction

The problem of a mass partition on the sphere S^2 by a configuration of three planes intersecting along the common line through the origin was first studied by V. Makeev [8] for the case of all equal parameters $q_1 = q_2 = q_3$. The generalized problem was stated and studied by the author and collaborators in [1], [2], [3]. It was proved that for parameters $(1, 1, 2)$, $(1, 1, 3)$, $(1, 2, 2)$ such partitions also exist. Following [3], the generalized problem can be stated as follows.

1.1. The problem

Consider a Borel probability measure μ on the sphere S^2 with the properties
(A) $\mu([a, b]) = 0$ for any circular arc $[a, b] \subset S^2$, and
(B) $\mu(U) > 0$ for each nonempty open set $U \subset S^2$.

A measure satisfying (A)–(B) on the sphere S^2 is a *proper* measure. Notice that there are no assumptions of any kind of symmetry on the measure μ .

PROBLEM 1. Determine all the six-tuples $\alpha = (q_1, \dots, q_6) \in \mathbb{N}^6$ with the property that for any proper Borel probability measure μ on the sphere S^2 there

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are three planes intersecting along a common line passing through the origin with angular sectors determined by hyperplanes having the prescribed amount of the measure $\mu(\sigma_i) = \frac{q_i}{q_1 + \dots + q_6}$, for all $i \in \{1, \dots, 6\}$. The six-tuples which satisfy these conditions are the *solutions* of the problem.

If the six-tuple $\alpha = (q_1, \dots, q_6)$ is a solution, then $q_1 = q_4$, $q_2 = q_5$, $q_3 = q_6$. Indeed, consider a uniform measure μ on the sphere S^2 . Then the measure of every angular sector bounded by two great semicircles is completely determined by the angle between tangent vectors to the boundary great semicircles. Since the sum of every three consecutive angles has to be π , then the first and the fourth, the second and the fifth, the third and the sixth angle must be the same. The equality of angles extends to the equality of measures since the measure μ is uniform. Thus the uniform measure, which is highly symmetric, provides necessary conditions on the set of parameters q_1, \dots, q_6 .

Since the only possible solutions are symmetric ones the notation can be simplified by listing only first three parameters q_1, q_2, q_3 .

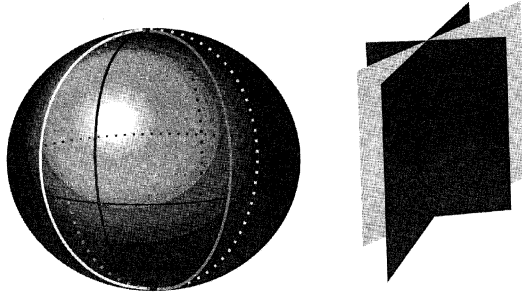


Fig. 1. Partition of the sphere measure into six angular sectors by three hyperplanes

1.2. Topological problem

We present a sketch of the configuration space/test map scheme presented in [1], [2], [3] to obtain a topological question which we address through computational obstruction theory. Let us fix a proper Borel probability measure μ on S^2 and a triple of parameters (q_1, q_2, q_3) .

A *k-fan* $(x; l_1, \dots, l_k)$ is a collection of a point $x \in S^2$ and k great semicircles l_1, \dots, l_k emanating from x . Instead of great semicircles one can also use the notation:

- (a) $(x; \sigma_1, \dots, \sigma_k)$ where σ_i is an open angular sector between l_i and l_{i+1} , $i = 1, \dots, k$; or
- (b) $(x; t_1, \dots, t_k)$ where $t_i \in T_x S^2$ is a tangent vector which is determined by the great semicircle curves l_i , $i = 1, \dots, k$. Here $T_x S^2$ denotes the tangent space at a point $x \in S^2$.

The space of all k -fans on the sphere S^2 is denoted by F_k .

For $n > 1$ consider the space

$$X_{\mu,n} = \{(x; t_1, \dots, t_n) \in F_n \mid \mu(\sigma_i) = \frac{1}{n} \text{ for every } i = 1, \dots, n\}.$$

Every n -fan $(x; t_1, \dots, t_n) \in X_{\mu,n}$ is determined by the pair $(x, t_1) \in S^2 \times T_x S^2$ and the measure μ . Thus $X_{\mu,n}$ is homeomorphic with the Stiefel manifold $V_2(\mathbb{R}^3)$.

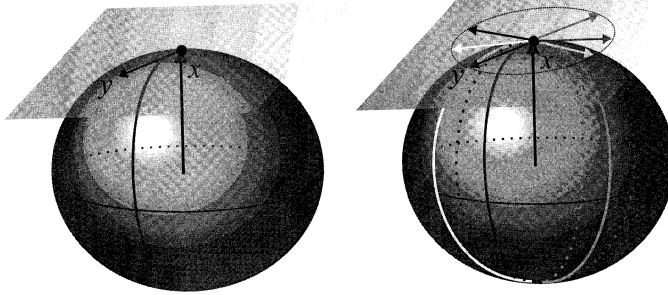


Fig. 2. Stiefel manifold and the tangent representation of a fan for $n = 6$

The test map associated to our problem is given by

$$\Phi : X_{\mu,n} \rightarrow W_n = \{\mathbf{y} \in \mathbb{R}^n \mid \sum_i y_i = 0\}, \Phi(x; t_1, \dots, t_n) = (\theta_1 - \frac{2\pi}{n}, \dots, \theta_n - \frac{2\pi}{n}),$$

where θ_i denotes the angle between tangent vectors t_i and t_{i+1} in the tangent plane $T_x S^2$. It is assumed that $t_{n+1} = t_1$. The map Φ is a continuous map because measure μ is a proper measure.

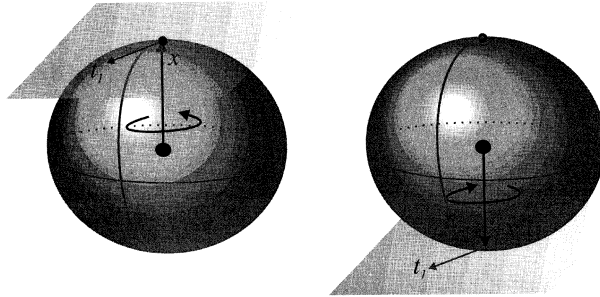


Fig. 3. Action of j on $X_{\mu,n}$

The dihedral group $D_{2n} = \langle j, \varepsilon \mid \varepsilon^n = j^2 = 1, \varepsilon j = j \varepsilon^{n-1} \rangle$ acts on the space $X_{\mu,n}$ and on the hyperplane W_n by

$$\begin{cases} \varepsilon(x; t_1, \dots, t_n) = (x; t_n, t_1, \dots, t_{n-1}) \\ j(x; t_1, \dots, t_n) = (-x; t_1, t_n, \dots, t_2) \end{cases} \quad \begin{cases} \varepsilon(y_1, \dots, y_n) = (y_n, y_1, \dots, y_{n-2}, y_{n-1}) \\ j(y_1, \dots, y_n) = (y_n, y_{n-1}, \dots, y_2, y_1) \end{cases} \quad (1)$$

where $(x; t_1, \dots, t_n) \in X_{\mu,n}$ and $(y_1, \dots, y_n) \in W_n$. The action of j on $X_{\mu,n}$ is derived from the fact that the pair of unit vectors $(-x, t_1)$ determines the opposite orientation of \mathbb{R}^3 to the one given by the pair (x, t_1) ; see Figure 3.

The action on $X_{\mu,n}$ is free and the test map Φ is an equivariant map.

The test subspace where possible solutions of the problem lie is the union $\bigcup \mathcal{A}_n \subset W_n$ of the smallest D_{2n} -invariant arrangement \mathcal{A}_n , which contains the linear subspace $L \subset W_n$ defined by the equalities

$$y_1 + \dots + y_{\frac{n}{2}} = y_{q_1+1} + \dots + y_{q_1+\frac{n}{2}} = y_{q_1+q_2+1} + \dots + y_{q_1+q_2+\frac{n}{2}} = 0. \quad (2)$$

Indeed, if $(x, t_1, \dots, t_n) \in \Phi(X_{\mu,n}) \cap (\bigcup \mathcal{A}_n)$ then

$$\begin{aligned} \theta_1 - \frac{2\pi}{n} + \dots + \theta_{\frac{n}{2}} - \frac{2\pi}{n} &= \theta_{q_1+1} - \frac{2\pi}{n} + \dots + \theta_{q_1+\frac{n}{2}} - \frac{2\pi}{n} \\ &= \theta_{q_1+q_2+1} - \frac{2\pi}{n} + \dots + \theta_{q_1+q_2+\frac{n}{2}} - \frac{2\pi}{n} = 0 \end{aligned}$$

where, again θ_i denotes the angle between tangent vectors t_i and t_{i+1} in the tangent plane $T_x S^2$. Consequently,

$$\theta_1 + \dots + \theta_{\frac{n}{2}} = \theta_{q_1+1} + \dots + \theta_{q_1+\frac{n}{2}} = \theta_{q_1+q_2+1} + \dots + \theta_{q_1+q_2+\frac{n}{2}} = \pi.$$

Therefore $l_1 \cup l_{\frac{n}{2}}$, $l_{q_1+1} \cup l_{q_1+\frac{n}{2}}$ and $l_{q_1+q_2+1} \cup l_{q_1+q_2+\frac{n}{2}}$ are great circles on the sphere S^2 and the fan $(x, t_1, \dots, t_n) \in \Phi(X_{\mu,n}) \cap (\bigcup \mathcal{A}_n)$ is a solution of the problem.

The basic proposition of the method holds (cf. [3], [1]).

PROPOSITION 2. *If there is no continuous D_{2n} -equivariant map $V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A}_n$, then for every proper Borel probability measure on the sphere S^2 , there exist three planes intersecting along the common line throughout the origin with angular sectors such that, for every $i \in \{1, \dots, 6\}$, $\mu(\sigma_i) = q_i/n$.*

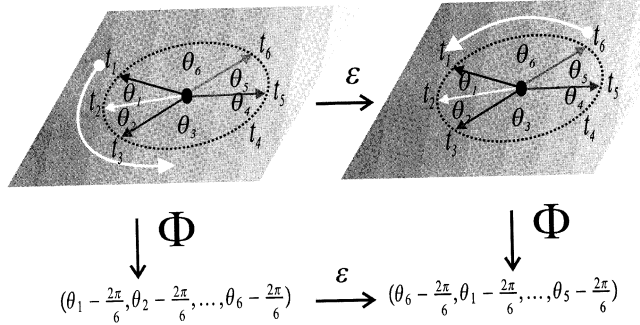


Fig. 4. Map Φ is an equivariant map

The following proposition is proved in [4]. Let Q_{4n} denote the generalized quaternion group $\langle \epsilon, j \rangle \subset S^3$ (cf. [5, p. 253] and [3, Appendix A.1, p. 2654]). The group Q_{4n} acts on S^3 as a subgroup, and on W_n by the quotient homomorphism $Q_{4n} \rightarrow Q_{4n}/\{1, -1\} \cong D_{2n}$ and already defined D_{2n} -action on W_n . The Q_{4n} -action on S^3 is free.

PROPOSITION 3. *The following continuous equivariant maps coexist*

$$D_{2n}\text{-map } V_2(\mathbb{R}^3) \rightarrow W_n \setminus \bigcup \mathcal{A}_n \quad \text{and} \quad Q_{4n}\text{-map } S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n.$$

In the light of Propositions 2 and 3 we obtained the following topological problem.

PROBLEM 4. Determine all symmetric six-tuples $\alpha = (q_1, q_2, q_3, q_1, q_2, q_3) \in \mathbb{N}^6$ such that there is no Q_{4n} -equivariant map $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n$, where the arrangement \mathcal{A}_n is the minimal Q_{4n} -invariant (D_{2n} -invariant) arrangement containing the linear subspace $L \subset W_n$ defined by the equations (2).

1.3. Statement of main result

In this paper we address Problem 4 through computational equivariant obstruction theory [3]. We give an explicit and exact algorithm (Section 2) which for the given parameters q_1 and q_2 establishes the nonexistence of a Q_{4n} -equivariant map $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n$ for a symmetric six-tuple $(q_1, q_2, q_2, q_1, q_2, q_2)$. The given algorithm is realized in the computational package `Mathematica` [7]. The results obtained so far are summarized in the following theorem.

THEOREM 5. *For all triples (q_1, q_2, q_2) such that $\frac{n}{2} = q_1 + 2q_2 \leq 40$, and any proper Borel probability measure μ on S^2 ,*

- (A) *there are no continuous Q_{4n} -equivariant maps $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n$, which implies that*
- (B) *there are three planes intersecting along the common line throughout the origin such that angular sectors determined by them have the $(q_1, q_2, q_2, q_1, q_2, q_2)$ proportion of the measure μ .*

The same algorithm can provide an answer for any set of parameters when $n > 80$, up to a computational time which grows. The assumption that $q_2 = q_3$ reflects the properties of the arrangement \mathcal{A}_n and surfaces in the equations (4). The algorithm given in [7] for given parameters q_1 and $q_2 = q_3$ proves the theorem by applying the procedure of computing the obstruction cocycle presented in Section 2 followed by a parity criterion (Theorem 9).

2. Computational obstruction theory

In this section we follow the method of the computational obstruction theory explained in [3, Section 1.5.] for setting up the algorithm for questioning the existence of a Q_{4n} -map $S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n$ when $q_2 = q_3$.

The assumptions for the application of the equivariant obstruction theory are satisfied because:

- $\dim S^3 - 2 = \text{conn}(W_n \setminus \bigcup \mathcal{A}_n) = 1$, $\text{codim}_{W_n}(\bigcup \mathcal{A}_n) = 3$,
- $W_n \setminus \bigcup \mathcal{A}_n$ is 2-simple, $\pi_1(W_n \setminus \bigcup \mathcal{A}_n) = 0$ acts trivially on $\pi_2(W_n \setminus \bigcup \mathcal{A}_n) \cong H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z})$,
- the action of Q_{4n} on S^3 is free.

Therefore the existence of an equivariant map depends on the *primary obstruction element* $[\mathfrak{o}] = [\mathfrak{o}(f)]$ living in the equivariant cohomology group

$$H_{Q_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z})).$$

The obstruction element does not depend on the particular Q_{4n} -map f from the 2-skeleton of the sphere $(S^3)^{(2)}$ to the complement $W_n \setminus \bigcup \mathcal{A}_n$.

Thus, the obstruction element can be determined by the explicit computation of the obstruction cocycle $\mathfrak{o}(f)$ for some nicely defined Q_{4n} -map $f : (S^3)^{(2)} \rightarrow W_n \setminus \bigcup \mathcal{A}_n$. Such a map is called a map *in general position* and should satisfy the properties in [3, Definition 1.5, p. 2639]. After obtaining the obstruction cocycle $\mathfrak{o}(f)$, its class in cohomology is the obstruction element and

$[\mathfrak{o}] = [\mathfrak{o}(f)] = 0$ if and only if there is a continuous Q_{4n} -map $f : S^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n$.

Since the action of Q_{4n} on the sphere S^3 is free it can be substituted by any other free Q_{4n} action on the sphere S^3 . This observation allows us to choose a particular action on the sphere in the process of computation of the obstruction cocycle.

The algorithm of computing the obstruction cocycle $\mathfrak{o} = \mathfrak{o}(f)$, for a map $f : S_{(2)}^3 \rightarrow W_n \setminus \bigcup \mathcal{A}_n$ in general position has following three steps (cf. [3]). It is followed by a parity criterion for the non-vanishing of the obstruction element $[\mathfrak{o}]$.

2.1. Equivariant simplicial structure on S^3

The sphere S^3 can be seen as the Q_{4n} simplicial complex $P_n^{(1)} * P_n^{(2)}$ where $P_n^{(i)}$, $i = 1, 2$, denotes a regular n -gon. We denote vertices of $P_n^{(1)}$ by a_0, \dots, a_{n-1} and vertices of $P_n^{(2)}$ by b_0, \dots, b_{n-1} . Then all tetrahedra in the join $P_n^{(1)} * P_n^{(2)}$ are of the form $[a_r, a_{r+1}, b_k, b_{k+1}]$. The action of Q_{4n} on $P_n^{(1)} * P_n^{(2)}$ is given on 0-skeleton in the following way: let t denote a_0 vertex then

$$a_i = \epsilon^i \cdot t \quad \text{and} \quad b_j = \epsilon^i j \cdot t.$$

The action extends to all skeleta equivariantly. In this way we equipped the sphere S^3 with a free Q_{4n} -action.

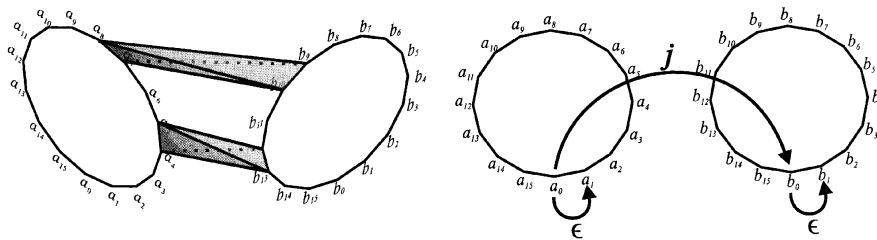


Fig. 5. Q_{4n} simplicial structure on S^3

This particularly chosen free Q_{4n} action on S^3 is not the one introduced in (1). The reason why the substitution of actions can be done, as we already indicated, is the consequence of the following lemma from the obstruction theory.

LEMMA 6. *Let G be a finite group and let the sphere S^n be equipped with two free G actions \cdot and \circ . Then there exist G -equivariant maps $(S^n, \cdot) \rightarrow (S^n, \circ)$ and $(S^n, \circ) \rightarrow (S^n, \cdot)$.*

Proof. The sphere S^n is $(n - 1)$ -connected space, both actions are free and therefore there is no obstruction for the existence of G map $(S^n, \cdot) \rightarrow (S^n, \circ)$ or $(S^n, \circ) \rightarrow (S^n, \cdot)$. ■

2.2. A map in a general position

We define a Q_{4n} -equivariant map $F : P_n^{(1)} * P_n^{(2)} \rightarrow W_n$ by defining its image on the vertex t :

$$F(t) := \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-1}, -\sum_{i=1}^{n-1} \frac{1}{i} \right)$$

and then extend it equivariantly. The requirement that F is piecewise affine determines the extension of F uniquely. The image of t should be chosen in such a way that it satisfies the property of being in general position toward the arrangement \mathcal{A}_n , [3, Definition 1.5, p. 2639]. Particularly, in this situation this means that there is no triangle τ in $P_n^{(1)} * P_n^{(2)}$ such that $F(\tau) \cap (\bigcup \mathcal{A}_n) \neq \emptyset$. This property is tested by the algorithm for every choice of parameters q_1, q_2 . If the property is not satisfied the algorithm reports and allows manual intervention in definition of $F(t)$.

Let $f = F|_{(S^3)^{(2)}}$ be a map from the second skeleton of the sphere S^3 . Since F is in general position f can be seen as a Q_{4n} -map $(S^3)^{(2)} \rightarrow W_n \setminus (\bigcup \mathcal{A}_n)$. Therefore it is enough to compute the obstruction cocycle of the map f .

2.3. Obstruction cocycle $\mathfrak{o}(f)$

The obstruction cocycle $\mathfrak{o}(f)$ is computed using [3, Proposition 1.6, p. 2640]. When $F : P_n^{(1)} * P_n^{(2)} \rightarrow W_n$ is a map in general position with respect to \mathcal{A}_n , then for every 3-simplex $\sigma = [a_i, a_{i+1}, b_j, b_{j+1}]$ of $P_n^{(1)} * P_n^{(2)}$ the value of the obstruction cocycle $\mathfrak{o}(f)$ on σ is given by:

$$\mathfrak{o}(f)(\sigma) = \sum_{x \in F^{-1}(F(\sigma) \cap (\bigcup \mathcal{A}_n))} I(\sigma, S_{F(x)}) \|f(x)\| \in H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z}). \quad (3)$$

Here $I(\sigma, S_{F(x)}) \in \mathbb{Z}$ denotes the intersection number of the image $F(\sigma)$ and the appropriate oriented element $S_{F(x)}$ of the arrangement \mathcal{A}_n . The homology class $\|F(x)\|$ is the so-called point class associated with the point $\|f(x)\|$ and the element of the arrangement $S_{F(x)}$. The notion of point classes was introduced in [4].

The formula (3) is the heart of the algorithm [7]. By computing all intersections

$$F([a_i, a_{i+1}, b_j, b_{j+1}]) \cap (\bigcup \mathcal{A}_n)$$

and testing whether they belong to more than one maximal element of the arrangement, the algorithm obtains an explicit formula for the obstruction cocycle of the map $f = F|_{(S^3)^{(2)}}$.

The essential consequence of the assumption $q_2 = q_3$ is that

$$\epsilon^{\frac{n}{2}}L = L \quad \text{and} \quad \epsilon^{-2q_2}jL = L. \quad (4)$$

This means that the arrangement \mathcal{A}_n has $\frac{n}{2}$ maximal elements. Therefore it is enough to determine the $n \times \frac{n}{2}$ intersections $f(P_n^{(1)} * P_n^{(2)}) \cap (\bigcup \mathcal{A}_n)$ of the form

$$f([a_i, a_{i+1}, b_0, b_1]) \cap \epsilon^r L = f([\epsilon^i t, \epsilon^{i+1} t] * [j t, \epsilon j t]) \cap \epsilon^r L$$

where $i \in \{0, \dots, n-1\}$ and $r \in \{0, \dots, \frac{n}{2} - 1\}$.

2.4. Parity criterion for $[\mathfrak{o}] \neq 0$

In this subsection we give an explicit criterion when the obstruction cocycle given by

$$\mathfrak{o}(f) \in C_{Q_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z}))$$

does not vanish while passing to cohomology into the obstruction element

$$[\mathfrak{o}(f)] \in H_{Q_{4n}}^3(S^3, H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z})).$$

An Economical Q_{4n} cell structure on S^3 . As presented in [5, p. 253] and [3, Appendix A, pp. 2654–2655], there is a Q_{4n} cell structure on S^3 with just one equivariant generator in dimension 3. The cellular complex has one Q_{4n} 0-cell a , two Q_{4n} 1-cells b and b' , two Q_{4n} 2-cells c and c' , and one Q_{4n} 3-cell e . The chain complex $\mathfrak{D} = \{D_i\}$ is given by

$$0 \rightarrow \mathbb{Z}[Q_{4n}]e \xrightarrow{\partial} \mathbb{Z}[Q_{4n}]c \oplus \mathbb{Z}[Q_{4n}]c' \xrightarrow{\partial} \mathbb{Z}[Q_{4n}]b \oplus \mathbb{Z}[Q_{4n}]b' \xrightarrow{\partial} \mathbb{Z}[Q_{4n}]e \rightarrow 0$$

where

$$\begin{aligned} \partial e &= (\epsilon - 1)c - (\epsilon j - 1)c', & \partial c &= (1 + \dots + \epsilon^{n-1})b - (j + 1)b', & \partial b &= (\epsilon - 1)a, \\ \partial c' &= (\epsilon j + 1)b + (\epsilon - 1)b', & \partial b' &= (j - 1)a. \end{aligned}$$

The just defined Q_{4n} complex can be cellularly mapped to the Q_{4n} simplicial complex $P_n^{(1)} * P_n^{(2)}$ such that in dimension 3 it is given by

$$e \longmapsto [a_0, a_1, b_0, b_1] + [a_1, a_2, b_0, b_1] + \dots + [a_{n-1}, a_n, b_0, b_1].$$

The following proposition is the consequence of the existence of just one equivariant generator in the Q_{4n} cell structure on S^3 .

PROPOSITION 7. *The obstruction cocycle, in terms of the Q_{4n} cell structure on S^3 , again denoted by $\mathfrak{o}(f)$, is completely determined by the value*

$$\mathfrak{o}(f)(e) = \sum_{i=0}^{n-1} \mathfrak{o}(f)([a_i, a_{i+1}, b_0, b_1]) \in H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z}). \quad (5)$$

Q_{4n} -module structure on $H_2(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z})$. Following the same argument as in [3, Eq. (20–24), pp. 2646–2647] we have the isomorphism of Q_{4n} -modules

$$\varphi: H_2\left(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z}\right) \rightarrow \text{Hom}\left(H_{n-4}\left(\bigcup \widehat{\mathcal{A}}_n, \mathbb{Z}\right), \mathbb{Z}\right)$$

given on the point class $\|y\|$ as the evaluation of the linking number

$$\varphi(\|y\|)(l) = \text{link}(l, \|y\|)$$

where $l \in H_{n-4}(\bigcup \widehat{\mathcal{A}}_n, \mathbb{Z})$. Here $\widehat{\mathcal{A}}_n$ denotes the one-point compactification of the arrangement \mathcal{A}_n . Moreover there is a Q_{4n} decomposition of modules

$$\begin{aligned} H_{n-4}\left(\bigcup \widehat{\mathcal{A}}_n, \mathbb{Z}\right) &\cong \bigoplus_{d=0}^{n-4} \left(\bigoplus_{V \in P: \dim V=d} \tilde{H}_{n-5-d}(\Delta(P_{<V}), \mathbb{Z}) \right) \\ &\cong \left(\bigoplus_{V \in P: \dim V=n-4} \mathbb{Z} \right) \oplus \left(\bigoplus_{d=0}^{n-5} \left(\bigoplus_{V \in P: \dim V=d} \tilde{H}_{n-5-d}(\Delta(P_{<V}), \mathbb{Z}) \right) \right) \end{aligned}$$

where P denotes the intersection poset of the arrangement $\widehat{\mathcal{A}}_n$. Thus there is an exact sequence of Q_{4n} -modules

$$0 \longrightarrow \bigoplus_{V \in P: \dim V=n-4} \underbrace{\tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z})}_{\cong \mathbb{Z}} \longrightarrow H_{n-4}\left(\bigcup \widehat{\mathcal{A}}_n, \mathbb{Z}\right)$$

which induces an exact sequence of Q_{4n} -modules

$$H_2\left(W_n \setminus \bigcup \mathcal{A}_n, \mathbb{Z}\right) \xrightarrow{\chi} \text{Hom}\left(\bigoplus_{V \in P: \dim V=n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right) \longrightarrow 0. \quad (6)$$

The map χ is geometrically interpreted as the computation of the linking numbers. The left exactness of the coinvariant functor, implies the exactness of the sequence

$$H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right)_{Q_{4n}} \xrightarrow{\chi^*} \text{Hom}\left(\bigoplus_{V \in P: \dim V=n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right)_{Q_{4n}} \longrightarrow 0. \quad (7)$$

The map χ^* is the summation of linking numbers along the orbit.

PROPOSITION 9. $\text{Hom}\left(\bigoplus_{V \in P: \dim V=n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right)_{Q_{4n}} \cong \mathbb{Z}/2$.

Proof. The conclusion follows along the same lines as in [3, Lemma 2.5, p. 2647] using (4). ■

The criterion. With all the assumptions and notions introduced in Section 2 we state the criterion.

THEOREM 10. *If the number of elements of the set $F^{-1}(F(e) \cap (\bigcup \mathcal{A}_n))$ is odd then $[\mathfrak{o}(f)] = [\mathfrak{o}] \neq 0$.*

Proof. Let $|F^{-1}(F(e) \cap (\bigcup \mathcal{A}_n))|$ be an odd number. Then by (3) the obstruction cocycle is

$$\mathfrak{o}(f)(e) = \sum_{x \in F^{-1}(F(e) \cap (\bigcup \mathcal{A}_n)) : \sigma \subset e} \mathbf{I}(\sigma, S_{f(x)}) \|f(x)\|$$

where the sum has odd number of summands and $\mathbf{I}(\sigma, S_{f(x)}) \in \{-1, +1\}$. Using Proposition 7 we can think of the obstruction element $\mathfrak{o}(f)$ as an element of the coefficient group $\mathfrak{o}(f)(e) \in H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})$. The obstruction element $[\mathfrak{o}(f)](e)$ is the image of $\mathfrak{o}(f)(e)$ along the quotient homomorphism (cf. [3, Proposition 1.6, p. 2640])

$$\zeta : H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right) \longrightarrow H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right)_{Q_{4n}}.$$

Here $H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})_{Q_{4n}}$ denotes the group of coinvariants of the Q_{4n} -module $H_2(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z})$.

The image of the obstruction cocycle $\mathfrak{o}(f)(e)$, mapped by the composition

$$\chi^* \circ \zeta : H_2\left(W_n \setminus \bigcup \mathcal{A}, \mathbb{Z}\right) \longrightarrow \text{Hom}\left(\bigoplus_{V \in P: \dim V = n-4} \tilde{H}_{n-5-d}(\Delta(P_{<V}); \mathbb{Z}), \mathbb{Z}\right)_{Q_{4n}}$$

is given by

$$\mathfrak{o}(f)(e) \xrightarrow{\zeta} [\mathfrak{o}(f)] \xrightarrow{\chi^*} \sum_{x \in F^{-1}(F(e) \cap (\bigcup \mathcal{A}_n)) : \sigma \subset e} \mathbf{I}(\sigma, S_{f(x)}).$$

Since the number of summands is odd, the obstruction element is different from zero. ■

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