

ON STRONGLY PRE-OPEN SETS AND A DECOMPOSITION OF CONTINUITY

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Abstract. In this paper some new concepts of generalized open sets and generalized continuous functions are studied. The notions of strongly preopen sets, S -precontinuous functions, δ -continuous functions are introduced and some of their properties are studied showing their behavior in comparison to other generalized structures already available in literature. The final result in this paper gives one new decomposition of continuity.

One major area of research in general topology during the last few decades that mathematicians have been pursuing is to investigate different types of generalized open sets, generalized continuous functions and study their structural properties. Moreover, these investigations lead to solve the problem of finding the continuity dual of some generalized continuous functions in order to have a decomposition of continuity.

Levine [15] in 1963, started the study of generalized open sets with the introduction of semi-open sets. Then Njastad [17] studied α -open sets; Mashour et al. [16] introduced preopen sets. Bourbaki [3] invented the concept of locally closed sets. Ganster [8], [9] in 1987 studied preopen sets in detail in connection with resolvable and irresolvable spaces which were introduced by Hewitt in 1943 [13]. Andrijević [1] introduced the notion of semi-preopen sets. Tong [18] defined and studied A -sets. Chattopadhyay and Bandyopadhyay [4] introduced the concept of δ -sets in 1991 and later on they studied on δ -sets to some extent in [5], [6]. Recently, Chattopadhyay [7] invented the idea of strongly semi-preopen sets which are found to be equivalent with the concept of b -open sets as defined by Andrijević [2]. This paper introduces the concept of strongly preopen sets (simply, S -preopen sets).

Let (X, τ) be a topological space and $A \subset X$. Let us denote the closure of A and the interior of A by $\text{cl } A$ and $\text{int } A$, respectively. By a space we will mean a topological space.

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DEFINITION 1. In a space (X, τ) , a subset A of X is called:

- (i) α -open if $A \subset \text{int cl int } A$;
- (ii) semi-open if $A \subset \text{cl int } A$;
- (iii) preopen if $A \subset \text{int cl } A$;
- (iv) locally closed if $A = U \cap F$ where U is open and F is closed;
- (v) and A -set if $A = U \cap F$ where U is open and F is regular closed;
- (vi) semi-open if $A \subset \text{cl int cl } A$;
- (vii) δ -set if $\text{int cl } A \subset \text{cl int } A$;
- (viii) strongly semi-preopen if $A \subset \text{int cl } A \cup \text{cl int } A$;
- (ix) S -preopen if A is pre-open and $A = U \cap B$ where U is open and $\text{int } B = \text{int cl int } B$.

Let us denote the collection of all α -open sets, semi-open sets, preopen sets, S -preopen sets, δ -sets in (X, τ) by $\alpha(\tau)$, $SO(\tau)$, $PO(\tau)$, $STPO(\tau)$ and $\delta(\tau)$, respectively. We have $\tau \subset \alpha(\tau) \subset SO(\tau) \subset \delta(\tau)$, also complements of δ -sets are δ -sets [4]. In particular, semi-closed sets and nowhere dense sets are δ -sets.

OBSERVATION 1. It follows from Theorem 2.10 in [1] that for a set B , $\text{int } B = \text{int cl int } B$ iff B is semi-preclosed. Therefore, from the definition of an S -preopen set it follows that an S -preopen set is a preopen set which is additionally the intersection of an open set and a semi-preclosed set.

OBSERVATION 2. Let A be preopen and preclosed. Then $A = \text{int cl } A \cap A$ and thus A is S -preopen. In particular, if A is preopen and codense (i.e., $\text{int } A = \emptyset$) then A is S -preopen.

OBSERVATION 3. Let $A = U \cap B$ where U is open and B is semi-preclosed. Then $\alpha\text{-int } A = \text{int } A$.

Proof. We have $\text{int cl int } A \subset \text{int cl int } B = \text{int } B$, since $A \subset B$ and B is semi-preclosed. Therefore, $\alpha\text{-int } A = A \cap \text{int cl int } A \subset A \cap \text{int } B = \text{int } A$. One always has that $\text{int } A \subset \alpha\text{-int } A$ and so $\alpha\text{-int } A = \text{int } A$.

OBSERVATION 4. Using the result $\text{pint sint } A = \alpha\text{-int } A$ from [1], it follows easily that for a δ -set A we have $\alpha\text{-int } A = \text{pint } A$ (where $\text{pint } A$, resp. $\text{sint } A$ denote the preinterior of A , resp. the semi-interior of A).

THEOREM 1. $\tau \subset STPO(\tau) \subset PO(\tau)$.

The inclusions are not reversible in general as shown by the following examples.

EXAMPLE 1. Let (\mathbf{R}, τ) be the space of real numbers with the usual topology τ . Then the set \mathbf{Q} of all rational numbers is an S -preopen set but not open.

EXAMPLE 2. Let X be a space having a nowhere dense subset N which is not closed. Let $A = X \setminus N$. Then A is α -open and thus preopen. Since A is not open, by Observation 3, A cannot be S -preopen.

THEOREM 2. *Let X be a space and $A \subset X$. The following statements are equivalent:*

- (i) A is an open set;
- (ii) A is an S -preopen set and a δ -set.

Proof. (i) \Rightarrow (ii). By Theorem 1, an open set is strongly preopen. Also, an open set is a δ -set.

(ii) \Rightarrow (i). Since A is preopen and a δ -set, by Observation 4, $\alpha\text{-int } A = \text{pint } A = A$. By Observation 3 we have that $\alpha\text{-int } A = \text{int } A = A$ and so A is open. ■

We now have the following conclusion.

THEOREM 3. *Let (X, τ) be a space and $A \subset X$. The following statements are equivalent:*

- (i) A is an open set;
- (ii) A is α -open and locally closed;
- (iii) A is preopen and locally closed;
- (iv) A is preopen and an A -set;
- (v) A is strongly preopen and a δ -set.

Equivalences (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) are due to Ganster and Reilly [10].

Regarding study of generalized continuous functions, there are different notions of generalized continuity already available in the literature, e.g., almost continuity [14], α -continuity [17], semi-continuity [15], A -continuity [18], pre-continuity [16], LC -continuity [12], β -continuity [12]. We shall introduce two new types of generalized continuous functions in this paper. One is termed as δ -continuous functions and the other as S -precontinuous functions.

DEFINITION 2. Let (X, τ) and (Y, σ) be two topological spaces. A function $f: X \rightarrow Y$ is said to be:

- (i) δ -continuous if $V \in \sigma \implies f^{-1}(V) \in \delta(\tau)$;
- (ii) S -precontinuous if $V \in \sigma \implies f^{-1}(V) \in STPO(\tau)$.

THEOREM 4. $f: X \rightarrow Y$ is δ -continuous iff for each $x \in X$ and for each open set V containing $f(x)$, there exists a semi-open set U containing x such that $U \cap \text{cl } f^{-1}(V) \subset f^{-1}(V)$.

Proof. Let f be δ -continuous. Let $x \in X$ and V be an open set containing $f(x)$. Then $f^{-1}(V) \in \delta(\tau)$. Since complement of a δ -set is a δ -set, $f^{-1}(V)$ can be written as $f^{-1}(V) = F \cap D$ where F is closed and $\text{int } D$ is dense in τ . Thus $f^{-1}(V) = \text{cl } f^{-1}(V) \cap D$, where D is semi-open in τ . Hence the condition is necessary.

Now suppose that the condition holds. Let $V \in \sigma$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists a semi-open set U_x in τ with $x \in U_x$ such that $U_x \cap$

$\text{cl } f^{-1}(V) \subset f^{-1}(V)$. Now varying x over $f^{-1}(V)$ it follows

$$\left(\bigcup_{x \in f^{-1}(V)} U_x \right) \cap \text{cl } f^{-1}(V) = f^{-1}(V).$$

Since $\bigcup_{x \in f^{-1}(V)} U_x$ is semi-open, it follows that $f^{-1}(V)$ is a δ -set in τ . Hence the condition is sufficient. ■

The following example shows that a pre-continuous function may not be δ -continuous. This example also serves construction of an S -precontinuous function which is not continuous.

EXAMPLE 3. Let X be resolvable, i.e., X has a dense subset D such that $X \setminus D$ is also dense. Define $f: X \rightarrow \mathbf{R}$ by $f(x) = \begin{cases} 0, & \text{if } x \in D, \\ 1, & \text{if } x \in X \setminus D. \end{cases}$ Then possible inverse images of open subsets of \mathbf{R} are $\emptyset, D, X \setminus D, X$. Thus f is clearly pre-continuous but fails to be δ -continuous since D is not a δ -set. Also, since D and $X \setminus D$ are both S -preopen sets, f is S -precontinuous. Clearly, f is not continuous.

The following example shows that a δ -continuous function may not be pre-continuous, semi-continuous, LC -continuous, A -continuous.

EXAMPLE 4. Let X be a space having a nowhere dense subset N which is not closed. Define $f: X \rightarrow \mathbf{R}$ by $f(x) = \begin{cases} 0, & \text{if } x \in N, \\ 1, & \text{if } x \in X \setminus N. \end{cases}$ Then possible inverse images of open subsets of \mathbf{R} are $\emptyset, N, X \setminus N, X$. Thus f is clearly δ -continuous. N is neither pre-open nor semi-open so f is neither pre-continuous nor semi-continuous. Since $X \setminus N$ is dense but not open, it cannot be locally closed and so f fails to be LC -continuous. Now since A -continuity implies LC -continuity [10], it follows that f is not A -continuous.

Now we shall have the following theorem which shows a new decomposition for continuous functions.

THEOREM 6. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following statements are equivalent:*

- (i) f is continuous;
- (ii) f is δ -continuous and S -precontinuous.

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (i) follows from the equivalent statements (i) and (v) in Theorem 3. ■

Finally we state

THEOREM 7. *Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then the following statements are equivalent:*

- (i) f is continuous iff f is α -continuous and LC -continuous [10];
- (ii) f is continuous iff f is pre-continuous and LC -continuous [10];
- (iii) f is continuous iff f is pre-continuous and A -continuous [10];
- (iv) f is continuous iff f is δ -continuous and S -continuous.

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