M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 10, 2004

Paper 8
Received: Dec 23, 2003, Revised: Aug 2, 2004, Accepted: Nov 9, 2004
Editor: J. Bellissard

Quantum Dynamical Systems with Quasi–Discrete Spectrum

Sławomir Klimek

Department of Mathematics
IUPUI
Indianapolis, IN 46205, USA

Abstract. We study totally ergodic quantum dynamical systems with quasi–discrete spectrum. We investigate the classification problem for such systems in terms of algebraic invariants. The results are noncommutative analogs of (a part of) the theory of Abramov.

Supported in part by the National Science Foundation under grant DMS–9801612

I. Introduction

Let (X, μ), $\mu(X) = 1$, be a standard Lebesgue space and let $\alpha : X \to X$ be an automorphism of (X, μ). Then α defines an unitary operator, called the Koopman operator [K], in $L^2(X, d\mu)$ and denoted by the same letter.

In the important papers [VN] and [HvN], von Neumann and Halmos classified all classical ergodic systems for which the Koopman operator has purely discrete spectrum. The main result of their analysis is that such systems are classified by the spectrum, which forms a discrete subgroup of $U(1)$, and each such a system is conjugate to a shift on a compact abelian group, the Pontriagin dual of the spectrum. Here, and throughout the paper, $U(1)$ is the group of complex numbers with absolute value 1 and discrete topology. For a clear account of that result, see e.g. [CFS], [W], or [Si].

This theory was extended to noncommutative setting by Olsen, Pedersen and Takesaki [OPT]. It turns out that noncommutative ergodic systems with discrete spectrum are classified by the spectrum of the automorphism, which as above is a discrete subgroup H of $U(1)$ and a second cohomology class of H. This theorem is stated more carefully in Section II.

The notions of quasi-eigenvalue and quasi-eigenfunction were introduced by von Neumann and Halmos [H]. They proved, using those concepts, that there exist spectrally equivalent but not conjugate automorphisms with mixed spectrum. Later Abramov [Ab] gave a complete classification of totally ergodic systems with quasi-discrete spectrum. A topological version of Abramov's theory for minimal systems was discussed in [HaP], [HoP].

Let us shortly describe what quasi-eigenvalues and quasi-eigenfunctions are and state the Abramov's theorem. With the above notation α is called totally ergodic if α^n is ergodic for every $n = 1, 2, \ldots$. Ordinary eigenvectors and eigenvalues of α are called, correspondingly, quasi-eigenvectors and quasi-eigenvalues of the first order. A function $f \in L^2(X, d\mu)$ is called a quasi-eigenvector of the second order if

$$\alpha(f) = \phi f,$$

where ϕ is a quasi-eigenvectors of the first order (i.e. an eigenvector) of α. In such a case ϕ is called a quasi-eigenvalue of the second order. Continuing this process one obtains quasi-eigenvectors and quasi-eigenvalues of arbitrary order - see Section II for a more precise definition. The crucial observation is that, if α is totally ergodic, quasi-eigenvectors corresponding to different quasi-eigenvalues are orthogonal. One considers then the situation when $L^2(X, d\mu)$ has a basis consisting of quasi-eigenvectors of α possibly of arbitrary order. If this is the case, then we say that α has purely quasi-discrete spectrum. The Abramov's theorem can be formulated as follows.

Theorem I.1. [Ab] *There is a one-to-one correspondence between the conjugacy classes of totally ergodic dynamical systems with purely quasi-discrete spectrum and the equivalence classes of pairs* (H, R) *where* H *is a discrete abelian group of the form* $H = \bigcup_{n=1}^{\infty} H_n$ *where* $H_1 \subset H_2 \subset \ldots$ *is an increasing sequence of discrete abelian groups,* $H_1 \subset U(1)$ *and* H_1 *has no non-trivial elements of finite order, and* R *is a homomorphism of* H *such that for every* $n = 1, 2, \ldots$ *the kernel of* R^n *is the group* H_n*.*

This paper contains an attempt to extend the Abramov theorem to the quantum mechanical context i.e. when the space X is replaced by a noncommutative von Neumann algebra. One case of this program that we were able to understand fairly completely is when the second order quasi-eigenvectors form a basis in the corresponding L^2-space. This assumption is satisfied in the original example that has motivated our work on the subject. The main results of the paper, Equivalence Theorem and Representation Theorem, show that such systems are classified by quadruples $(H_1, H_2, [r], k)$, called quantum quasi-spectra, where H_1 and H_2 are groups, $k : H_2 \mapsto H_2$ is an isomorphism and $[r]$ is (essentially) a k invariant second cohomology class of H_2. Such quadruples are also required to satisfy a number of conditions described in Section IV.

It seems that the classification problem in full generality leads to an excessively complicated system of algebraic invariants and is left for future investigation. In what follows we present a detailed account of the classification theory under the above mentioned additional assumption.

Our proofs and organization of the material follow closely that of Abramov's with several important differences. Among them are:

• The set of quasi-eigenvalues forms a group but not with respect to operator multiplication but rather a twisted version of it denoted by $*$ in this paper.

• We introduce a natural concept of a normalized basis of quasi-eigenvectors which simplifies proofs of the Equivalence Theorem and the Representation Theorem.

The paper is organized as follows. In Section II we introduce a fairly general setup and precisely formulate the problem. In Section III we show how to construct group-theoretic invariants for totally ergodic quantum dynamical systems with purely quasi-discrete spectrum (of the second order). We prove the equivalence theorem in Section IV, and the representation theorem in Section V. Finally, Section VI contains a simple example of such a quantum dynamical system.

II. Quantum Ergodic Systems

We begin by reviewing the basic concepts which are used throughout the paper. We will work within the von Neumann algebra framework, see e.g. [BR], as this is the natural setup for noncommutative (quantum) ergodic theory. We will adopt the following definition of a quantum dynamical system.

Definition II.1. A quantum dynamical system is a quadruple $(\mathfrak{A}, G, \alpha, \tau)$ with the following properties:

(i) $\mathfrak{A}$ is a von Neumann algebra with a separable predual.
(ii) G is a locally compact abelian group.
(iii) $\alpha : G \to \mathrm{Aut}\,(\mathfrak{A})$ is an action of G on $\mathfrak{A}$ by von Neumann algebra automorphisms.
(iv) τ is a G-invariant, normal, faithful state on $\mathfrak{A}$.

Since locally compact abelian groups are amenable, it allows one to define the time average of an observable and prove ergodic theorems, see e.g. [L], [J], and references therein. The most relevant are the groups $G = \mathbb{Z}$ (in which case the system is called a quantum map) and $G = \mathbb{R}$ (in which case the system is called a quantum flow).

We will denote by $\mathcal{K} = L^2\,(\mathfrak{A}, \tau)$ the GNS representation space of $\mathfrak{A}$ associated with the state τ. Since $\mathfrak{A}$ has a separable predual, $\mathcal{K}$ is a separable Hilbert space. It is natural to think of $\mathcal{K}$ as a quantum version of the classical Koopman space. The automorphisms α_g extend to unitary operators of the $\mathcal{K}$-spaces. By a slight abuse of notation, we continue to denote them by α_g.

Definition II.2. Two quantum dynamical systems $(\mathfrak{A}, G, \alpha, \tau)$ and $(\mathfrak{B}, G, \beta, \omega)$ are conjugate if there exists an isomorphism of von Neumann algebras $\Phi : \mathfrak{A} \to \mathfrak{B}$ such that

(i) $\Phi \circ \alpha = \beta \circ \Phi$;
(ii) $\omega \circ \Phi = \tau$.

A non-zero element $U \in \mathcal{K}$ is an eigenvector of α if for every $g \in G$ we have $\alpha_g(U) = \lambda(g)U$, where $\lambda(g) \in U(1)$. Clearly, each $g \to \lambda(g)$ is a character of the group G. The set $\mathrm{Spec}_p\,(\alpha)$ of all such characters is called the point spectrum of α.

Definition II.3. A quantum dynamical system $(\mathfrak{A}, G, \alpha, \tau)$ is called a system with purely discrete spectrum if $\mathcal{K}$ has an orthonormal basis consisting of eigenvectors of α.

As a consequence of the separability assumption, $\mathrm{Spec}_p\,(\alpha)$ is a countable subset of the dual group $\widehat{G}$.

Ergodic theory of von Neumann algebras has been studied by many authors. For references and a variety of results, see e.g. [C], [KL1,2], [KLMR], [L] and [J]. For our purposes, the following definition of quantum ergodicity will be sufficient.

Definition II.4. A quantum dynamical system $(\mathfrak{A}, G, \alpha, \tau)$ is called ergodic if the only G-invariant elements of $\mathcal{K}$ are scalar multiples of I.

Equivalently, the joint eigenspace of α_g's corresponding to the eigenvalue 1 is one dimensional and consists of the scalar multiples of the identity operator. For quantum ergodic systems, the time and ensemble averages of an observable are equal. Also one has the following classification theorem due to Olsen, Pedersen and Takesaki [OPT].

Theorem II.5. [OPT] *There is a one-to-one correspondence between the conjugacy classes of ergodic quantum dynamical systems with purely discrete spectrum and the family of pairs (H, σ) where $H \subset \widehat{G}$ is a discrete group and σ is a second cohomology class of H.*

In fact, in analogy with the commutative theory, every quantum dynamical system is conjugate to a shift on the noncommutative deformation of $\widehat{H}$ determined by σ - see **[OPT]**.

Definition II.6. A quantum dynamical system $(\mathfrak{A}, G, \alpha, \tau)$ is called totally ergodic if for every $g \in G$ individually, the only elements of $\mathcal{K}$ invariant under α_g, are scalar multiples of I.

For an example of ergodic but not totally ergodic quantum dynamical system see Section VI.

We shall call the eigenvectors of α quasi-eigenvectors of the first order. Similarly, eigenvalues of α are called quasi-eigenvalues of the first order. The set of normalized quasi-eigenvectors of the first order is denoted by G_1 while the set of all quasi-eigenvalues of the first order is denoted by H_1. We define the set G_n of normalized quasi-eigenvectors of n-th order and the set H_n of quasi-eigenvalues of n-th order inductively. Suppose that G_n and H_n are defined.

Definition II.7. With the above notation, a non-zero element $U \in \mathcal{K}$ is called a quasi-eigenvector of order $n+1$ of α if $\alpha_g(U) = \lambda(g)U$, where $\lambda(g) \in \mathfrak{A} \cap G_n$. Then λ is called a quasi-eigenvalue of order $n+1$.

Definition II.8. A quantum dynamical system $(\mathfrak{A}, G, \alpha, \tau)$ is called a system with purely quasi-discrete spectrum if $\mathcal{K}$ has an orthonormal basis consisting of quasi-eigenvectors of α of possibly arbitrary orders.

The subject of this paper is the classification problem for (noncommutative) totally ergodic systems with quasi-discrete spectrum. This is to be solved by constructing a complete set of algebraic invariants of such systems.

III. Classification of Quasi-Discrete Systems

In this paper we tackle the program described in the previous section under the following additional assumptions:

1. We consider only $G = \mathbb{Z}$, i.e. quantum maps. The automorphism α_1 corresponding to the generator 1 of $\mathbb{Z}$ will simply be denoted by α.
2. We asume that $\mathcal{K}$ has an orthonormal basis consisting of the second order quasi-eigenvectors of α.
3. We require that τ is a normalized trace.

Additionally, throughout the rest of the paper we assume that the system $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ is ergodic. We do explicitly mention when total ergodicity is used.

With extra effort the classification program can be presumably carried out for arbitrary abelian locally compact groups and, what is most challenging, arbitrary quasi-discrete spectrum. The trace assumption is used in the proof of unitarity in the following proposition and possibly is not really needed. In any case it seems likely that ergodicity and discreteness of the quasi-spectrum will force any invariant state to be a trace.

Every constant is an eigenvector belonging to the eigenvalue $\lambda = 1$, and therefore $H_1 \subset G_1$. Moreover, obviously:

$$H_1 \subset H_2 \subset G_1 \subset G_2. \tag{III.1}$$

Proposition III.1. *Let λ be an eigenvalue of α. If $U_\lambda \in \mathcal{K}$ is a normalized second order quasi-eigenvector of α:*

$$\alpha\left(U_\lambda\right) = \lambda U_\lambda\ , \tag{III.2}$$

then $U_\lambda \in \mathfrak{A}$ and U_λ is unitary.

Proof. This needs a little von Neumann algebras theory from [Ar]. Let $P^\natural \subset L^2(\mathfrak{A}, \tau)$ be the closure of $\mathfrak{A}_+ 1$, where $\mathfrak{A}_+$ is the positive part of $\mathfrak{A}$ and where $1 \in \mathfrak{A} \subset L^2(\mathfrak{A}, \tau)$ is the unit in $\mathfrak{A}$. It follows from this definition that $P^\natural$ is invariant under α. It is known that every $x \in L^2(\mathfrak{A}, \tau)$ has a unique decomposition:

$$x = u\,|x|,$$

where $u \in \mathfrak{A}$ is a partial isometry and $|x| \in P^\natural$. Write $U_\lambda = u\,|U_\lambda|$ in (III.2). Then:

$$\alpha(u)\alpha(|U_\lambda|) = (\lambda u)\,|U_\lambda|$$

It follows that $|U_\lambda|$ is an invariant vector for α and so, by ergodicity, it is equal to 1. But that means that $U_\lambda \in \mathfrak{A}$. Applying the ergodicity assumption to $U_\lambda^* U_\lambda$ we see that $U_\lambda^* U_\lambda = 1$.

Since $1 - U_\lambda U_\lambda^*$ is positive and $\tau\left(1 - U_\lambda U_\lambda^*\right) = \tau\left(1 - U_\lambda^* U_\lambda\right) = 0$ we see that U_λ is unitary. □

Proposition III.2. *If $U, V \in G_2$ belong to the same quasi-eigenvalue λ then there is a constant C, $|C| = 1$, such that $U = CV$.*

Proof. Applying α to $U^{-1}V$ yields:

$$\alpha(U^{-1}V) = U^{-1}\lambda^{-1}\lambda V = U^{-1}V.$$

It follows from ergodicity of α that $U^{-1}V$ is a constant. □

Let us recall from [OPT] the following structural result about G_1.

Proposition III.3. *For each pair $\lambda, \mu \in H_1$, we have*

$$U_\lambda U_\mu = \sigma(\lambda, \mu) U_\mu U_\lambda, \tag{III.3}$$

where $U_\lambda, U_\mu \in G_1$ are the corresponding eigenvectors and $\sigma : H_1 \times H_1 \to U(1)$. Furthermore, σ has the following properties:

$$\sigma(\lambda, \lambda) = 1, \tag{III.4}$$

$$\sigma(\lambda, \mu\nu) = \sigma(\lambda, \mu) \sigma(\lambda, \nu), \tag{III.5}$$

and

$$\sigma(\mu, \lambda) = \sigma(\lambda, \mu)^{-1}. \tag{III.6}$$

A map $\sigma : H_1 \times H_1 \to U(1)$ satisfying (III.4), (III.5), (III.6) is called a symplectic bicharacter.

The following lemma deals with effects of noncommutativity of $\mathfrak{A}$ on the classification problem.

Lemma III.4.

(i) If $U_\lambda \in G_2$ belongs to quasi-eigenvalue $\lambda \in H_2$ then there exist a number $\phi(\lambda) \in U(1)$ such that

$$U_\lambda^{-1} \lambda U_\lambda = \phi(\lambda)\lambda.$$

(ii) If $U \in G_2$ and $V \in G_1$ then $UVU^{-1} \in G_1$.

Proof. We verify by direct calculation that $U_\lambda^{-1}\lambda U_\lambda$ and λ belong to the same eigenvalue of α. Consequently, Proposition III.2 implies item (i).

If $U \in G_2$ belongs to $\lambda \in H_2$, $\lambda \in H_2 \subset G_1$ belongs to $R(\lambda) \in H_1$, and $V \in G_1$ belongs to $\mu \in H_1$, then we compute:

$$\begin{aligned} \alpha(UVU^{-1}) = \lambda U \mu V U^{-1} \lambda^{-1} &= \frac{\mu}{\phi(\lambda)} \lambda U V \lambda^{-1} U^{-1} \\ &= \mu U \lambda V \lambda^{-1} U^{-1} = \mu \sigma(R(\lambda), \mu) U V U^{-1} \end{aligned} \tag{III.7}$$

which proves (ii). In the above calculation we used (i) twice as well as Proposition III.3. □

If $\lambda, \mu \in H_2$ and $U_\lambda \in G_2$ is a quasi-eigenvector belonging to λ we define the following product on H_2:

$$\lambda * \mu := \lambda U_\lambda \mu U_\lambda^{-1}. \tag{III.8}$$

Proposition III.5. *Each of the sets H_1, G_1, G_2 is a group under operator multiplication while H_2 is a group under $*$ multiplication. Moreover $H_1 \subset H_2$ is a subgroup.*

Proof. The fact that H_1 and G_1 are groups follows from [OPT] so we need to concentrate on H_2 and G_2. We first verify that the right hand side of (III.8) is in G_1:

$$\alpha(\lambda U_\lambda \mu U_\lambda^{-1}) = R(\lambda)R(\mu)\sigma(R(\lambda), R(\mu)) \cdot \lambda U_\lambda \mu U_\lambda^{-1} \tag{III.9}$$

by (III.7). Here $R(\lambda)$ and $R(\mu)$ are eigenvalues corresponding to eigenvectors λ and μ. Additionally:

$$\alpha(U_\lambda U_\mu) = \lambda U_\lambda \mu U_\mu = \lambda U_\lambda \mu U_\lambda^{-1} \cdot U_\lambda U_\mu = \lambda * \mu \cdot U_\lambda U_\mu \tag{III.10}$$

so that $\lambda * \mu \in H_2$. Consequently the $*$- product is well defined. The identity operator $1 \in \mathfrak{A}$ is the unit for this multiplication. Since

$$\alpha(U_\lambda^{-1}) = \frac{\lambda^{-1}}{\phi(\lambda)} \cdot U_\lambda^{-1}$$

the $*$ inverse of λ is

$$I(\lambda) := \frac{\lambda^{-1}}{\phi(\lambda)}$$

with λ^{-1} the operator multiplication inverse. Associativity of the $*$ multiplication follows from (III.9) which also shows that G_2 is a group under operator multiplication. Finally if $\lambda, \mu \in H_1$ then $\lambda * \mu = \lambda\mu$. $\square$

We define a map $R : G_2 \to H_2$ by $R(U) := \lambda$ if $\alpha(U) = \lambda U$. In other words, R assigns to a quasi-eigenvector the corresponding quasi-eigenvalue. Clearly R maps $G_1 \subset G_2$ into $H_1 \subset H_2$. Also R maps $H_2 \subset G_1$ into H_1.

Proposition III.6. *The mapping $R : H_2 \to H_1$ has the following properties:*

(i) For every $\lambda \in H_2$ and $\mu \in H_1$ we have $\mu\,\sigma\,(\mu, R(\lambda)) \in H_1$ and

$$\lambda * \mu * I(\lambda) = \mu\,\sigma\,(R(\lambda), \mu)\,. \tag{III.11}$$

In particular, H_1 is a normal subgroup of H_2.

(ii) R is a "twisted" homomorphism:

$$R(\lambda * \mu) = R(\lambda) * \lambda * R(\mu) * I(\lambda) = R(\lambda)R(\mu)\sigma(R(\lambda), R(\mu)). \tag{III.12}$$

(iii) The kernel of R is the group H_1.

Proof. Item (i) is just a rephrasing of (III.7) and item (ii) follows directly from (III.9). Item (iii) is a consequence of ergodicity of α, as eigenvectors corresponding to eigenvalue $\lambda = 1$ are proportional to the identity. $\square$

Let N :=Image of $R \subset H_1$. Equip N with the following product:

$$n_1 * n_2 := n_1 n_2 \sigma(n_1, n_2) \in N,$$

where the last inclusion follows from Proposition III.6, item (i). It is easy to see that N is a group with respect to this product and $R : H_2 \mapsto R$ is a homomorphism. Consequently, we have the following short exact sequence of groups:

$$1 \longrightarrow H_1 \longrightarrow H_2 \xrightarrow{R} N \longrightarrow 1\,. \tag{III.13}$$

This sequence is an extension with abelian kernel, and the N-module structure on H_1 is given by (III.11), see [B].

Proposition III.7. *The group H_2 is at most countable, and, assuming that α is totally ergodic, H_1 has no nontrivial elements of finite order.*

Proof. Since α is assumed to be totally ergodic no nontrivial elements of finite order in H_1 can exist. Also H_1 is at most countable as a consequence of separability of $\mathcal{K}$. Since R defines a one-to-one map $H_2/H_1 \mapsto H_1$, the group H_2 is at most countable. $\square$

If U belongs to $\lambda \in H_2$ then $\alpha(U)$ belongs to $R(\lambda) * \lambda$. Thus it makes sense to study the properties of the map:

$$k(\lambda) := R(\lambda) * \lambda. \tag{III.14}$$

Proposition III.8. *The map k defined by (III.14) is an isomorphism of H_2. Moreover $k(\lambda) * I(\lambda) \in H_1$ and $k(\lambda) = \lambda$ iff $\lambda \in H_1$.*

Proof. k is a homomorphism since

$$\begin{aligned} k(\lambda * \mu) &= R(\lambda * \mu) * \lambda * \mu = R(\lambda) * \lambda * R(\mu) * I(\lambda) * \lambda * \mu \\ &= R(\lambda) * \lambda * R(\mu) * \mu = k(\lambda) * k(\mu) \end{aligned}$$

by Proposition III.6. The inverse of k is $k^{-1}(\lambda) = R(\lambda)^{-1} * \lambda$. Next $k(\lambda) * I(\lambda) = R(\lambda)$ so it is in H_1. Finally $k(\lambda) = \lambda$ iff $R(\lambda) = 1$ so $\lambda \in H_1$. $\square$

Proposition III.9. *If the automorphism α is totally ergodic, then quasi-eigenvectors belonging to different quasi-eigenvalues are orthogonal in $\mathcal{K}$.*

Proof. The statement is true for ordinary eigenvectors. Let $\mathcal{K}_1$ be the closed subspace of $\mathcal{K}$ spanned by G_1, and let $\mathcal{K}_2$ be its orthogonal complement. The assumption of total ergodicity of α is used in the following lemma which says that quasi-eigenvector which is not an eigenvector can not be a linear combination of eigenvectors.

Lemma III.10. *Suppose $U \in G_2$ is not in G_1 and belongs to $\lambda \in H_2$. Then $U \notin \mathcal{K}_1$.*

Proof. Assume that

$$U = \sum_{\mu \in H_1} a_\mu U_\mu. \tag{III.15}$$

We can compute $\alpha^n(U)$ in two different ways. First use (III.15) and apply α^n to each U_μ. This yields:

$$\alpha^n(U) = \sum_{\mu \in H_1} a'_\mu U_\mu,$$

where a'_μ differs from a_μ by a phase. Secondly, use $\alpha(U) = \lambda U$ n-times and then expand:

$$\alpha^n(U) = \sum_{\mu \in H_1} a''_\mu U_{R(\lambda)^n \mu},$$

where, as before, a''_μ differs from a_μ by a phase. By Proposition III.7 $R(\lambda)^n\mu$ are all different. Consequently, for any μ there is an infinite number of coefficients in (III.15) equal, up to a phase, to a_μ, and so they must be zero. □

Returning to the proof of Proposition III.9, if $U \in G_2$ and not in G_1, then we claim that U is in $\mathcal{K}_2$. In fact, let $U = U_1 + U_2$ be the orthogonal decomposition of U with respect to $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. It follows from Lemma III.10 that $U_2 \neq 0$. Since α is unitary, $\alpha(U_1) \in \mathcal{K}_1$ and $\alpha(U_2) \in \mathcal{K}_2$. Moreover $\lambda U_1 \in \mathcal{K}_1$ because G_1 forms a group. For the same reason $\lambda U_2 \in \mathcal{K}_2$ as:

$$(\mu, \lambda U_2) = (\lambda^{-1}\mu, U_2) = 0,$$

for $\mu \in G_1$. Consequently we have $\alpha(U_1) = \lambda U_1$ and $\alpha(U_2) = \lambda U_2$ which implies, in view of Proposition III.2, that $U_1 = CU_2$. This can happen only if $C = 0$ as U_1 and U_2 belong to perpendicular subspaces of $\mathcal{K}$.

It remains to prove that if $U, V \in G_2$ are not in G_1 and belong to different quasi-eigenvalues $\lambda, \mu \in H_2$ then U, V are orthogonal. But this is the same as proving that $U^{-1}V$ is orthogonal to $1 \in \mathcal{K}_1$. Since G_2 is a group with respect to operator multiplication, $U^{-1}V \in G_2$ and belongs to quasi-eigenvalue $I(\lambda) * \mu$. If $U^{-1}V$ is not in G_1 then the orthogonality follows from the previous argument. It remains to consider the case when $U^{-1}V \in G_1$. But two elements of G_1 are orthogonal unless they belong to the same eigenvalue, and, since $\lambda \neq \mu$, $I(\lambda) * \mu \neq 1$. □

Corollary III.11. *For every $\lambda \in H_2$ we have:*

$$\tau(U_\lambda) = \begin{cases} 1 & \textit{if } \lambda = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is a direct consequence of Proposition III.9 and $\tau(U_\lambda) = (1, U_\lambda)$. □

IV. Equivalence Theorem

In this section we spell out the complete set of group theoretic invariants for totally ergodic quantum dynamical systems with quasi-discrete spectrum of the second order. The equivalence theorem proved here says that if two such systems have the same set of invariants then they are conjugate.

If H is a group, then a function $r : H \times H \to U(1)$ is called a 2-cocycle if

$$r(\lambda, \mu)\, r(\lambda\mu, \nu) = r(\lambda, \mu\nu)\, r(\mu, \nu), \tag{IV.1}$$

for all $\lambda, \mu, \nu \in H$. A 2-cocycle r is called *trivial* if there is a function $d : H \to U(1)$, such that $r(\lambda, \mu) = d(\lambda\mu)/d(\lambda)\, d(\mu)$. The set of equivalence classes of 2-cocycles mod trivial 2-cocycles is the second cohomology group $H^2(H)$ of group H (with values in $U(1)$).

Lemma IV.1. *Let $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ be a totally ergodic quantum dynamical system with purely quasi-discrete spectrum of the second order. Choose an orthonormal basis $\{U_\lambda\}$, $\lambda \in H_2$, in $\mathcal{K}$, consisting of quasi-eigenvalues of α and such that $U_1 = 1$. Then for each pair $\lambda, \mu \in H_2$,*

$$U_\lambda U_\mu = r(\lambda, \mu)\, U_{\lambda * \mu}, \tag{IV.2}$$

where $r(\lambda, \mu)$ is a 2-cocycle on H_2. Moreover, any other orthonormal basis of $\mathcal{K}$ consisting of quasi-eigenvectors of α leads to a cohomologous r and $\mathfrak{A}$ is linearly spanned by $\{U_\lambda\}$.

Proof. (IV.2) is a consequence of Proposition III.2, (III.10). The associativity of the operator multiplication implies that r is a cocycle. If $\{V_\lambda\}$ is any other orthonormal basis of $\mathcal{K}$ consisting of quasi-eigenvectors of α then $V_\lambda = d(\lambda)U_\lambda$, $d(\lambda) \in U(1)$, and $d(\lambda)$ gives the equivalence of the corresponding cocycles. Finally, since U_λ is a basis in $\mathcal{K}$ it follows that $\mathcal{A}$ is a σ-weakly closure of the linear span of $\{U_\lambda\}$. □

Since $H_2 \subset G_1$, given a choice of a basis in $\mathcal{K}$ we can write for any $\lambda \in H_2$:

$$\lambda = C(\lambda) U_{R(\lambda)}, \tag{IV.3}$$

where $C(\lambda) \in U(1)$. The main properties of the coefficients $C(\lambda)$ are summarized in the following lemma.

Lemma IV.2. *With the above notation we have:*

$$C(\lambda * \mu) = C(\lambda)C(\mu)\frac{r(\lambda, R(\mu))\, r(R(\lambda), \lambda * R(\mu) * I(\lambda))}{r(\lambda * R(\mu) * I(\lambda), \lambda)}. \tag{IV.4}$$

Additionally, if $\lambda \in H_1$ then $C(\lambda) = \lambda$.

Proof. Proof is a straightforward calculation using (IV.2), (IV.3), and Proposition III.6 which we omit. □

Let $D(\lambda)$ be the following $U(1)$-valued function on H_2:

$$D(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in H_1 \\ 1 & \text{otherwise.} \end{cases} \tag{IV.5}$$

We shall show below that one can choose a basis $\{U_\lambda\}$, $\lambda \in H_2$, in $\mathcal{K}$, consisting of quasi-eigenvalues of α, such that the matrix elements of α are particularly simple.

Proposition IV.3. *There is a basis $\{U_\lambda\}$, $\lambda \in H_2$, in $\mathcal{K}$, consisting of quasi-eigenvalues of α, such that*

$$\alpha(U_\lambda) = D(\lambda) U_{k(\lambda)}. \tag{IV.6}$$

Such a basis will be called a normalized *basis.*

Proof. Notice that (IV.6) says that $\alpha(U_\lambda) = \lambda U_\lambda$ is $\lambda \in H_1$, which is always true, and $\alpha(U_\lambda) = U_{k(\lambda)}$ if $\lambda \notin H_1$. Consider the orbits of k. If $\lambda \in H_1$ then $k(\lambda) = \lambda$ and H_1 is the set of fixed points for k. If $\lambda \notin H_1$ then $k^n(\lambda) = R(\lambda)^n * \lambda$ and, as H_1 has no elements of finite order, all $k^n(\lambda)$ are different for different $n \in \mathbb{Z}$. Choose one element $s(\lambda)$ from each orbit $k^n(\lambda)$, so that each λ can be uniquely written as $\lambda = k^n(s(\lambda))$. Choose $U_{s(\lambda)}$ arbitrarily and set

$$U_\lambda := \alpha^n\left(U_{s(\lambda)}\right).$$

Since $U_{k(\lambda)} = \alpha^{n+1}\left(U_{s(\lambda)}\right)$, (IV.6) is clearly satisfied. □

Let $\{U_\lambda\}$ be a normalized basis and let $r(\lambda, \mu)$ be the corresponding 2-cocycle on H_2. Applying α to (IV.2) we infer that

$$\frac{r(k(\lambda), k(\mu))}{r(\lambda, \mu)} = \frac{D(\lambda * \mu)}{D(\lambda)D(\mu)}. \tag{IV.7}$$

Such a cocycle will be called a *normalized* cocycle. If $V_\lambda = d(\lambda)U_\lambda$, $d(\lambda) \in U(1)$ is another normalized basis then

$$d(k(\lambda)) = d(\lambda). \tag{IV.8}$$

By $H^2_k(H_2)$ we denote the set of equivalence classes of normalized 2-cocycles on H_2 modulo k-invariant coboundaries (IV.8).

Remark. If H_2 is abelian the set $H_k^2(H_2)$ can be alternatively described as follows. Let $\tilde{D}$ be a homomorphism of H_2 into $U(1)$ extending the natural embedding $H_1 \subset U(1)$. Such an extension is always possible for abelian groups [Ab]. Then, just like in Proposition (IV.5), a basis $\tilde{U}_\lambda$ can be constructed satisfying $\alpha(\tilde{U}_\lambda) = \tilde{D}(\lambda)\tilde{U}_{k(\lambda)}$. The corresponding 2-cocycle $\tilde{r}$ on H_2 is then k-invariant by an analog of (IV.7), and cohomologous to r by Lemma IV.1. So, in this case, $H_k^2(H_2)$ is the second group of k-invariant cohomologies of H_2. In general, when H_2 is not necessarily abelian, it is desirable to have a better description of $H_k^2(H_2)$.

Let us denote by $[r]$ the cohomology class of r in $H_k^2(H_2)$. When restricted to H_1 the conditions (IV.7) and (IV.8) are void. Moreover, since H_1 is abelian, there is a one-to-one correspondence between the second cohomology classes $[r]$ and symplectic bicharacters σ, see Proposition III.3. The correspondence is given by:

$$r(\lambda, \mu) = \sigma(\lambda, \mu)\, r(\mu, \lambda), \tag{IV.9}$$

see [OPT].

So far to a totally ergodic system with purely quasi-discrete spectrum of the second order we have associated the following algebraic structure:

1. A countable abelian group $H_1 \subset U(1)$ which has no nontrivial elements of finite order.
2. A countable group H_2, such that $H_1 \subset H_2$ is a normal subgroup.
3. An isomorphism $k : H_2 \mapsto H_2$ such that $k(\lambda) * \lambda^{-1} \in H_1$ and $k(\lambda) = \lambda$ iff $\lambda \in H_1$.
4. A cohomology class $[r]$ in $H_k^2(H_2)$.

Definition IV.4. A quadruple $(H_1, H_2, [r], k)$ satisfying conditions 1-4 above is called a quantum quasi-spectrum.

Definition IV.5. Two quantum quasi-spectra $(H_1, H_2, [r], k)$ and $(H_1', H_2', [r'], k')$ are called isomorphic if

(i) $H_1 = H_1'$.
(ii) There exists an isomorphism ϕ of the groups H_2 and H_2' leaving fixed all the elements of the group $H_1 = H_1'$ and such that

$$k = \phi^{-1} k' \phi, \qquad [r] = \phi^*[r'],$$

where ϕ^* is the induced isomorphism of the cohomology groups.

We are now prepared to prove the following theorem which is the main result of the section.

Theorem IV.6. (Equivalence Theorem) *Let* $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ *and* $(\mathfrak{B}, \mathbb{Z}, \beta, \omega)$ *be two totally ergodic quantum dynamical systems with purely quasi-discrete spectrum of the second order, and let* $(H_1(\alpha), H_2(\alpha), [r_\alpha], k_\alpha)$ *and* $(H_1(\beta), H_2(\beta), [r_\beta], k_\beta)$ *denote the corresponding quantum quasi-spectra. The following statements are equivalent:*

(i) The quantum quasi-spectra $(H_1(\alpha), H_2(\alpha), [r_\alpha], k_\alpha)$ *and* $(H_1(\beta), H_2(\beta), [r_\beta], k_\beta)$ *are isomorphic;*

(ii) $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ *and* $(\mathfrak{B}, \mathbb{Z}, \beta, \omega)$ *are conjugate.*

Proof. Only $(i) \to (ii)$ is non trivial. Let $\mathcal{K}(\alpha)$ and $\mathcal{K}(\beta)$ be the corresponding GNS Hilbert spaces. We are going to construct a conjugation $\Phi : \mathfrak{A} \mapsto \mathfrak{B}$ as an isomorphism implemented by a unitary map $Q : \mathcal{K}(\alpha) \mapsto \mathcal{K}(\beta)$. Let $\{U_\lambda\}$ and $\{V_\mu\}$ be normalized orthonormal basis in $\mathcal{K}(\alpha)$ and $\mathcal{K}(\beta)$ correspondingly, consisting of quasi-eigenvectors. Set:

$$Q\,(U_\lambda) := V_{\phi(\lambda)}, \tag{IV.10}$$

where ϕ is an isomorphism of $H_2(\alpha)$ and $H_2(\beta)$. By Lemma IV.1 we have $U_{\lambda_1}U_{\lambda_2} = r_\alpha(\lambda_1, \lambda_2)U_{\lambda_1 * \lambda_2}$ and $V_{\mu_1}V_{\mu_2} = r_\beta(\mu_1, \mu_2)V_{\mu_1 * \mu_2}$. Since r_α and $\phi^* r_\beta$ are cohomologous, we may assume, renormalizing V_μ if necessary, that

$$r_\alpha(\lambda_1, \lambda_2) = r_\beta\left(\phi(\lambda_1), \phi(\lambda_2)\right). \tag{IV.11}$$

We can deduce from (IV.11) that $\Phi(U_\lambda) := QU_\lambda Q^{-1} = V_{\phi(\lambda)}$ as follows:

$$\begin{aligned} QU_{\lambda_1}Q^{-1}V_{\phi(\lambda_2)} &= QU_{\lambda_1}U_{\lambda_2} = r_\alpha(\lambda_1, \lambda_2)QU_{\lambda_1 * \lambda_2} = r_\alpha(\lambda_1, \lambda_2)V_{\phi(\lambda_1 * \lambda_2)} \\ &= r_\beta\left(\phi(\lambda_1), \phi(\lambda_2)\right)V_{\phi(\lambda_1) * \phi(\lambda_2)} = V_{\phi(\lambda_1)}V_{\phi(\lambda_2)} \end{aligned}$$

But $\mathfrak{A}$ and $\mathfrak{B}$ are linearly generated by, correspondingly, U_λ and V_μ and so Φ extends to an isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$. A straightforward calculation verifies that $\Phi \circ \alpha = \beta \circ \Phi$:

$$\begin{aligned} (\Phi \circ \alpha)\,U_\lambda &= D(\lambda)\,\Phi\left(U_{k_\alpha(\lambda)}\right) = D(\lambda)\,V_{\phi(k_\alpha(\lambda))} = D(\lambda)\,V_{k_\beta(\phi(\lambda))} \\ &= \beta\left(V_{\phi(\lambda)}\right) = (\beta \circ \Phi)\,U_\lambda \end{aligned}$$

Also $\omega(\Phi(U_\lambda)) = \tau(U_\lambda)$ by Corollary III.11. It follows that $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ and $(\mathfrak{B}, \mathbb{Z}, \beta, \omega)$ are conjugate. $\square$

V. Representation Theorem

In this section we prove a representation theorem which says that for any system of invariants (i.e. a quantum quasi-spectrum) there is a corresponding quantum dynamical system with exactly that system of invariants. Consequently, the correspondence between the conjugacy classes of totally ergodic systems with purely quasi-discrete spectrum and the isomorphism classes of quantum quasi-spectra is onto.

Theorem V.1. (Representation Theorem) *Let $(H_1, H_2, [r], k)$ be a quantum quasi-spectrum. There exists a totally ergodic quantum dynamical system $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ with purely quasi-discrete spectrum such that its quantum quasi-spectrum is isomorphic to $(H_1, H_2, [r], k)$*

Proof. Consider $\mathcal{K} := l^2(H_2)$ and let $\{\phi_\lambda\}$ be the canonical basis in $\mathcal{K}$. Define $\mathfrak{A}$ to be the von Neumann algebra generated by the following operators U_λ:

$$U_\lambda \phi_\mu := r(\lambda, \mu)\phi_{\lambda * \mu}, \tag{V.1}$$

where $r(\lambda, \mu)$ is a normalized 2-cocycle on H_2 corresponding to $[r]$. For any $f \in \mathcal{K}$ we obtain

$$U_\lambda f(\mu) = r(\lambda, I(\lambda) * \mu) f(I(\lambda) * \mu).$$

It follows that

$$U_\lambda U_\mu = r\,(\lambda, \mu)\, U_{\lambda * \mu},$$

Then set

$$\beta \phi_\lambda := D(\lambda)\phi_{k(\lambda)}, \tag{V.2}$$

where $D(\lambda) \in U(1)$ was defined in (IV.5). Equivalently, or any $f \in \mathcal{K}$ we have

$$\beta f(\lambda) = D\left(k^{-1}(\lambda)\right) f\left(k^{-1}(\lambda)\right) = D\,(\lambda)\, f\left(k^{-1}(\lambda)\right), \tag{V.3}$$

since $D(\lambda)$ is k invariant. β is a unitary operator in $\mathcal{K}$ with the inverse given by

$$\beta^{-1}\phi_\lambda = \frac{1}{D\left(k^{-1}(\lambda)\right)}\, \phi_{k^{-1}(\lambda)},$$

or, equivalently, for any $f \in \mathcal{K}$

$$\beta^{-1} f(\lambda) = \frac{1}{D(\lambda)}\, f(k(\lambda)).$$

Conjugation with β gives an automorphism α of $\mathfrak{A}$ since one verifies that

$$\alpha(U_\lambda) := \beta U_\lambda \beta^{-1} = D(\lambda) U_{k(\lambda)}. \tag{V.4}$$

In fact,

$$\beta U_\lambda \beta^{-1}\phi_\mu = \frac{1}{D\left(k^{-1}(\mu)\right)}\,\beta U_\lambda \phi_{k^{-1}(\mu)} = \frac{r\left(\lambda, k^{-1}(\mu)\right)}{D\left(k^{-1}(\mu)\right)}\,\beta\phi_{\lambda * k^{-1}(\mu)}$$
$$= \frac{r\left(\lambda, k^{-1}(\mu)\right) D(\lambda * k^{-1}(\mu))}{D\left(k^{-1}(\mu)\right)}\,\phi_{k(\lambda * k^{-1}(\mu))} = \frac{r\left(\lambda, k^{-1}(\mu)\right) D(\lambda * k^{-1}(\mu))}{D\left(k^{-1}(\mu)\right)}\,\phi_{k(\lambda)*\mu}.$$

Notice that by (V.1) we have

$$U_{k(\lambda)}\phi_\mu = r(k(\lambda),\mu)\,\phi_{k(\lambda)*\mu}.$$

Consequently,

$$\beta U_\lambda \beta^{-1}\phi_\mu = \frac{r\left(\lambda, k^{-1}(\mu)\right) D(\lambda * k^{-1}(\mu))}{D\left(k^{-1}(\mu)\right) r(k(\lambda),\mu)}\,U_{k(\lambda)}\phi_\mu = D(\lambda)U_{k(\lambda)}\phi_\mu$$

by (IV.7).

Define

$$\tau(A) := (\phi_1, A\phi_1)$$

Since $\beta\phi_1 = \phi_1$, the state τ is α invariant. Moreover vector ϕ_1 is cyclic and separating for $\mathfrak{A}$ and so the GNS Hilbert space of state τ is canonically identified with $\mathcal{K}$. In this identification U_λ is mapped to ϕ_λ and the unitary operator in $\mathcal{K}$ defined by α is simply β. Also τ is a trace since $\mathfrak{A}$ is linearly generated by U_λ's.

We need to verify that the system $(\mathfrak{A}, \mathbb{Z}, \alpha, \tau)$ is totally ergodic and that its quantum quasi-spectrum $(H_1(\alpha), H_2(\alpha), [r_\alpha], k_\alpha)$ is isomorphic to $(H_1, H_2, [r], k)$. It follows from (V.3) that the spectrum of β (and equivalently of α) is H_1 with ϕ_λ, $\lambda \in H_1$ being the corresponding eigenvectors. Also

$$\beta^n f(\lambda) = D\left(\lambda\right)^n f\left(R(\lambda)^{-n} * \lambda\right),$$

where $R(\lambda) := k(\lambda) * \lambda^{-1} \in H_1$. As H_1 has no nontrivial elements of finite order, ϕ_1 is the only invariant vector for β^n and α is totally ergodic. Next observe that

$$\alpha(U_\lambda) = \frac{D(\lambda)U_{R(\lambda)}}{r(R(\lambda),\lambda)}\,U_\lambda,$$

and so that $H_2(\alpha)$ consists of the operators of the form $\frac{D(\lambda)U_{R(\lambda)}}{r(R(\lambda),\lambda)}$. They are different for different λ's as they correspond to different quasi-eigenvectors of an ergodic system. The map

$$H_2 \ni \lambda \mapsto \phi(\lambda) := \frac{D(\lambda)U_{R(\lambda)}}{r(R(\lambda),\lambda)} \in H_2(\alpha)$$

is consequently bijective. ϕ is a homomorphism as a consequence of the following calculation:

$$\begin{aligned}\phi(\lambda*\mu)U_{\lambda*\mu} &= \alpha(U_{\lambda*\mu}) = \frac{1}{r(\lambda,\mu)}\alpha(U_\lambda U_\mu) = \frac{1}{r(\lambda,\mu)}\alpha(U_\lambda)\alpha(U_\mu)\\ &= \frac{1}{r(\lambda,\mu)}\phi(\lambda)U_\lambda\phi(\mu)U_\mu = \frac{1}{r(\lambda,\mu)}\phi(\lambda)*\phi(\mu)U_\lambda U_\mu = \phi(\lambda)*\phi(\mu)U_{\lambda*\mu}\end{aligned}$$

Since $R(\phi(\lambda)) = R(\lambda)$, it follows that $k = \phi^{-1}k_\alpha\phi$. Finally, as $\{\phi_\lambda\}$ is a normalized basis in $\mathcal{K}$ consisting of quasi-eigenvectors of α, formula (V.1) implies that $[r] = \phi^*[r_\alpha]$. $\square$

VI. Examples: Quantum Torus

In this section we consider examples of systems, defined on on quantum tori, illustrating our theory. The first example is a system satisfying all the assumptions of our classification scheme. Interestingly, it appears as a quantization of a kicked rotor in [BB].

Recall that the algebra $\mathfrak{A}$ of observables on a quantum torus is defined as the universal von Neumann algebra generated by two unitary generators U, V satisfying the relation [R]:

$$UV = e^{2\pi ih}VU\ .$$

One can think of the elements of $\mathfrak{A}$ as series of the form $a = \sum a_{n,m}U^nV^m$. A natural trace on $\mathfrak{A}$ is simply given by $\tau(a) = a_{0,0}$. The automorphism α is defined on generators by:

$$\alpha(U) := e^{2\pi i\omega}U, \quad \alpha(V) := UV.$$

It extends to an automorphism of $\mathfrak{A}$. If ω is irrational, then α is totally ergodic. In fact, the eigenvectors of α are just powers of U:

$$\alpha(U^n) = e^{2\pi in\omega}U^n.$$

Consequently $H_1 = \{e^{2\pi in\omega},\ n\in\mathbb{Z}\} \cong \mathbb{Z}$ and the spectrum is simple which proves total ergodicity if ω is irrational. Moreover

$$\alpha(U^nV^m) = e^{2\pi i(n\omega+hm(m-1)/2)}U^m\cdot U^nV^m,$$

which shows that U^nV^m are quasi-eigenvectors of the second order for α. Since they form an orthonormal basis in $L^2(\mathfrak{A},\tau)$ we see that $(\mathfrak{A},\mathbb{Z},\alpha,\tau)$ is a totally ergodic system with purely quasi-discrete spectrum of the second order.

We can identify $H_2 \cong \mathbb{Z}^2$ as groups and H_1 is simply the subgroup $\mathbb{Z} \times \{0\} \subset \mathbb{Z}^2$. The mapping R is given by

$$R(n, m) = m \in \mathbb{Z} \cong H_1,$$

and the isomorphism k is

$$k(n, m) = (n + m, m).$$

Define

$$C(n, m) := \begin{cases} 1 & \text{if } m = 0 \\ e^{\pi i \left(h(nm-n) + \omega(n^2/m - n)\right)} & \text{otherwise.} \end{cases}$$

Then a simple calculation shows that $C(n, m)U^nV^m$ is a normalized basis for this ergodic system.

In this simple example the group H_2 is abelian. We can identify $H_k^2(H_2)$ with $H^2(H_2)$, the second cohomology group of H_2. The later group is identified with the set of symplectic bicharacters by (IV.9). A simple calculation shows that the following symplectic bicharacter represents $[r]$ in our example.

$$\sigma\left((n, m), (n', m')\right) = e^{2\pi i h(nm' - n'm)},$$

Notice that σ is trivial on H_1 and k-invariant.

The above example can be easily extended to give systems with a basis consisting of quasi-eigenvectors of arbitrary order. Here is one way to do it. Consider the algebra as before but with an extra generator W which we assume for simplicity to commute with U and V. As before define a trace τ such that $\tau(U^nV^mW^k) = 0$ unless $n = m = k = 0$. Finally extend the automorphism α by $\alpha(W) = UVW$. Then one easily verifies that quasi-eigenvectors of order one are powers of U, quasi-eigenvectors of the second order are U^nV^m and quasi-eigenvectors of the third order are $U^nV^mW^k$. The last expressions form a basis in the corresponding Hilbert space.

Systems that are ergodic but not totally ergodic are usually associated with elements of the finite order. For example in an algebra generated by two unitary generators U, V satisfying the relations $UV = e^{2\pi i/N}VU$, and $V^N = 1$ consider an automorphism α given by

$$\alpha(U) := e^{2\pi i\omega}U, \quad \alpha(V) := e^{2\pi i/N}V.$$

Here N is a positive integer and ω is assumed to be irrational. The eigenvectors of α are U^nV^m, $0 \leq m \leq N-1$. This system is ergodic but not totally ergodic since $\alpha^N(V^m) = V^m$ for any m.

Acknowledgment. I would like to thank Andrzej Lesniewski for inspiration, Jerry Kaminker for his enthusiasm and an anonymous referee for invaluable suggestions.

References

[Ab] Abramov, L. M.: Metric automorphisms with quasi-discrete spectrum, *Izv. Akad. Nauk USSR*, **26**, 513–530 (1962)

[Ar] Araki, H.: Positive cones for von Neumann algebras, in Operator Algebras and its Applications, *Proc. of Symp. in Pure Math of the AMS*, vol 38, AMS (1982)

[BB] J. Bellissard, J., Barelli, A.: *Dynamical localization: a Mathematical Framework* in Quantum Chaos, Quantum Measurements, edited by P. Cvitanovic, I.C. Percival & A. Wirzba, Kluwer Publ. (1992)

[B] Brown, K. S.: *Cohomology of Groups*, Springer Verlag (1982)

[BR] Bratteli, O., and Robinson, D.: *Operator Algebras and Quantum Statistical Mechanics*, Vol. 1, Springer Verlag (1979)

[C] Connes, A.: *Noncommutative Geometry*, Academic Press (1994)

[CFS] Cornfeld, I. P., Fomin, S. V., Sinai, Ya.: *Ergodic Theory*, Springer Verlag (1982)

[G] Greenleaf, F. P.: *Invariant Means on Topological Groups and Their Applications,* Van Nostrand Reinhold Co. (1969)

[HaP] Hahn, F., Parry, W.: Minimal dynamical systems with quasi-discrete spectrum, *J. London Math. Soc.*, **40**, 309–323 (1965)

[H] Halmos, P.: Measurable transformations, *Bull. AMS*, **55**, 1015–1034 (1949)

[HvN] Halmos, P., von Neumann, J.: Operator methods in classical mechanics, II, *Ann. Math.*, **43**, 332 - 350 (1942)

[HoP] Hoare, H., Parry, W.: Affine transformations with quasi-discrete spectrum I, II, *J. London Math. Soc.*, **41**, 88–96 and 529–530 (1966)

[HR] Hewitt, E., Ross, K. A.: *Abstract Harmonic Analysis* , Vol. I, Springer Verlag (1963)

[J] Jajte, R.: *Strong Limit Theorems in Non-Commutative Probability, Lecture Notes in Mathematics,* **1110**, Springer Verlag (1985)

[K] Koopman, B.: Hamiltonian Systems and Transformations in Hilbert Spaces, *Proc. Nat. Acad. Sci.*, **17**, 315–318 (1931)

[KL1] Klimek, S., Leśniewski, A.: Quantum Maps, in *Contemporary Mathematics*, **214**, Proceedings of the 1996 Joint Summer Research Conference on Quantization, edited by L. Coburn and M. Rieffel, AMS (1997)

[KL2] Klimek, S., Leśniewski, A.: Quantum Ergodic Theorems, in *Contemporary Mathematics*, **214**, Proceedings of the 1996 Joint Summer Research Conference on Quantization, edited by L. Coburn and M. Rieffel, AMS (1997)

[KLMR] Klimek, S., Leśniewski, A., Maitra, N., Rubin, R.: Ergodic properties of quantized toral automorphisms, *J. Math. Phys*, **38**, 67–83 (1997)

[L] Lance, C. E.: Ergodic theorems for convex sets and operator algebras, *Inv. Math.,* **37**, 201 - 214 (1976)

[OPT] Olsen, D., Pedersen, G., Takesaki, M.: Ergodic actions of compact abelian groups, *J. Oper. Theory*, **3**, 237–269 (1980)

[R] Rieffel, M.: Non-commutative tori - a case study of non-commutative differentiable manifolds, *Contemp. Math.*, **105** , 191 – 211 (1990)

[Si] Sinai, Ya.: *Topics in Ergodic Theory*, Princeton University Press (1994)

[VN] Von Neumann, J.: Zur Operathorenmethode in der klassischen Mechanik, *Ann. Math.,* **33**, 587 - 642 (1932)
[W] Walters, P.: *An Introduction to Ergodic Theory*, Springer Verlag (1982)