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RADICALS AND COMPLETE DISTRIBUTIVITY IN RELATIVELY NORMAL LATTICES

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Abstract. Lattices in the class $\mathcal{IRN}$ of algebraic, distributive lattices whose compact elements form relatively normal lattices are investigated. We deal mainly with the lattices in $\mathcal{IRN}$ the greatest element of which is compact. The distributive radicals of algebraic lattices are introduced and for the lattices in $\mathcal{IRN}$ with the sublattice of compact elements satisfying the conditional join-infinite distributive law they are compared with two other kinds of radicals. Connections between complete distributivity of algebraic lattices and the distributive radicals are described. The general results can be applied e.g. to MV-algebras, GMV-algebras and unital ℓ-groups.

Keywords: relatively normal lattice, algebraic lattice, complete distributivity, closed element, radical

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1. Introduction

Let L be an algebraic lattice with least element 0 and greatest element 1, and let its join-subsemilattice of compact elements be denoted by $\mathrm{Com}(L)$. If $c \in \mathrm{Com}(L)$, then an element $p \in L$ is called a *value* of c if p is maximal with respect to not exceeding c. Let us denote by $\mathrm{Val}(c)$ the set of all values of c.

Let $\mathrm{Val}(L)$ be the set of all values of all compact elements in L. Recall that an element $p < 1$ is called *completely meet-irreducible* if, whenever $p = \bigwedge_{\alpha\in\Gamma} x_\alpha$, then there is $\alpha_0 \in \Gamma$ such that $p = x_{\alpha_0}$. By [20], p. 43, $\mathrm{Val}(L)$ is exactly the set of all completely meet-irreducible elements in L. Since every element of an algebraic lattice is the meet of a set of completely meet-irreducible elements, every element in L is the meet of a set of values.

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If $a \in L$, let us set $(a] = \{x \in L;\ x \leqslant a\}$. An element $y \in (a]$ is called *maximal in* $(a]$ if $y < a$ and if $y \leqslant z \in (a]$ implies $z = y$ or $z = a$. If L is any algebraic lattice and $c \in \mathrm{Com}(L)$ then for any $x \in (c]$, $x \neq c$, there is an element m maximal in $(c]$ such that $x \leqslant m$. (See [13], p. 248.)

This particularly means that, if the greatest element 1 of an algebraic lattice L is compact, then L is dually atomic. In such a case let us denote by $\mathrm{rad}(L)$ the meet of all dual atoms, i.e. the maximal elements in $(1] = L$. The element $\mathrm{rad}(L)$ will be called *the radical* of the lattice L.

Another kind of a radical in L, which is in certain cases in connection with $\mathrm{rad}(L)$ (e.g. in the theory of MV-algebras and GMV-algebras), has been introduced in [8]. Let us recall that if L is an algebraic, distributive lattice and $a \in L$ then a is called *essential* if there exists $0 \neq x \in \mathrm{Com}(L)$ such that $\sup \mathrm{Val}(x) \leqslant a$. Denote by $r(L)$ the meet of the set of essential elements in L.

A poset P is called a *root-system* provided the principal upper set $[a) = \{x \in P;\ a \leqslant x\}$ is a chain for all $a \in P$. A lower-bounded distributive lattice is called *relatively normal* (see e.g. [20]) provided the set of its prime ideals is a root-system under set-inclusion. Recall (see also [20]) that the term "relatively normal" is suggested by topological considerations. A bounded distributive lattice is called *normal* if each of its prime ideals is contained in a unique maximal ideal. By [14], [15], a topological space is normal if and only if the lattice of its open sets is normal. A lower-bounded distributive lattice is relatively normal if and only if each of its closed subintervals is a normal lattice. By [14], [15], a topological space is hereditarily normal if and only if its open sets form a relatively normal lattice.

The class of all algebraic, distributive lattices whose compact elements form a relatively normal lattice is denoted by $\mathcal{IRN}$. By [20], Corollary 3.2, if L is an algebraic, distributive lattice such that $\mathrm{Com}(L)$ is a sublattice (i.e., it is closed under binary suprema and infima) of L, then L is a member of $\mathcal{IRN}$ if and only if the (finitely) meet-prime elements of L form a root-system. (An element $p < 1$ is called *meet-prime* if, whenever $x \wedge y \leqslant p$, then $x \leqslant p$ or $y \leqslant p$. If arbitrary meets are allowed in the definition then we have the notion of a *completely meet-prime* element.) The structure properties of the lattices which belong to $\mathcal{IRN}$ have been studied in [20] and subsequently in [8] and [12].

In this paper we deal with radicals of lattices in the class $\mathcal{IRN}$ in which the greatest element 1 is compact. Further, we introduce the notion of a closed element of an algebraic lattice L. This enables us to define the distributive radical of L and we describe connections among three kinds of radicals of lattices in the class $\mathcal{IRN}$ which have the sublattice of compact elements satisfying the conditional join-infinite distributive law (CJIP). Connections between complete distributivity of algebraic lattices and distributive radicals of algebraic, distributive lattices satisfying (CJIP) are found.

R e m a r k. Various important lattices belong to the class $\mathcal{IRN}$. For instance, if G is a lattice ordered group (an ℓ-group) then the lattice $\mathcal{C}(G)$ of convex ℓ-subgroups of G is a member of $\mathcal{IRN}$. Similarly for the lattice $\mathcal{I}(G)$ of ℓ-ideals (i.e., kernels of ℓ-homomorphisms) of any ℓ-group G which is a subdirect product of linearly ordered groups. (See e.g. [1], [3], [11].) Furthermore, if A is an MV-algebra (see [5], [7]) or a GMV-algebra then the lattice $\mathcal{I}(A)$ of ideals of A is also a member of $\mathcal{IRN}$. (GMV-algebras have been introduced recently by the author in [17] and, independently, under the name pseudo MV-algebras, by G. Georgescu and A. Iorgulescu in [9] and [10], as a non-commutative generalization of MV-algebras which have been defined by Chang in [4] as an algebraic counterpart of the Łukasiewicz infinite valued propositional logic.)

Hence many important structure results of these theories are special instances of those for members of $\mathcal{IRN}$ which are formulated in purely lattice-theoric terms.

2. Radicals in lattices with compact greatest elements

In this section we will describe connections between radicals $\operatorname{rad}(L)$ and $r(L)$ in lattices L with 1 compact which belong to the class $\mathcal{IRN}$.

R e m a r k. Recall that if $G = (G, +, \vee, \wedge)$ is an ℓ-group and $0 < u \in G$, then u is called a *strong unit* of G provided the principal convex ℓ-subgroup generated by u is equal to G. An ℓ-group G is called *unital* if G contains a strong unit. Since the compact elements in the lattice $\mathcal{C}(G)$ are exactly the principal convex ℓ-subgroups, each unital ℓ-group G is a compact element in $\mathcal{C}(G)$. Analogously, if A is an MV-algebra or a GMV-algebra, then A is a compact element in the lattice $\mathcal{I}(A)$.

Theorem 1. *If $L \in \mathcal{IRN}$, $1 \in \operatorname{Com}(L)$ and L has a finite number of dual atoms, then $r(L) \leqslant \operatorname{rad}(L)$.*

P r o o f. Since L is an algebraic lattice and, by assumption, $1 \in \operatorname{Com}(L)$, every element in L different from 1 is less than or equal to some maximal element of L. Moreover, $L \in \mathcal{IRN}$, and hence by [20], Lemma 3.5, for any $c \in \operatorname{Com}(L)$ we have that $\operatorname{Val}(c)$ is finite if and only if every value of c is a completely meet-prime element. Therefore in our case every maximal element in L is completely meet-prime.

By [20], p. 43, for any completely meet-prime element p there is $0 \neq c \in \operatorname{Com}(L)$ such that $\operatorname{Val}(c) = \{p\}$, and thus p is essential. Hence under our assumption every maximal element in L is essential. From this we get $r(L) \leqslant \operatorname{rad}(L)$. $\square$

Let us recall that an algebraic lattice is called *finite-valued* if $\mathrm{Val}(x)$ is finite for each $x \in \mathrm{Com}(L)$.

Corollary 2. *If $L \in \mathcal{IRN}$, $1 \in \mathrm{Com}(L)$ and L is finite-valued, then $r(L) \leqslant \mathrm{rad}(L)$.*

An algebraic lattice L is said to be *archimedean* if for each compact element $c \in L$ the meet of the elements which are maximal below c is 0. (See [13].)

Theorem 3. *If $L \in \mathcal{IRN}$, $1 \in \mathrm{Com}(L)$, L has a finite number of dual atoms and L is archimedean, then $r(L) = \mathrm{rad}(L)$.*

Proof. From the assumption it follows that the meet of the set of maximal elements in L is 0. Hence $r(L) \leqslant \mathrm{rad}(L) = 0$, and so $r(L) = \mathrm{rad}(L)$. □

3. Closed elements and complete distributivity

Now, we will introduce the notion of a closed element of an algebraic lattice L. Let a be an element in L. Then a is called a *closed element* of L if it satisfies the following condition: If $B \subseteq (a] \cap \mathrm{Com}(L)$ and $\sup_{\mathrm{Com}(L)} B$ exists then $\sup_{\mathrm{Com}(L)} B \in (a]$.

Lemma 4. *The meet of any set of closed elements of L is a closed element of L.*

If $a \in L$, denote by $\bar{a}$ the meet of the set of closed elements of L exceeding a. The element $\bar{a}$ will be called the *closure of a*.

As is known, if L is an algebraic, distributive lattice, then L satisfies the join-infinite distributive law (JID)

$$a \wedge \bigvee_{\alpha \in \Gamma} b_\alpha = \bigvee_{\alpha \in \Gamma} (a \wedge b_\alpha).$$

Hence L is a Heyting lattice (see e.g. [2]), that means, for any elements a and b in L, the set $\{x \in L;\ a \wedge x \leqslant b\}$ has a greatest element $a \to b$ (called *the relative pseudo-complement of a with respect to b*). In particular, the element $a^* = a \to 0$ is called *the pseudo-complement of a in L.*

Lemma 5. *If L is a Heyting lattice and $z, x_\alpha \in L$, $\alpha \in \Gamma$, then*

$$\left(\bigvee_{\alpha\in\Gamma} x_\alpha\right) \to z = \bigwedge_{\alpha\in\Gamma} (x_\alpha \to z)$$

provided both sides of the above equality exist.

P r o o f. $x_\beta \leqslant \bigvee_{\alpha\in\Gamma} x_\alpha$, thus $\left(\bigvee_{\alpha\in\Gamma} x_\alpha\right) \to z \leqslant x_\beta \to z$ for any $\beta \in \Gamma$, and hence

$$\left(\bigvee_{\alpha\in\Gamma} x_\alpha\right) \to z \leqslant \bigwedge_{\alpha\in\Gamma} (x_\alpha \to z).$$

On the other side,

$$\left(\bigvee_{\alpha\in\Gamma} x_\alpha\right) \wedge \left(\bigwedge_{\beta\in\Gamma} (x_\beta \to z)\right) = \bigvee_{\alpha\in\Gamma} \left(x_\alpha \wedge \left(\bigwedge_{\beta\in\Gamma} (x_\beta \to z)\right)\right) \leqslant \bigvee_{\alpha\in\Gamma} (x_\alpha \wedge (x_\alpha \to z)) \leqslant z,$$

therefore

$$\bigwedge_{\alpha\in\Gamma} (x_\alpha \to z) \leqslant \left(\bigvee_{\alpha\in\Gamma} x_\alpha\right) \to z.$$

□

Now, we will generalize the condition (JID) to any lattice (which need not be complete). Let A be a lattice. We say that A *satisfies the conditional join-infinite distributive law* (CJID) if the following condition holds: If $y, x_\alpha \in A$, $\alpha \in \Gamma$, and if $\bigvee_{\alpha\in\Gamma} x_\alpha$ in A exists, then also $\bigvee_{\alpha\in\Gamma} (y \wedge x_\alpha)$ exists and

$$y \wedge \bigvee_{\alpha\in\Gamma} x_\alpha = \bigvee_{\alpha\in\Gamma} (y \wedge x_\alpha).$$

(For example, the underlying lattice of any non-trivial ℓ-group satisfies the condition (CJID) but not (JID).)

Proposition 6. *Let L be an algebraic, distributive lattice, let* $\mathrm{Com}(L)$ *be a sublattice of L and let the lattice* $\mathrm{Com}(L)$ *satisfy* (CJID). *Then $a \to c$ is closed for each $a \in L$ and $c \in \mathrm{Com}(L)$.*

P r o o f. a) Let first $a \in \mathrm{Com}(L)$ and $c \in \mathrm{Com}(L)$. Let $b_\alpha \in (a \to c] \cap \mathrm{Com}(L)$, $\alpha \in \Gamma$, and let $b = \bigvee_{\mathrm{Com}(L)}\{b_\alpha;\ \alpha \in \Gamma\}$ exist. Then

$$a \wedge b = a \wedge \bigvee\nolimits_{\mathrm{Com}(L)}\{b_\alpha;\ \alpha \in \Gamma\} = \bigvee\nolimits_{\mathrm{Com}(L)}\{a \wedge b_\alpha;\ \alpha \in \Gamma\}.$$

Since $a \wedge b_\alpha \leqslant a \wedge (a \to c) \leqslant c$ for each $\alpha \in \Gamma$, we have $a \wedge b \leqslant c$, hence $b \leqslant a \to c$.

b) Now, let $a \in L$ be arbitrary and $c \in \mathrm{Com}(L)$. The lattice L is algebraic, thus $a = \bigvee_L \{e;\ e \in (a] \cap \mathrm{Com}(L)\}$. Therefore by Lemma 5,

$$a \to c = \Big(\bigvee_L \{e;\ e \in (a] \cap \mathrm{Com}(L)\}\Big) \to c = \bigwedge_L \{e \to c;\ e \in (a] \cap \mathrm{Com}(L)\},$$

and so, by Lemma 4, $a \to c \in \mathrm{Com}(L)$. □

Corollary 7. *If L is an algebraic, distributive lattice such that $\mathrm{Com}(L)$ is a sublattice of L and $\mathrm{Com}(L)$ satisfies (CJID), then the pseudo-complement a^* is a closed element for any $a \in L$.*

Theorem 8. *Let L be an algebraic, distributive lattice such that $\mathrm{Com}(L)$ is a sublattice of L satisfying (CJID). If $a \in L$ then its closure $\bar{a}$ is equal to the join in L of all elements from $\mathrm{Com}(L)$ which are joins in $\mathrm{Com}(L)$ of sets of elements belonging to $(a] \cap \mathrm{Com}(L)$.*

Proof. Let $a \in L$ and let $\bar{a}$ be its closure. Let

$$\Big\{x_\alpha;\ x_\alpha = \bigvee\nolimits_{\mathrm{Com}(L)} \{x_{\alpha\beta};\ x_{\alpha\beta} \in (a] \cap \mathrm{Com}(L),\ \alpha \in \Gamma,\ \beta \in \Delta_\alpha\}\Big\}$$

be the set of existing joins in $\mathrm{Com}(L)$ of elements of $(a] \cap \mathrm{Com}(L)$. Denote by b the join of this set in L . Since $x_\alpha \leqslant \bar{a}$ for any $\alpha \in \Gamma$, $b \leqslant \bar{a}$.

Let $y_\gamma \in (b] \cap \mathrm{Com}(L)$, $\gamma \in \Sigma$, and let $\bigvee_{\mathrm{Com}(L)} \{y_\gamma;\ \gamma \in \Sigma\}$ exist. Then

$$y_\gamma \leqslant b = \bigvee\nolimits_L \Big\{x_\alpha;\ x_\alpha = \bigvee\nolimits_{\mathrm{Com}(L)} x_{\alpha\beta};\ x_{\alpha\beta} \in (a] \cap \mathrm{Com}(L)\Big\},$$

and thus there are $\alpha_1, \ldots, \alpha_n \in \Gamma$ ($n \in \mathbb{N}$) such that

$$y_\gamma \leqslant x_{\alpha_1} \vee_L \ldots \vee_L x_{\alpha_n} = x_{\alpha_1} \vee_{\mathrm{Com}(L)} \ldots \vee_{\mathrm{Com}(L)} x_{\alpha_n} = x_{\bar{\gamma}}.$$

Hence $x_{\bar{\gamma}}$ is the join in $\mathrm{Com}(L)$ of elements from $(a] \cap \mathrm{Com}(L)$.

Let $x_{\bar{\gamma}} = \bigvee_{\mathrm{Com}(L)} \{x_{\bar{\gamma}\delta};\ \delta \in \Omega\}$, where $x_{\bar{\gamma}\delta} \in (a] \cap \mathrm{Com}(L)$ for each $\delta \in \Omega$. Then

$$y_\gamma = y_\gamma \wedge x_{\bar{\gamma}} = y_\gamma \wedge \bigvee\nolimits_{\mathrm{Com}(L)} \{x_{\bar{\gamma}\delta};\ \delta \in \Omega\} = \bigvee\nolimits_{\mathrm{Com}(L)} \{y_\gamma \wedge x_{\bar{\gamma}\delta};\ \delta \in \Omega\},$$

where $y_\gamma \wedge x_{\bar{\gamma}\delta} \in (a] \cap \mathrm{Com}(L)$.

Hence $\bigvee_{\mathrm{Com}(L)} \{y_\gamma;\ \gamma \in \Sigma\} \leqslant b$, thus b is a closed element for which $a \leqslant b$. Therefore $b = \bar{a}$. □

Let us denote by r_x the join of the set $\mathrm{Val}(x)$ for any $0 \neq x \in \mathrm{Com}(L)$.

Proposition 9. *Let $L \in \mathcal{IRN}$ and let* $\mathrm{Com}(L)$ *be a sublattice of L satisfying* (CJIP). *If an element $a \in L$ is essential and if $y \nleqslant a$ holds for every $y \in \mathrm{Com}(L)$ such that $r_y \leqslant a$, then a is a closed element.*

P r o o f. Let $a \in L$ be an essential element and let $0 \neq x \in \mathrm{Com}(L)$ be such that $r_x \leqslant a$. If $0 \neq z \in \mathrm{Com}(L)$ and $z \nleqslant x^*$, then $0 \neq x \wedge z \leqslant x$, thus $r_{x\wedge z} \leqslant r_x \leqslant a$. Therefore by assumption $x \wedge z \nleqslant a$, so $z \nleqslant a$, and hence $a \leqslant x^*$.

Conversely, a is essential, and since (by [20], Corollary 3.2) elements exceeding prime elements are prime, a is also prime. We assume $x \nleqslant a$, so we get $x^* \leqslant a$. Hence $a = x^*$ and that means by Corollary 7 that a is a closed element of L. $\square$

If L is an algebraic lattice, denote by $d(L)$ the meet of all closed prime elements in L. Then $d(L)$ will be called the *distributive radical* of the lattice L.

R e m a r k. The notion of distributive radical of an algebraic lattice is a generalization of that of an ℓ-group (see e.g. [1], [3], [11]), of an MV-algebra ([6]) and of a GMV-algebra ([18], [19]).

Theorem 10. *Let $L \in \mathcal{IRN}$ and let* $\mathrm{Com}(L)$ *be a sublattice of L satisfying* (CJIP). *Let L satisfy the following condition: If a is an essential element in L for which there exists $0 \neq y \in \mathrm{Com}(L)$ such that $y \leqslant a$ and $r_y \leqslant a$, then a is closed.*

a) *Then $d(L) \leqslant r(L)$.*

b) *If, moreover, $1 \in \mathrm{Com}(L)$, then $d(L) \leqslant r(L) \leqslant \mathrm{rad}(L)$.*

P r o o f. The assertions of the theorem are immediate consequences of Proposition 9 and Corollary 2. $\square$

N o t e. In [8], Proposition 2.7, it is proved that if $L \in \mathcal{IRN}$ and L is finite-valued, then $r(L) = \{0\}$. Hence, if L satisfies, moreover, the assumptions of Theorem 10, then $d(L) = \{0\}$.

Let us recall that a lattice A is called *completely distributive* if for any $a_{\alpha\beta} \in A$ ($\alpha \in \Gamma$, $\beta \in \Delta$),

$$\bigwedge_{\alpha\in\Gamma} \bigvee_{\beta\in\Delta} a_{\alpha\beta} = \bigvee_{f\in\Delta^\Gamma} \bigwedge_{\alpha\in\Gamma} a_{\alpha f(\alpha)},$$

whenever all joins and meets on both sides exist.

Now, we will study connections between the complete distributivity of the lattices $\mathrm{Com}(L)$ and the distributive radicals of algebraic lattices L.

Let L be an algebraic lattice, $0 \neq c \in \mathrm{Com}(L)$. Then an element $c' \in \mathrm{Com}(L)$ is said to be *subordinate to* c if $c = \bigvee_{\mathrm{Com}(L)}\{c_\alpha;\ \alpha \in \Gamma\}$ implies the existence of $\beta \in \Gamma$ such that $c' \leqslant c_\beta$. Denote by $S(c)$ the set of elements in $\mathrm{Com}(L)$ subordinate to c.

Note that for complete lattices the relation *totally below* (notation $\lll$) has been defined as follows (see e.g. [16]): If P is a complete lattice and $x, y \in P$ then

$$x \lll y :\Leftrightarrow \forall A \subseteq P \Big(y \leqslant \bigvee A \Rightarrow \exists a \in A; x \leqslant a \Big).$$

One can introduce a generalization of the relation totally below for any poset if one supposes validity of the implication in the definition for all existing $\bigvee A$. Hence in our case, c' is subordinate to c if and only if $c' \lll c$ and $S(c) = \{x; x \lll c\}$ in $\mathrm{Com}(L)$.

Proposition 11. *Let L be an algebraic, distributive lattice such that* $\mathrm{Com}(L)$ *is a sublattice of L satisfying* (CJIP). *If $c \in (d(L)] \cap \mathrm{Com}(L)$, then $S(c) = \{0\}$.*

P r o o f. Suppose that $0 \neq c \in (d(L)] \cap \mathrm{Com}(L)$ and that there is $0 \neq c' \in S(c)$. Let $a \in \mathrm{Val}(c')$. Obviously $d(L) \leqslant \bar{a}$ (because $\bar{a}$ is a closed prime element in L), thus $c \leqslant \bar{a}$. Hence by Theorem 8, $c = \bigvee_{\mathrm{Com}(L)}\{d_\alpha;\ \alpha \in \Gamma\}$, where $d_\alpha \in (a] \cap \mathrm{Com}(L)$ for each $\alpha \in \Gamma$. Therefore there exists $\beta \in \Gamma$ such that $c' \leqslant d_\beta$, so $c' \leqslant a$, a contradiction. $\square$

Proposition 12. *Let P be a completely distributive lattice with a least element 0. Then for any nonzero element $c \in P$ there is a nonzero element $c' \in P$ such that $c' \lll c$.*

P r o o f. Consider $0 \neq c \in P$. Let $\{d_{\alpha\beta};\ \beta \in \Delta_\alpha\}$ $(\alpha \in \Gamma)$ be exactly all subsets of P satisfying $c = \bigvee\{d_{\alpha\beta};\ \beta \in \Delta_\alpha\}$. We can suppose that $\Delta_\alpha = \Delta$ for each $\alpha \in \Gamma$. Hence $c = \bigvee\{d_{\alpha\beta};\ \beta \in \Delta\}$ for any $\alpha \in \Gamma$. If $\bigwedge\{d_{\alpha f(\alpha)};\ \alpha \in \Gamma\} = 0$ for every $f \in \Delta^\Gamma$, then we get

$$c = \bigwedge_{\alpha \in \Gamma} \bigvee_{\beta \in \Delta} d_{\alpha\beta} = \bigvee_{f \in \Delta^\Gamma} \bigwedge_{\alpha \in \Gamma} d_{\alpha f(\alpha)} = 0,$$

a contradiction.

Therefore there is $f \in \Delta^\Gamma$ such that 0 is not the meet of the set $\{d_{\alpha f(\alpha)};\ \alpha \in \Gamma\}$. Hence there exists $c' \in P$ with $0 < c' \leqslant d_{\alpha f(\alpha)}$ for any $\alpha \in \Gamma$, that means $c' \lll c$. $\square$

The following theorem is now an immediate consequence of Propositions 11 and 12.

Theorem 13. *Let L be an algebraic, distributive lattice such that* $\operatorname{Com}(L)$ *is a sublattice of L satisfying* (CJIP). *If* $\operatorname{Com}(L)$ *is completely distributive then* $d(L) = 0$.

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