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PRECOBALANCED AND COBALANCED SEQUENCES OF MODULES OVER DOMAINS

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Abstract. The class of pure submodules ($\mathcal{P}$) and torsion-free images ($\mathcal{R}$) of finite direct sums of submodules of the quotient field of an integral domain were first investigated by M. C. R. Butler for the ring of integers (1965). In this case $\mathcal{P} = \mathcal{R}$ and short exact sequences of such modules are both prebalanced and precobalanced. This does not hold for integral domains in general. In this paper the notion of precobalanced sequences of modules is further investigated. It is shown that as in the case for abelian groups the exact sequence $0 \to M \to L \to T \to 0$ with torsion T is precobalanced precisely when it is cobalanced and in this case will split if M is torsion-free of rank 1. It is demonstrated that containment relationships between $\mathcal{P}$ and $\mathcal{R}$ for a domain R are intimately related to the issue of when pure submodules of Butler modules are precobalanced. An analogous statement is made regarding the dual question of when torsion-free images of Butler modules are prebalanced.

Keywords: precobalanced sequence, cobalanced sequence, torsion-free image, pure submodule

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1. Introduction

In 1965 M. C. R. Butler presented a class of abelian groups that were later termed *Butler* groups [1]. An abelian group is called *completely decomposable* if it is a finite direct sum of subgroups of the group of rational numbers, and a Butler group is a torsion-free image of a completely decomposable group. In the case of Butler groups, any pure subgroup is *precobalanced* with a *prebalanced* factor group. These two notions will be studied here. In particular, we will investigate, when pure submodules of Butler modules are precobalanced, and when the factor module is prebalanced, for Butler modules over an integral domain R.

Given a domain R, the modules isomorphic to a submodule of the quotient field Q of R are said to be *rank one*. A *completely decomposable* module is a finite direct sum of rank one modules. The class $\mathcal{R}$ denotes the class of torsion-free modules that occur as epimorphic images of completely decomposable modules. A submodule K of a torsion-free module M is said to be *pure* (in the sense of Matlis), if M/K is again torsion-free. The class $\mathcal{P}$ represents the class of pure submodules of completely decomposable modules.

In the case for the integers, Butler showed that $\mathcal{P}$ and $\mathcal{R}$ coincide. This is not the case for general domains, and we find that containment relationships between $\mathcal{P}$ and $\mathcal{R}$ for a domain R are intimately related to the issue of when pure submodules of Butler modules are precobalanced. An analogous statement can be made regarding the dual question of when torsion-free images of Butler modules are prebalanced.

2. A notion of precobalanced

A sequence

$$0 \to L \to M \to N \to 0$$

is called *precobalanced*, if for every homomorphism $f\colon L \to X$ where X is torsion-free of rank 1, there is a finite rank completely decomposable module C, a pure embedding $\alpha\colon X \to C$, and a map $\beta\colon M \to C$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "f"] & M \arrow[d, "\beta"] \\
0 \arrow[r] & X \arrow[r, "\alpha"] & C
\end{tikzcd}$$

is commutative.

The definition of precobalanced given here is the homological dual to the definition of prebalanced given in [6], as such the requirement that α be a pure map is imposed. Without the requirement that α be pure, we could take $C = Q$ and every sequence would qualify.

Lemma 1. *Pushouts of precobalanced sequences are precobalanced.*

P r o o f. Let $0 \to L \to M \to N \to 0$ be precobalanced, and let $f\colon L \to K$ be given. Given the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "g"] & M \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & M' \arrow[r] & N \arrow[r] & 0,
\end{tikzcd}$$

we must show that the bottom row is precobalanced.

Given a map $f\colon K \to X$, where X is torsion-free rank 1, there is a completely decomposable C, a pure embedding $\alpha\colon X \to C$, and a map $\beta\colon M \to C$ such that $\beta|_L = \alpha f g$. Referring to the diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, "g"] & M \arrow[d] \\
0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & M' \\
0 \arrow[r] & X \arrow[r, "\alpha"'] & C.
\end{tikzcd}$$

By the pushout property of M', there is a unique map from $\beta'\colon M' \to C$ such that $\beta'|_K = \alpha f$. $\square$

Lemma 2. *If X is torsion-free rank 1 and T is torsion, then any precobalanced $0 \to X \to H \to T \to 0$ is split.*

P r o o f. We will show that the sequence $0 \to X \to H \to T \to 0$ splits. There is a completely decomposable C and maps (with α pure) such that the following is commutative

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"'] \arrow[d, "1_X"] & H \arrow[r, "g"'] \arrow[d, "\beta"] & T \arrow[r] \arrow[d, "z"] & 0 \\
0 \arrow[r] & X \arrow[r, "\alpha"'] & C \arrow[r] & C/\alpha(X) \arrow[r] & 0.
\end{tikzcd}$$

The map $z\colon T \to C/\alpha(X)$ is zero because, $C/\alpha(X)$ is torsion-free. Therefore the image of β is contained in the image of α and since α is one-to-one, the map $\alpha^{-1}\beta$ is a splitting of $0 \to X \to H \to T \to 0$. $\square$

We view $\operatorname{Ext}(N, M)$ as equivalences classes of exact sequences

$$0 \to M \to L \to N \to 0.$$

Since the equivalence classes of precobalanced sequences

$$0 \to M \to L \to N \to 0$$

for a proper class, the collection of these equivalence classes, $\operatorname{Pcbext}(N, M)$ forms a submodule of $\operatorname{Ext}(N, M)$. Likewise, the collection of equivalence classes of cobalanced sequences $0 \to M \to L \to N \to 0$ is a submodule of $\operatorname{Ext}(N, M)$ which is denoted by $\operatorname{Cbext}(N, M)$.

Theorem 3. *If M is any module and T is torsion, then*

$$\operatorname{Pcbext}(T, M) = \operatorname{Cbext}(T, M).$$

P r o o f. First consider the case that M is torsion-free. Let

$$0 \to M \to N \to T \to 0$$

be a precobalanced sequence, and let $f\colon M \to X$ be a homomorphism with X rank 1. Consider the pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & N \arrow[r] & T \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r] & H \arrow[r] & T \arrow[r] & 0.
\end{tikzcd}$$

By Lemmas 1 and 2, the pushout sequence $0 \to X \to H \to T \to 0$ is split, implying that the sequence $0 \to M \to N \to T \to 0$ is cobalanced.

For general M, suppose

$$E\colon\ 0 \to M \to N \to T \to 0$$

is precobalanced. For $\pi\colon M \to M/t(M)$ equal to the natural map, the bottom row of the push-out diagram below is precobalanced:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d, "\pi"] & G \arrow[r] \arrow[d, "\pi'"] & T \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M/t(M) \arrow[r, "f'"'] & G' \arrow[r] & T \arrow[r] & 0.
\end{tikzcd}$$

For any map $\alpha\colon M \to X$ with X rank 1, $t(M) \subseteq \ker\alpha$ and so α induces a unique map $\alpha'\colon M/t(M) \to X$. Since the bottom row is precobalanced, by what we have shown above, there is a map $\delta\colon G' \to X$ such that $\delta f' = \alpha'$. Then, $\alpha = \delta\pi' f$ by commutativity of the diagram. $\square$

3. When are pure submodules of Butler modules precobalanced?

Theorem 4. *The following are equivalent:*

(1) $\mathcal{R} \subseteq \mathcal{P}$.

(2) *Any pure submodule of a Butler module is precobalanced.*

(3) *Any pure rank 1 submodule of a Butler module is precobalanced.*

(4) *Any pure submodule of a finite rank completely decomposable module is precobalanced.*

Proof. We will first assume that $\mathcal{R} \subseteq \mathcal{P}$ and show that (2) holds. Let B be a given Butler module and K a pure submodule of C. Given a rank 1 module X and a map $f\colon K \to X$, consider the diagram with the bottom row a pushout:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\beta"] & U \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & X \arrow[r] & B' \arrow[r] & U \arrow[r] & 0. \end{tikzcd}$$

Note, B' is a Butler module since it is an epimorphic image of $B \oplus X$, and so, by assumption B' is a pure submodule of a finite rank completely decomposable module C. Since purity is transitive, the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "\beta"] & U \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X \arrow[r] & C \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

satisfies the criterion for precobalanced.

The implication (2) $\to$ (3) is obvious. To show (3) $\to$ (1), a given Butler module B can be expressed as an image of a finite rank completely decomposable $C = Y_1 \oplus \ldots \oplus Y_n$; let $f\colon C \to B$ be an epimorphism. Set X_j equal to the pure submodule of B generated by $f(Y_j)$ (assume that $f(Y_j) \neq 0$ for all j), and note that the natural (summation) map from $\bigoplus_{j=1}^{n} X_j \to B$ is an epimorphism.

By assumption, for each j, there is a finite rank completely decomposable module C_j containing X_j as a pure submodule, and a map $\beta_j\colon B \to C_j$ such that

$$\begin{tikzcd} 0 \arrow[r] & X_j \arrow[r] \arrow[d, "1"] & B \arrow[d, "\beta_j"] \\ 0 \arrow[r] & X_j \arrow[r] & C_j \end{tikzcd}$$

is commutative.

Define $\beta\colon B \to \bigoplus_{j=1}^{n} C_j$ by $\beta(b) = (\beta_1(b), \dots, \beta_n(b))$. That β is an embedding follows from the fact that the X_j's generate B, and the purity of β can be established from the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_j X_j \arrow[r] \arrow[d, "\mathrm{nat}"] & \bigoplus_j C_j \arrow[r, "\alpha"'] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "\beta"'] & \bigoplus_j C_j \arrow[r] & U \arrow[r] & 0.
\end{tikzcd}$$

Observe, the middle vertical map is an isomorphism, W is torsion-free, and the map from $\bigoplus_j X_j \to B$ is an epimorphism. By the Snake Lemma, $U \cong W$, so β has a torsion-free cokernel.

The implication (2) $\to$ (4) is obvious. To see that (4) implies (2), let B be a Butler module with K a pure submodule of B. There is a finite rank completely decomposable module C and an epimorphism $f\colon C \to B$. Consider the pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "h'"'] \arrow[d, "g"] & C \arrow[r] \arrow[d, "f"] & C/H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "h"'] & B \arrow[r] & B/K \arrow[r] & 0.
\end{tikzcd}$$

Then H is a pure submodule of C and by (4) is precobalanced in C. Given a map $\alpha\colon K \to X$ with X rank 1, there is a completely decomposable module D, a pure monomorphism $\delta\colon X \to D$, and a map $\gamma\colon C \to D$ such that $\gamma h' = \delta\alpha g$. By diagram chasing (or the Snake Lemma), $\ker f = \ker g$ and so $\gamma(\ker f) = 0$. Therefore, γ induces a map $\gamma'\colon B \to D$. To see that $\gamma' h = \delta\alpha$, take $x \in K$ and $y \in H$ such that $g(y) = x$ (g is an epimorphism). Then $\gamma' h(x) = \gamma' h(g(y)) = \gamma' f(h'(y)) = \gamma h'(y) = \delta\alpha g(y) = \delta\alpha(x)$. $\square$

Note that while proofing (3) $\to$ (1) we showed that any Butler Module whose pure rank 1's are precobalanced is necessarily a pure submodule of a finite rank completely decomposable. Thus the following examples from [7] and [6], respectively, hold.

E x a m p l e 5. Let R be a 1-dimensional noetherian domain whose integral closure $\overline{R}$ inside Q is not finitely generated as an R-module. Then, there exists a Butler module B over R such that some rank 1 pure submodule of B is not precobalanced.

E x a m p l e 6. Let R be a subring of an algebraic number field such that for some integral prime p, $R_p \neq \overline{R}_p$ and $p\overline{R}$ splits into a product of at least 2 distinct prime ideals of $\overline{R}$. Then, there exists a Butler module B over R such that some rank 1 pure submodule of B is not precobalanced.

Theorem 7. *The following are equivalent:*
(1) $\mathcal{P} \subseteq \mathcal{R}$.
(2) *Given $A \in \mathcal{P}$, any pure submodule of A is prebalanced in A.*
(3) *Given $A \in \mathcal{P}$, any corank 1 pure submodule of A is prebalanced in A.*
(4) *Any pure submodule of a completely decomposable module A is prebalanced in A.*

Proof. (1) → (2). Let B be a pure submodule of $A \in \mathcal{P}$ and denote A/B by G. Given $f\colon X \to G$ with X rank 1, form the pull-back:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "1"] & A' \arrow[r, "\alpha"'] \arrow[d, "\beta"] & X \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & B \arrow[r] & A \arrow[r] & G \arrow[r] & 0. \end{tikzcd}$$

By a characterization of the pull-back, A' is a pure submodule of $X \oplus A$ and so $A' \in \mathcal{R}$. There exists a completely decomposable module C and an epimorphism $g\colon C \to A'$. The diagram

$$\begin{tikzcd} C \arrow[r, "\alpha g"'] \arrow[d, "\beta g"] & X \arrow[r] \arrow[d, "f"] & 0 \\ A \arrow[r] & G \arrow[r] & 0 \end{tikzcd}$$

reveals that B is a prebalanced submodule of A.

The implication (2) → (3) is obvious. We will prove (3) → (1) by induction on the rank of $A \in \mathcal{P}$. Note that every rank 1 module is in $\mathcal{R}$. Given $A \in \mathcal{P}$, of rank $n > 1$, let K be a rank $n - 1$ pure submodule of A. Then $A/K = X$ is rank 1 and since K is prebalanced in A, there is a completely decomposable module C, an epimorphism $f\colon C \to X$, and a map $g\colon C \to A$ such that $\pi g = f$ where $\pi\colon A \to X$ is the natural map. We have that A is an epimorphic image of $C \oplus K$. Since $K \in \mathcal{R}$ and $\mathcal{R}$ is closed under epimorphic images and finite direct sums, $A \in \mathcal{R}$.

The implication (2) → (4) is obvious. To see (4) → (2), let $A \in \mathcal{P}$ and B a pure submodule of A. There is a completely decomposable module C and a pure monomorphism $\alpha\colon A \to C$. Consider the push-out

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\pi"] & C \arrow[r, "\alpha"'] \arrow[d, "\beta"] & H \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & A/B \arrow[r, "\delta"'] & G \arrow[r] & H \arrow[r] & 0, \end{tikzcd}$$

where $\pi\colon A \to A/B$ is the natural map.

Let $f\colon X \to A/B$ where X is rank 1. Since β is onto and $\ker\beta$ is prebalanced in C, there is a completely decomposable C', an epimorphism $g\colon C' \to X$, and a map $\mu\colon C' \to C$ such that $\beta\mu = \delta fg$. A diagram chase shows that $\alpha\mu = 0$ so μ induces a map $\mu'\colon C' \to A$ such that $\pi\mu' = fg$. □

Note that the analogous statement after Theorem 4 holds. That is, in the proof of (3) → (1), we showed that any pure submodule of a finite rank completely decomposable whose pure corank 1's are prebalanced is necessarily a Butler Module.

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