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ON THE PERIODS OF 2-STEP GENERAL FIBONACCI SEQUENCES IN DIHEDRAL GROUPS

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Abstract. In this paper, we investigate the simply periodic cases of 2-step general Fibonacci sequences in dihedral groups D_n, and we also find the period of the sequences if the sequences are simply periodic.

1. Introduction

The study of Fibonacci sequences in groups began with the earlier work of Wall [9] where the ordinary Fibonacci sequences in cyclic groups were investigated. In the mid-eighties, Wilcox extended the problem to abelian groups [10]. Prolific co-operation of Campbell, Doostie and Robertson expanded the theory to some finite simple groups [3]. Aydin and Smith proved in [2] that the lengths of ordinary 2-step Fibonacci sequences are equal to the lengths of ordinary 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 4 and a prime exponent. The theory has been generalized in [4,5] to the ordinary 3-step Fibonacci sequences in finite nilpotent groups. Then, it is shown in [1] that the period of 2-step general Fibonacci sequence is equal to the length of fundamental period of the 2-step general recurrence constructed by two generating elements of the group of exponent p and nilpotency class 2. Karaduman and Yavuz showed that the periods of the 2-step Fibonacci recurrences in finite nilpotent groups of nilpotency class 5 and a prime exponent are $p \cdot k(p)$, for $2 < p \leqslant 2927$, where p is prime and $k(p)$ are the periods of ordinary 2-step Fibonacci sequences [7]. The theory has been generalized in [6] to the 2-step general Fibonacci sequences in finite nilpotent groups of nilpotency class 4 and exponent p. Knox proved that periods of k-nacci (k-step Fibonacci) sequences in dihedral group were equal to $2k+2$ [8].

This paper is related to period of 2-step general Fibonacci sequences in dihedral groups. It not only addresses to the question whether the sequences are simply periodic or not, but it also gives an upper bound for period of the sequences when n is odd.

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A k-nacci sequence in a finite group is a sequence of group elements x_0, x_1, x_2, ... , x_n, ... for which, given an initial (seed) set x_0, x_1, x_2, ... , x_{j-1}, each element is defined by

$$x_n = \begin{cases} x_0x_1x_2\cdots x_{n-1}, & \text{for } j \leqslant n < k, \\ x_{n-k}x_{n-k+1}\cdots x_{n-1}, & \text{for } n \geqslant k. \end{cases}$$

We also require that the initial elements of the sequence, x_0, x_1, x_2, ... , x_{j-1}, generate a group, thus forcing the k-nacci sequence to reflect the structure of the group. The k-nacci sequence of a group generated by x_0, x_1, x_2, ... x_{j-1} is denoted by $f_k(G;\, x_0, x_1, \dots, x_{j-1})$.

2-step Fibonacci sequence in the integers modulo m can be written as $F_2(Z_m;\, 0, 1)$. We call a 2-step Fibonacci sequence of a group elements a Fibonacci sequence of a finite group. A finite group G is k-nacci sequenceable if there exists a k-nacci sequence of G such that every element of the group appears in the sequence.

A sequence of group elements is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is called period of the sequence. For example, the sequence a, b, c, d, e, b, c, d, e, ... is periodic after the initial element a and has period 4. A sequence of group elements is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence a, b, c, d, e, f, a, b, c, d, e, f, ... is simply periodic with period 6.

A group D_n is dihedral if

$$D_n = \langle a, b : a^n = e,\ b^2 = e,\ ab = ba^{-1}\rangle.$$

The order of the group D_n is $2n$. Note that in a dihedral group generated by a and b,

$$(ab)^2 = abab = aa^{-1}bb = e.$$

THEOREM 1.1. [8] *A k-nacci sequence in a finite group G is simply periodic.*

THEOREM 1.2. [8] *For some $n \geqslant 3$, $P_k(D_n;\, a, b) = 2k + 2$.*

2.Main results and proof

Firstly, we give a lemma which will be used in proofs of our theorems.

LEMMA 2.1. *In the dihedral group*

$$D_n = \langle a, b : a^n = e,\ b^2 = e,\ ab = ba^{-1}\rangle,$$

(i) $(a^xb)^y = e$*, for* $x = 1, 2, \dots, n-1$*, where* y *is any even integer.*

(ii) $(a^xb)^z = (a^xb)$*, for* $x = 1, 2, \dots, n-1$*, where* z *is any odd integer.*

Proof. (i) Let y be an even integer. We prove this fact by induction argument on y. If $y = 2$, then we have

$$(a^xb)^2 = a^xba^xb = a^xa^{-1}ba^{x-1}b = a^xa^{-2}ba^{x-2}b = \cdots = a^xa^{-x}bb = e.$$

So, our argument holds for $y = 2$. Now, we assume that

$$(a^x b)^{2t} = e.$$

Multiplying both sides by $(a^x b)^2$, we have

$$(a^x b)^{2t}(a^x b)^2 = (a^x b)^{2t+2} = e(a^x b)^2 = ee = e.$$

Hence, we are done.

(ii) Let $z = 2t + 1$ be an odd integer. By (i), we have $(a^x b)^{2t} = e$. Thus,

$$(a^x b)^{2t+1} = (a^x b)^{2t}(a^x b) = e(a^x b) = (a^x b).$$

So, we are done. ■

Let $n \geqslant 3$. Consider the dihedral group D_n with generators a, b. The 2-step general Fibonacci sequences in D_n are defined as

$$x_0 = a, \quad x_1 = b \quad \text{and for } i \geqslant 2, \quad x_i = x_{i-2}^m x_{i-1}^l,$$

where m, l are integers.

Theorem 2.2. *In D_n, we have*

(i) *If $m \equiv 0 \pmod{n}$, then 2-step general Fibonacci sequences in D_n are not simply periodic for any l.*

(ii) *If m is any even integer, then 2-step general Fibonacci sequences in D_n are not simply periodic for any l.*

(iii) *If $m \mid n$ such that $m \neq 1$, then 2-step general Fibonacci sequences in D_n are not simply periodic for any l.*

Proof. (i) Let $m \equiv 0 \pmod{n}$. Then, there exists an $x \in \mathbf{Z}$ such that $m = nx$. Thus, the sequence is

$$x_0 = a, \quad x_1 = b, \quad x_2 = b^l, \quad x_3 = b^m b^{l^2}, \quad \ldots$$

Since, the elements succeeding x_2, x_3 do not depend on a, b and the cycle does not begin again with a, b, the sequence is not simply periodic. We are done.

(ii) Let m be an even integer such that $m = 2t$, $t \in \mathbf{Z}$. If l is an even integer, then the sequence will be

$$x_0 = a, \quad x_1 = b, \quad x_2 = a^{2t}, \quad x_3 = a^{2tl}, \quad \ldots$$

Since $b^l = e$, the elements succeeding x_2, x_3 does not include b and the cycle does not begin again with a, b, the sequence is not simply periodic. If l is an odd integer, then the sequence will be

$$x_0 = a, \quad x_1 = b, \quad x_2 = a^{2t}b, \quad x_3 = a^{2t}b, \quad \ldots$$

Since $(a^{2t}b)^m = e$, $b^l = b$ and $(a^{2t}b)^l = a^{2t}b$, from Lemma 1,(i) and (ii), the cycle does not begin again with a, b, so the sequence is not simply periodic. We are done.

(iii) Let $m \mid n$ such that $m \neq 1$. Then, there exists an $x \in \mathbf{Z}$ such that $n = mx$. Notice that m is odd. If $x = 1$, then $m = n$ and the sequence will not be simply periodic, from Theorem 2.2.(i). Therefore, two cases occur, which are: l is odd and l is even.

Case 1. Let l be an odd integer. We know that $(a^m b)^l = a^m b$ and $b^l = b$, from Lemma 1,(ii). Hence, the sequence will be

$$x_0 = a, \quad x_1 = b, \quad x_2 = a^m b, \quad x_3 = a^{n-m}, \quad \ldots$$

Notice that we have $n - m = m$ or $n - m = mt$ $(t \in \mathbf{Z})$.

I. If $n - m = m$, then the sequence will be

$$x_0 = a, \quad x_1 = b, \quad x_2 = a^m b, \quad x_3 = a^m, \quad x_4 = a^m b a^{ml}, \quad \ldots$$

Notice that, the cases

$$(1) \quad ml \equiv 0 \pmod{n}, \qquad (2) \quad ml \equiv mt \pmod{n}$$

occur. Also, in the subcases (1) and (2), the cases $mm \equiv 0 \pmod{n}$ and $mm \equiv m \pmod{n}$ occur. Similarly,

II. If $n - m = mt$ $(t \in \mathbf{Z})$, then the cases

$$(1') \quad (mt)l \equiv 0 \pmod{n}, \qquad (2')(mt)l \equiv mt \pmod{n}, \ \ t \in \mathbf{Z}$$

occur. Also, in the cases (1') and (2'), the subcases, $(mt)m \equiv 0 \pmod{n}$ and $(mt)m \equiv mt \pmod{n}$ occur.

Now, for $n - m = m$, if $ml \equiv 0 \pmod{n}$, then the sequence reduces to

$$x_0 = a, \ \ x_1 = b, \ \ x_2 = a^m b, \ \ x_3 = a^m, \ \ x_4 = a^m b, \ \ x_5 = a^m b, \ \ x_6 = e, \ \ \ldots$$

$(mm \equiv 0 \pmod{n})$ or

$$x_0 = a, \ x_1 = b, \ x_2 = a^m b, \ x_3 = a^m, \ x_4 = a^m b, \ x_5 = a^{2m} b, \ x_6 = a^m, \ x_7 = a^m b,$$

$\ldots$ $(mm \equiv m \pmod{n})$. Thus, it can be easily seen that the cycle does not begin again with a, b in any of the cases. Therefore, the sequence is not simply periodic.

Now, for $n - m = m$, if $ml \equiv m \pmod{n}$, then the sequence reduces to

$$x_0 = a, \ \ x_1 = b, \ \ x_2 = a^m b, \ \ x_3 = a^m, \ \ x_4 = b, \ \ x_5 = b, \ \ x_6 = e, \ \ \ldots$$

$(mm \equiv 0 \pmod{n})$ or

$$x_0 = a, \ x_1 = b, \ x_2 = a^m b, \ x_3 = a^m, \ x_4 = b, \ x_5 = a^m b, \ x_6 = a^m,$$

$\ldots$ $(mm \equiv m \pmod{n})$. Thus, it can be easily seen that the cycle does not begin again with a, b, in both cases. Therefore, the sequence is not simply periodic.

Similarly, we study case II.

Now, for $n - m = mt$ $(t \in \mathbf{Z})$, if $(mt)l \equiv 0 \pmod{n}$, then the sequence reduces to

$$x_0 = a, \ \ x_1 = b, \ \ x_2 = a^m b, \ \ x_3 = a^{mt}, \ \ x_4 = a^m b,$$
$$x_5 = a^m b, \ \ x_6 = e, \ \ x_7 = a^m b, \ \ x_8 = a^m b, \ \ x_9 = e, \ \ \ldots \ \ ((mt)m \equiv 0 \pmod{n})$$

or

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m b,\ \ x_3 = a^{mt},\ \ x_4 = a^m b,\ \ x_5 = a^{mt+m} b,$$
$$x_6 = a^{-mt},\ \ x_7 = a^{mt+m} b,\ \ x_8 = a^{mt+m} b,\ \ x_9 = e,\ \ \ldots\ \ ((mt)m \equiv mt \pmod{n}).$$

Thus, it can be easily seen that the cycle does not begin again with a, b in both cases. Therefore, the sequence is not simply periodic.

Now, for $n - m = mt$, if $(mt)l \equiv mt \pmod{n}$, then the sequence reduces to

$$x_0 = a,\ x_1 = b,\ x_2 = a^m b,\ x_3 = a^{mt},\ x_4 = a^{m-mt} b,\ x_5 = a^{m-mt} b,\ x_6 = e,\ \ldots$$

$((mt)m \equiv 0 \pmod{n})$ or

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m b,\ \ x_3 = a^{mt},\ \ x_4 = a^{m-mt} b,$$
$$x_5 = a^m b,\ \ x_6 = a^{-mt},\ \ x_7 = a^{mt+m} b,\ \ \ldots\ \ ((mt)m \equiv mt \pmod{n}).$$

Thus, it can be easily seen that the cycle does not begin again with a, b in both cases. Therefore, the sequence is not simply periodic.

Case 2. Let l be an even integer. We know that $(a^m b)^l = e$ and $b^l = e$, from Lemma 1,(i). Hence, the sequence will be

$$x_0 = a, \quad x_1 = b, \quad x_2 = a^m, \quad x_3 = a^{n-ml} b, \quad \ldots$$

In this case, two subcases, which are $n - ml \equiv 0 \pmod{n}$ and $n - ml \equiv mt \pmod{n}$ occur. In other words, we have $n - ml = 0$ or $n - ml = mt$ ($t \in \mathbf{Z}$). For each of these cases, the possible subcases are $mm \equiv 0 \pmod{n}$, $mm \equiv m \pmod{n}$ and $mm \equiv mv \pmod{n}$.

If $ml \equiv 0 \pmod{n}$, then the sequence reduces to

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = b,\ \ x_4 = e,\ \ \ldots \text{for } mm \equiv 0 \pmod{n},$$
$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = b,\ \ x_4 = a^m,\ \ \ldots \text{for } mm \equiv m \pmod{n},$$
$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = b,\ \ x_4 = a^{m^2},\ \ x_5 = a^{n-m^2 l},\ \ x_6 = a^{m^3},$$
$$\ldots \quad \text{for } mm \equiv mv \pmod{n},\ \ (v \in \mathbf{Z}).$$

For the first two subcases, it is clear that the sequence is not simply periodic. Notice that, since $m \mid n$, after certain steps $m^v \equiv m \pmod{n}$ and $(n - m^2 l - m^3 l - \cdots - m^u l) \equiv 0 \pmod{n}$ in the third subcase. Thus, the sequence reduces to

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = b,\ \ x_4 = a^{m^2},\ \ x_5 = a^{n-m^2 l},$$
$$x_6 = a^{m^3},\ \ \ldots x_\eta = b,\ \ x_{\eta+1} = a^m,\ \ \ldots$$

which is not simply periodic.

If $ml \equiv mt \pmod{n}$, the sequence reduces to

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = a^{n-ml} b,\ \ x_4 = e,\ \ x_5 = a^{n-ml} b,\ \ \ldots$$

for $mm \equiv 0 \pmod{n}$, which is not simply periodic. On the other hand, when $ml \equiv mt \pmod{n}$ if $mm \equiv m \pmod{n}$, we have,

$$x_0 = a,\ x_1 = b,\ x_2 = a^m,\ x_3 = a^{n-ml}b,\ x_4 = a^m,\ x_5 = a^{n-2ml}b,\ x_6 = a^m,\ \ldots$$

for $mm \equiv m \pmod{n}$. Since $m \mid n$, after certain steps, we have $(n - m^2 l - m^3 l - \cdots - m^u l) \equiv 0 \pmod{n}$. Thus, the sequence reduces to

$$\begin{aligned} &x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = a^{n-ml}b,\ \ x_4 = a^m,\\ &x_5 = a^{n-2ml}b,\ \ \ldots,\ \ x_\mu = b,\ \ x_{\mu+1} = a^m,\ \ \ldots \quad \text{for } mm \equiv m \pmod{n}, \end{aligned}$$

which is not simply periodic.

Now, when $ml \equiv mt \pmod{n}$, if $mm \equiv mv \pmod{n}$ the sequence reduces to

$$\begin{aligned} &x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = a^{n-ml}b,\ \ x_4 = a^{m^2},\\ &x_5 = a^{n-ml-m^2 l}b,\ \ x_6 = a^{m^3},\ \ \ldots \end{aligned}$$

Since $m \mid n$, in third subcase, after certain steps there are $v, u \in \mathbf{Z}$ such that $m^v \equiv m \pmod{n}$ and $(n - ml - m^2 l - \cdots - m^u l) \equiv 0 \pmod{n}$. Thus, the sequence reduces to

$$\begin{aligned} &x_0 = a,\ \ x_1 = b,\ \ x_2 = a^m,\ \ x_3 = a^{n-ml}b,\ \ x_4 = a^{m^2},\ \ x_5 = a^{n-ml-m^2 l}b,\\ &x_6 = a^{m^3},\ \ \ldots x_\zeta = b,\ \ x_{\zeta+1} = a^m,\ \ \ldots \end{aligned}$$

which is not simply periodic.

Hence, we can say that the Theorem 1.1 does not hold for 2-step general Fibonacci sequences in dihedral groups. ■

Corollary. *If* $m \equiv d \pmod{n}$ *such that* $d \mid n$*, then 2-step general Fibonacci sequences in* D_n *are not simply periodic.*

Theorem 2.3. *Let* $n \geqslant 3$*. Consider the dihedral group* D_n *with generators* a*,* b*.*

(i) *If* $m \equiv 1 \pmod{n}$ *and* $l = [n, 2]t$*,* $t \in \mathbf{Z}$*, then 2-step general Fibonacci sequences in* D_n *are simply periodic with period 2, where* $[n, 2]$ *denotes the least common multiple of* n *and 2.*

(ii) *If* $m \equiv (n-1) \pmod{n}$ *and* $l \equiv (n-1) \pmod{n}$ *such that* l *is an odd integer, then 2-step general Fibonacci sequences in* D_n *are simply periodic with period 3.*

(iii) *If* $m \equiv 1 \pmod{n}$ *and* $l \equiv 1 \pmod{n}$ *such that* l *is an odd integer, then 2-step general Fibonacci sequences in* D_n *are simply periodic with period 6.*

Proof. (i) Let $m \equiv 1 \pmod{n}$ and $l = [n, 2]t$, $t \in \mathbf{Z}$. Then, there exists an $x \in \mathbf{Z}$ such that $m = nx + 1$. Notice that m is odd and l is even. Thus, the sequence reduces to

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a,\ \ x_3 = b,\ \ \ldots$$

Since the elements succeeding x_2, x_3 depend on a and b for their values, the cycle begins again with the 2nd element. Thus, the 2-step general Fibonacci sequences in D_n are simply periodic with period 2.

(ii) Let $m \equiv (n-1) \pmod{n}$ and $l \equiv (n-1) \pmod{n}$ such that l is an odd integer. Notice that m, l are odd. Thus, the sequence reduces to

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^{n-1}b,\ \ x_3 = ba^{n-1}b = a,\ \ x_4 = a^{n-1}ba^{n-1} = b,\ \ \ldots$$

Since the elements succeeding x_3, x_4 depend on a and b for their values, the cycle begins again with the 3rd element. Hence, the 2-step general Fibonacci sequences in D_n are simply periodic with period 3.

Notice that the conditions $m \equiv (n-1) \pmod{n}$, $l \equiv (n-1) \pmod{n}$ and l – odd, must hold at the same time. Otherwise, the period of the sequence may not be equal to 3. For example, if $m = 13 \equiv 6 \pmod{7}$ and $l = 6 \equiv 6 \pmod{7}$, the period of the 2-step general Fibonacci sequences in D_7 is 4.

(iii) Let $m \equiv 1 \pmod{n}$ and $l \equiv 1 \pmod{n}$ such that l is an odd integer. The sequence reduces to

$$\begin{aligned}&x_0 = a,\ \ x_1 = b,\ \ x_2 = ab,\ \ x_3 = a^{-1} = a^{n-1},\ \ x_4 = ab(a^l)^{-1} = a^2b,\\ &x_5 = (a^m)^{-1}a^2b = ab,\ \ a_6 = a,\ \ x_7 = b,\ \ \ldots\end{aligned}$$

Since the elements succeeding x_6, x_7 depend on a and b for their values, the cycle begins again with the 6th element. Thus, the 2-step general Fibonacci sequences in D_n are simply periodic with period 6. ∎

THEOREM 2.4. *Let n be an odd integer and $n \geqslant 3$. Consider the dihedral group D_n with generators a, b.*

(i) *If $m \equiv 1 \pmod{n}$ and l is even such that $l \neq [n,2]t$, $t \in \mathbf{Z}$, then 2-step general Fibonacci sequences in D_n are simply periodic with period $2n$, where $[n,2]$ denotes the least common multiple of n and 2.*

(ii) *If $m \equiv (n-1) \pmod{n}$ and $l \not\equiv (n-1) \pmod{n}$ such that l is an odd integer, then 2-step general Fibonacci sequences in D_n are simply periodic with period $3n$.*

Proof. (i) Let $m \equiv 1 \pmod{n}$ and l is even such that $l \neq [n,2]t$, $t \in \mathbf{Z}$. Then, the sequence reduces to

$$\begin{aligned}&x_0 = a,\ \ x_1 = b,\ \ x_2 = a,\ \ x_3 = a^{n-l}b,\ \ x_4 = a,\ \ x_5 = a^{n-2l}b,\ \ \ldots,\\ &x_{n-2} = a^{n-(\frac{n-1}{2}-1)l}b,\ \ x_{n-1} = a,\ \ x_n = a^{n-\frac{n-1}{2}l}b,\ \ \ldots, x_{2n-2} = a,\\ &x_{2n-1} = a^{n-(n-1)l}b,\ \ x_{2n} = a,\ \ x_{2n+1} = b,\ \ \ldots\end{aligned}$$

Since the elements succeeding x_{2n}, $x_{2n=1}$ depend on a and b for their values, the cycle begins again with the $2n$-th element. Thus, 2-step general Fibonacci sequences in D_n are simply periodic with period $2n$.

(ii) Let $m \equiv (m-1) \pmod n$ and $l \not\equiv (n-1) \pmod n$ such that l is an odd integer. Then, $l \equiv 0, 1, \ldots (n-2) \pmod n$. So, 2-step general Fibonacci sequences in D_n reduce to

$$x_0 = a,\ \ x_1 = b,\ \ x_2 = a^{n-1}b,\ \ x_3 = a,\ \ x_4 = a^{n-1-l}b,\ \ x_5 = a^{2n-2-l}b,$$
$$x_6 = a,\ \ \ldots,\ \ x_{3n-1} = a^{n-n-(n-1)l}b,\ \ x_{3n} = a,\ \ x_{3n+1} = b,\ \ \ldots$$

Since the elements succeeding x_{3n}, x_{3n+1} depend on a and b for their values, the cycle begins again with the $3n$-th element. 2-step general Fibonacci sequences in D_n are simply periodic with period $3n$. ■

The periods of 2-step general Fibonacci sequences in some D_n are given in tables 1, 2.

Period	D_3	D_4	D_5
2	$m \equiv 1 \pmod 3$, $l = [3,2]t$	$m \equiv 1 \pmod 4$, $l = [4,2]t$	$m \equiv 1 \pmod 5$, $l = [5,2]t$
3	$m \equiv 2 \pmod 3$, $l \equiv 2 \pmod 3$ where m, l are odd	$m \equiv 3 \pmod 4$, $l \equiv 3 \pmod 4$ where m, l are odd	$m \equiv 4 \pmod 5$, $l \equiv 4 \pmod 5$ where m, l are odd
4	$m \equiv 2 \pmod 3$, l is even	1. $m \equiv 3 \pmod 4$, $l = [4,2]t$ 2. m is odd, l is even, but $l \neq [4,2]t$	$m \equiv 4 \pmod 5$, l is even
6	$m \equiv 1 \pmod 3$, $l \neq [3,2]t$	1. m is odd, $l \equiv 1 \pmod 4$ 2. $m \equiv 1 \pmod 4$, $l \equiv 1 \pmod 4$	$m \equiv 1 \pmod 5$, l is odd
8			$m \equiv 2, 3 \pmod 5$, l is even
9	$m \equiv 2 \pmod 3$, $l \not\equiv 2 \pmod 3$, l is odd		
10			$m \equiv 1 \pmod 5$ $l \neq [5,2]t$, l is even
12			$m \equiv 2, 3 \pmod 5$, l is odd
15			$m \equiv 4 \pmod 5$, $l \not\equiv 4 \pmod 5$, l is odd

Table 1.

Period	D_6	D_7	D_8
2	$m \equiv 1 \pmod 6$, $l = [6,2]t$	$m \equiv 3 \pmod 7$, $l = [7,2]t$	$m \equiv 1 \pmod 8$, $l = [8,2]t$
3	$m \equiv 5 \pmod 6$, $l \equiv 5 \pmod 6$ where m, l are odd	$m \equiv 6 \pmod 7$, $l \equiv 6 \pmod 7$ where m, l are odd	$m \equiv 7 \pmod 8$, $l \equiv 7 \pmod 8$ where m, l are odd
4	$m \equiv 5 \pmod 6$, l is even	$m \equiv 6 \pmod 7$, l is even	1. $m \equiv 3, 7 \pmod 8$, $l \equiv 2, 6 \pmod 8$ 2. $m \equiv 1, 3, 5, 7 \pmod 8$ $l \equiv 4 \pmod 8$ 3. $m \equiv 3, 5, 7 \pmod 8$ $l \equiv 0 \pmod 8$
6	$m \equiv 1 \pmod 6$, $l \neq [6,2]t$	1. $m \equiv 1 \pmod 7$, l is odd 2. $m \equiv 2 \pmod 7$, l is even	1. $m \equiv 1, 5 \pmod 8$, $l \equiv 1, 5 \pmod 8$ 2. $m \equiv 1, 3, 5, 7 \pmod 8$ $l \equiv 3 \pmod 8$ 3. $m \equiv 1, 3, 5 \pmod 8$, $l \equiv 7 \pmod 8$
8			$m \equiv 1, 5 \pmod 8$, $l \equiv 2, 6 \pmod 8$
9	$m \equiv 5 \pmod 6$, $l \equiv 1, 3 \pmod 6$	$m \equiv 3, 5 \pmod 7$, l is odd	
10			
12		$m \equiv 3, 5 \pmod 7$, l is even	$m \equiv 3, 5 \pmod 8$ $l \equiv 1, 5 \pmod 8$
14		$m \equiv 1 \pmod 7$ $l \neq [7,2]t$, l is even	
18		$m \equiv 2, 4 \pmod 7$, l is odd	
21		$m \equiv 6 \pmod 7$, $l \not\equiv 6 \pmod 7$, l is odd	

Table 2.

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