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INEQUALITY OF POINCARÉ-FRIEDRICH'S TYPE ON L^p SPACES

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Abstract. In this paper it is demonstrated that the inequality

$$\left(\int_G |u|^p \, dx\right)^{1/p} \leqslant A_p \left(\int_D |\nabla u|^p \, dx\right)^{1/p}, \quad u|_{\partial D} = 0, 1 \leqslant p \leqslant \infty$$

holds, where $G \subset D \subset \mathbf{R}^2$, D is a convex domain and constant A_p is expressed in terms of areas of G and D.

1. Introduction

It is well known that the inequality

$$\int_D |u|^2 \, dx \leqslant c \int_D |\nabla u|^2 \, dx \qquad \text{(Friedrich's inequality)} \tag{1}$$

holds, where the function u satisfies the following conditions: $u \in C^1(\overline{D})$ and $u|_{\partial D} = 0$ and D is a domain in $\mathbf{R}^n$. The constant c depends only on the domain D.

Inequalities of the form (1) have received considerable attention in the literature, because of their fundamental role in the theory of Partial Differential Equations and various applications. For details we refer to the books by Courant and Hilbert [3], Friedman [5], Ladyzhenskaya and Ural'tseva [6], Mihlin [7].

In this paper we consider the case $n = 2$, i.e., $D \subset \mathbf{R}^2$. In this case, the best possible constant c is $1/\lambda_1(D)$, where $\lambda_1(D)$ is the smallest eigenvalue of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u, \\ u|_{\partial D} &= 0. \end{aligned}$$

In some situations one needs to estimate $\int_G |u|^p \, dx$ in terms of $\int_D |\nabla u|^p \, dx$, where $G \subset D \subset \mathbf{R}^2$ is a simply connected domain.

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It will be demonstrated how constant A_p (mentioned in the Abstract) depends on the areas of G and D.

2. Result

Let G and D be bounded simply connected domains in $\mathbf{R}^2$ with piecewise smooth boundaries, $G \subset D$ and let D be convex.

THEOREM 1. *If $f \in C^1(\overline{D})$ and $f|_{\partial D} = 0$ then*

$$\left(\int_G |f|^p \, dA(z)\right)^{1/p} \leqslant A_p \left(\int_D \left|\frac{\partial f}{\partial \bar{z}}\right|^p dA(z)\right)^{1/p}, \tag{2}$$

where

$$A_p = \begin{cases} \frac{2}{j_0^{2(1-\frac{1}{p})}} \left(\frac{|D|}{\pi}\right)^{\frac{1}{2p}} \left(\frac{|G|}{\pi}\right)^{\frac{1}{2}-\frac{1}{2p}}, & 1 \leqslant p \leqslant 2, \\ \frac{2}{j_0^{2/p}} \left(\frac{|D|}{\pi}\right)^{\frac{1}{2}-\frac{1}{2p}} \left(\frac{|G|}{\pi}\right)^{\frac{1}{2p}}, & 2 \leqslant p \leqslant +\infty. \end{cases}$$

Here, j_0 is the smallest positive zero of Bessel function J_0, $|G|$ and $|D|$ denote the areas of G and D, respectively, $dA(z)$ is Lebesgue measure and $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$.

Proof. If X and Y are normed spaces and S is a bounded operator from X to Y, the norm of S will be denoted by $\|S\colon X \to Y\|$. Consider the operator $T\colon L^2(D) \to L^2(G)$ defined by

$$Tf(z) = -\frac{1}{\pi}\int_D \frac{f(\xi)}{\xi - z} \, dA(\xi).$$

It follows from [4] that

$$\|T\colon L^2(D) \to L^2(G)\| \leqslant \frac{2}{\sqrt[4]{\lambda_1(D)\lambda_1(G)}}, \tag{3}$$

where $\lambda_1(D)$ and $\lambda_1(G)$ are the smallest eigenvalues of the boundary value problems

$$\begin{aligned} -\Delta u &= \lambda u, \\ u|_{\partial D} &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} -\Delta v &= \lambda v, \\ v|_{\partial G} &= 0, \end{aligned}$$

respectively.

From (3), using Faber-Krahn inequality [1]

$$\lambda_1(G) \geqslant \frac{\pi j_0^2}{|G|}, \qquad \lambda_1(D) \geqslant \frac{\pi j_0^2}{|D|},$$

we obtain

$$\|T\colon L^2(D) \to L^2(G)\| \leqslant \frac{2}{j_0}\sqrt[4]{\frac{|G|\cdot|D|}{\pi^2}}. \tag{4}$$

Let us now estimate $\|T\colon L^1(D) \to L^1(G)\|$ and $\|T\colon L^\infty(D) \to L^\infty(G)\|$.

It is easy to see that

$$\|T\colon L^1(D)\to L^1(G)\| \leqslant \sup_{z\in D}\frac{1}{\pi}\int_G \frac{dA(\xi)}{|\xi-z|} \leqslant \sup_{z\in D}\frac{1}{\pi}\int_D \frac{dA(\xi)}{|\xi-z|}.$$

Let $z\in D$. Since D is a convex domain, parametrization of the boundary ∂D can be done in the following way

$$\xi = z+\rho(\theta)e^{i\theta},\qquad 0\leqslant\theta\leqslant 2\pi.$$

and so

$$\frac{1}{\pi}\int_D \frac{dA(\xi)}{|\xi-z|} = \frac{1}{\pi}\int_0^{2\pi} d\theta\int_0^{\rho(\theta)}\frac{r\,dr}{r} = \frac{1}{\pi}\int_0^{2\pi}\rho(\theta)\,d\theta \leqslant$$
$$\leqslant \frac{1}{\pi}\sqrt{\int_0^{2\pi} d\theta}\sqrt{\int_0^{2\pi}\rho^2(\theta)\,d\theta} = \frac{\sqrt{2\pi}}{\pi}\left(2\cdot\frac{1}{2}\int_0^{2\pi}\rho^2(\theta)\,d\theta\right)^{1/2} = 2\sqrt{\frac{|D|}{\pi}}.$$

Therefore $\|T\colon L^1(D)\to L^1(G)\|\leqslant 2\sqrt{\frac{|D|}{\pi}}$ and similarly $\|T\colon L^\infty(D)\to L^\infty(G)\|\leqslant 2\sqrt{\frac{|D|}{\pi}}$. Then from (4), applying Riesz-Torin theorem [2], we get

$$\|T\colon L^p(D)\to L^p(G)\| \leqslant \frac{2}{j_0^{2(1-\frac{1}{p})}}\left(\frac{|D|}{|pi}\right)^{\frac{1}{2p}}\left(\frac{|G|}{\pi}\right)^{\frac{1}{2}-\frac{1}{2p}},\quad 1\leqslant p\leqslant 2$$

and

$$\|T\colon L^p(D)\to L^p(G)\| \leqslant \frac{2}{j_0^{2/p}}\left(\frac{|D|}{|pi}\right)^{\frac{1}{2}-\frac{1}{2p}}\left(\frac{|G|}{\pi}\right)^{\frac{1}{2p}},\quad 2\leqslant p\leqslant +\infty.$$

Putting

$$A_p = \begin{cases} \frac{2}{j_0^{2(1-\frac{1}{p})}}\left(\frac{|D|}{\pi}\right)^{\frac{1}{2p}}\left(\frac{|G|}{\pi}\right)^{\frac{1}{2}-\frac{1}{2p}}, & 1\leqslant p\leqslant 2,\\ \frac{2}{j_0^{2/p}}\left(\frac{|D|}{\pi}\right)^{\frac{1}{2}-\frac{1}{2p}}\left(\frac{|G|}{\pi}\right)^{\frac{1}{2p}}, & 2\leqslant p\leqslant +\infty. \end{cases}$$

we have

$$\|T\colon L^p(D)\to L^p(G)\|\leqslant A_p,\qquad 1\leqslant p\leqslant +\infty. \tag{5}$$

Let $f\in C^1(\overline{D})$ and $f|\partial D=0$. According to Cauchy-Green formula [8] we get (for $z\in G$) $f(z)=T(\frac{\partial f}{\partial\bar z})$ and from (5) we obtain

$$\left(\int_G |f|^p\,dA\right)^{1/p} \leqslant A_p\left(\int_D \left|\frac{\partial f}{\partial\bar z}\right|^p\,dA\right)^{1/p},$$

i.e.

$$\left(\int_G |f|^p\,dx\,dy\right)^{1/p} \leqslant \frac{A_p}{2}\left(\int_D |\nabla f|^p\,dx\,dy\right)^{1/p}. \quad \blacksquare$$

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