THE INDUCED PATHS IN A CONNECTED GRAPH AND A TERNARY RELATION DETERMINED BY THEM

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Abstract. By a ternary structure we mean an ordered pair (X_0, T_0), where X_0 is a finite nonempty set and T_0 is a ternary relation on X_0. By the underlying graph of a ternary structure (X_0, T_0) we mean the (undirected) graph G with the properties that X_0 is its vertex set and distinct vertices u and v of G are adjacent if and only if

$$\{x \in X_0\,;\ T_0(u,x,v)\} \cup \{x \in X_0\,;\ T_0(v,x,u)\} = \{u,v\}.$$

A ternary structure (X_0, T_0) is said to be the B-structure of a connected graph G if X_0 is the vertex set of G and the following statement holds for all $u, x, y \in X_0$: $T_0(x,u,y)$ if and only if u belongs to an induced $x-y$ path in G. It is clear that if a ternary structure (X_0, T_0) is the B-structure of a connected graph G, then G is the underlying graph of (X_0, T_0). We will prove that there exists no sentence σ of the first-order logic such that a ternary structure (X_0, T_0) with a connected underlying graph G is the B-structure of G if and only if (X_0, T_0) satisfies σ.

Keywords: connected graph, induced path, ternary relation, finite structure

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Introduction

The letters i, j, k, m and n will be reserved for denoting integers.

By a graph we mean here a graph in the sense of [2], i.e. a finite undirected graph without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively.

Let G be a graph, let $v_0, \ldots, v_n \in V(G)$, and let

$$P\colon\ v_0, \ldots, v_n$$

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be a path in G. We say that P is an *induced path* in G if $v_i v_j \notin E(G)$ for all $i, j \in \{0, \dots, n\}$ such that $|i-j| \neq 1$. Note that instead of the term "induced path" the term "minimal path" is sometimes used. If G is a connected graph, then we say that P is a *geodesic* in G, if $d(v_0, v_n) = n$, where d denotes the distance function of G. Instead of the term "geodesic" the term "shortest path" is sometimes used.

Let P and P' be induced paths in a graph G; we will say that P and P' are disjoint if no vertex of G belongs both to P and to P'; we will say that P and P' are non-adjacent in G if there exists no pair of vertices u and u' such that u belongs to P, u' belongs to P' and u and u' are adjacent in G.

Part 1

By a *ternary structure* we mean an ordered pair (X_0, T_0), where X_0 is a *finite* nonempty set and T_0 is a ternary relation on X_0.

Let (X_1, T_1) and (X_2, T_2) be ternary structures. By a *partial isomorphism* from (X_1, T_1) to (X_2, T_2) we mean such an injective mapping q that $\mathrm{Def}(q) \subseteq X_1$, $\mathrm{Im}(q) \subseteq X_2$ and

$$T_1(x, u, y) \text{ if and only if } T_2(q(x), q(u), q(y))$$

for all $u, x, y \in \mathrm{Def}(q)$. (Note that the notion of a partial isomorphism from a ternary structure to a ternary structure is a special case of the notion of a partial isomorphism in the sense of [4], p. 15). Let (X_0, T_0) be a ternary structure. By the *pseudointerval function* of (X_0, T_0) we mean the mapping J of $X_0 \times X_0$ into 2^{X_0} defined as follows:

$$J(x, y) = \{u \in X_0;\ T_0(x, u, y)\}$$

for all $x, y \in X_0$.

Let (X_0, T_0) be a ternary structure, and let J denote its pseudointeval function. By the *underlying graph* of (X_0, T_0) we mean the graph G defined as follows: $V(G) = X_0$ and

$$E(G) = \{uv;\ u, v \in X_0, u \neq v \text{ and } J(u, v) \cup J(v, u) = \{u, v\}\}.$$

We will say that (X_0, T_0) is *connected* if its underlying graph is connected.

Let G be a connected graph, and let $\mathbf{P}_0$ be a subset of the set of all paths in G. By the $\mathbf{P}_0$-*structure* of G we mean the ternary structure (X_0, T_0) such that $X_0 = E(G)$ and

$T_0(x, u, y)$ if and only if

there exists an $x - y$ path P in G such that $P \in \mathbf{P}_0$ and u belongs to P

for all $u, x, y \in X_0$. Let (X_0, T_0) be the $\mathbf{P}_0$-structure of G. If $\mathbf{P}_0$ is the set of all paths in G, the set of all induced paths in G, or the set of all geodesics in G, then we say that (X_0, T_0) is the A-structure of G, the B-structure of G, or the Γ-structure of G, respectively.

Let G be a connected graph, and let d denote its distance function. By the Σ-*structure* of G we mean the ternary structure (X_0, T_0) such that $X_0 = V(G)$ and

$$T_0(x, u, y) \text{ if and only if } d(x, u) = 1 \text{ and } d(u, y) = d(x, y) - 1$$

for all $u, x, y \in X_0$.

Let (X_0, T_0) be a ternary structure, and let $\mathbf{Z}$ stand for A, B, Γ or Σ. We say that (X_0, T_0) is a $\mathbf{Z}$-structure if there exists a connected graph G such that (X_0, T_0) is the $\mathbf{Z}$-structure of G.

Let (T_0, X_0) be a ternary structure, and let J denote its pseudointerval function. We will say that (X_0, T_0) satisfies condition C1, C1′, C2 or C3 if

(C1) $\quad J(x, x) = \{x\}$ for all $x \in X_0$,

(C1′) $\quad J(x, x) = \emptyset$ for all $x \in X_0$,

(C2) $\quad J(x, y) = J(y, x)$ for all $x, y \in X_0$, or

(C3) $\quad x \in J(x, y)$ for all $x, y \in X_0$,

respectively. It is obvious that all A-structures, B-structures and Γ-structures satisfy conditions C1, C2 and C3 and that all Σ-structures satisfy condition C1′.

Let $\mathbf{Z}$ stand for B, Γ or Σ. It is easy to see that if (X_0, T_0) is a $\mathbf{Z}$-structure, then it is the $\mathbf{Z}$-structure of exactly one connected graph, namely of the underlying graph of (X_0, T_0). This means that all B-structures, all Γ-structures and all Σ-structures are connected. However, this is not the case with A-structures. The underlying graph of the A-structure of a complete graph with at least three vertices has no edges.

Let (X_0, T_0) be a ternary structure, and let J denote its pseudointerval function. We will say that (X_0, T_0) is *scant* if (a) it satisfies conditions C1 and C2, and (b) the following statement holds for all distinct $x, y \in X_0$: if $J(x, y) \neq \{x, y\}$, then $J(x, y) = X_0$. Clearly, every scant ternary structure is determined by its underlying graph. It is not difficult to see that if the Γ-structure of a connected graph G is scant, then the diameter of G does not exceed two. This is not the case with B-structures. It is obvious that the B-structure of every cycle is scant. Thus, for every $n \geqslant 3$ there exists a connected graph G of diameter n such that the B-structure of G is scant.

Let (X_0, T_0) be a ternary structure, let J denote its pseudointerval function, and let G denote the underlying graph of (X_0, T_0). If J satisfies conditions C1, C2 and C3, then J is a transit function on G in the sense of Mulder [7]. Recall that if (X_0, T_0)

is a Γ-structure or a B-structure, then it is respectively the Γ-structure or the B-structure of G. If (X_0, T_0) is a Γ-structure, then J is called the interval function of G; cf. Mulder [6], where the interval function of a connected graph was studied widely. If (X_0, T_0) is a B-structure, then J is called the induced path function or the minimal path function on G in [7]. The induced path function on a connected graph was studied by Duchet [3] and by Morgana and Mulder [5].

The pseudointerval functions of A-structures were characterized in Changat, Klavžar and Mulder [1] while the pseudointerval functions of Γ-structures were characterized by the present author in [8], [10] and [12]. These characterizations can be reformulated easily as characterizations of A-structures and of Γ-structures by a finite set of axioms or, more strictly, by a unique axiom.

The result obtained for Σ-structures by the present author in [9] and [11] is not too strong: Σ-structures were characterized as connected ternary structures satisfying a finite set of axioms. This result could be reformulated as follows: there exists an axiom σ in a language of the first order logic such that a connected ternary structure (X_0, T_0) is a Σ-structure if and only if (X_0, T_0) satisfies σ.

In the present paper we will prove that a similar result does not hold for B-structures. To prove this, we will need a certain portion of mathematical logic; for precise formulations and further details the reader is referred to Ebbinghaus and Flum [4], p. 1–12. (Especially, the explanation of the term "satisfy", which will be used in Theorem 1, can be found in [4], p. 6).

Let T be the symbol for a ternary relation. By an atomic formula of the first-order logic of vocabulary $\{T\}$ (shortly: by an atomic formula) we mean an expression

$$x = y,$$

where x and y are variables, or an expression

$$T(x, u, y),$$

where u, x and y are variables. The formulae of the first-order logic of vocabulary $\{T\}$ (shortly: the formulae) will be defined as follows:

every atomic formula is a formula;

if α is a formula, then $\neg\alpha$ is a formula;

if α_1 and α_2 are formulae, then $\alpha_1 \vee \alpha_2$ is a formula;

if α is a formula and x is a variable, then $\exists x\alpha$ is a formula;

no other expressions are formulae.

Following [4] we define the *quantifier rank* $qr(\alpha)$ of a formula α:

if α is atomic, then $qr(\alpha) = 0$;
if α is $\neg\beta$, where β is a formula, then $qr(\alpha) = qr(\beta)$;
if α is $\beta_1 \vee \beta_2$, where β_1 and β_2 are formulae, then $qr(\alpha) = \max(qr(\beta_1), qr(\beta_2))$;
if α is $\exists x\beta$, where β is a formula and x is a variable, then $qr(\alpha) = qr(\beta) + 1$.

The most important formulae are sentences: a formula α is called a sentence if for every atomic subformula β of α, every variable belonging to β is in the scope of the corresponding quantifier.

The next theorem, which is a special case of Fraïssé's Theorem, will be an important tool for us:

Theorem 1. *Let (X_1, T_1) and (X_2, T_2) be ternary structures, and let $n \geqslant 1$. Then the following statements* (A) *and* (B) *are equivalent:*

(A) *(X_1, T_1) and (X_2, T_2) satisfy the same sentences σ with $qr(\sigma) \leqslant n$.*

(B) *There exist nonempty sets $\mathbf{Q}_0, \ldots, \mathbf{Q}_n$ of partial isomorphisms from (X_1, T_1) to (X_2, T_2) such that for each m, $1 \leqslant m < n$, we have*
   (I) *for every $q \in \mathbf{Q}_{m+1}$ and every $x \in X_1$ there exists $r \in \mathbf{Q}_m$ such that $q \subseteq r$ and $x \in \mathrm{Def}(r)$;*
   (II) *for every $q \in \mathbf{Q}_{m+1}$ and every $x \in X_2$ there exists $r \in \mathbf{Q}_m$ such that $q \subseteq r$ and $x \in \mathrm{Im}(r)$.*

For the proof of Fraïssé's Theorem (and further closely related results) the reader is referred to [4], Chapter 1.

Part 2

Assume that an infinite sequence

$$u_0, w_0, u_1, w_1, u_2, w_2, \ldots$$

of mutually distinct vertices is given.

Let $k \geqslant 3$. By F_k we denote the graph with vertices

$$u_0, w_0, u_1, w_1, \ldots, u_{6k-1}, w_{6k-1}$$

and with edges

$$u_0u_1, u_1u_2, \dots, u_{3k-2}u_{3k-1}, u_{3k-1}u_0,$$
$$u_{3k}u_{3k+1}, u_{3k+1}u_{3k+2}, \dots, u_{6k-2}u_{6k-1}, u_{6k-1}u_{3k},$$
$$w_0w_1, w_1w_2, \dots, w_{3k-2}w_{3k-1}, w_{3k-1}w_0,$$
$$w_{3k}w_{3k+1}, w_{3k+1}w_{3k+2}, \dots, w_{6k-2}w_{6k-1}, w_{6k-1}w_{3k},$$
$$u_0w_0, u_1w_1, \dots, u_{6k-1}w_{6k-1},$$
$$u_0u_{3k}, u_ku_{4k}, u_{2k}u_{5k}.$$

A diagram of F_3 is presented in Fig. 1.

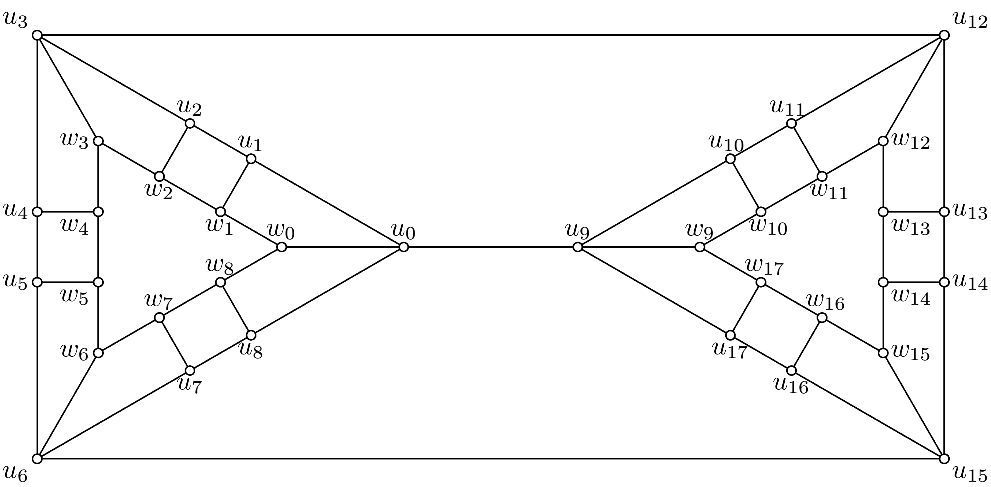

Fig. 1.

Lemma 1. *Let $k \geqslant 3$. Then the B-structure of F_k is scant.*

P r o o f. Let $x \in V(F_k)$. Then there exists exactly one i, $0 \leqslant i \leqslant 6k-1$, such that $x = u_i$ or $x = w_i$; we define $\operatorname{ind}(x) = i$. For every $y \in V(F_k)$ we define y^L and y^R as follows:

if $\operatorname{ind}(y) \in \{0, k, 2k, 3k, 4k, 5k\}$, then $y^L = y^R = u_{\operatorname{ind}(y)}$;
if $jk < \operatorname{ind}(y) < (j+1)k$, where $j \in \{0, 1, 3, 4\}$, then $y^L = u_{jk}$ and $y^R = u_{(j+1)k}$;
if $2k < \operatorname{ind}(y) < 3k$, then $y^L = u_{2k}$ and $y^R = u_0$;
if $5k < \operatorname{ind}(y) < 6k$, then $y^L = u_{5k}$ and $y^R = u_{3k}$.

Let J denote the pseudointerval function of the B-system of F_k. Consider arbitrary $x, y \in V(F_k)$ such that $d(x,y) \geqslant 2$, where d denotes the distance function of F_k. We want to prove that $J(x,y) = V(F_k)$.

Denote $V_1 = \{v \in V(F_k);\ 0 \leqslant \operatorname{ind}(v) \leqslant 3k-1\}$ and $V_2 = V(F_k) \setminus V_1$. Without loss of generality we assume that $x \in V_1$. We distinguish two cases.

Case 1. Let $y \in V_1$. It is clear that $V_1 \subseteq J(x,y)$ and

$$V_2 \subseteq J(u_0,u_k) \cap J(u_k,u_{2k}) \cap J(u_{2k},u_0).$$

Recall that $d(x,y) \geqslant 2$. We can see that there exist $x_1 \in \{x^L, x^R\}$ and $y_1 \in \{y^L, y^R\}$ such that $x_1 \neq y_1$ and there exist an induced $x - x_1$ path P_x in F_k and an induced $y_1 - y$ path P_y in F_k with the property that P_x and P_y are disjoint and non-adjacent in F_k. This implies that $J(x,y) = V(F_k)$.

Case 2. Let $y \in V_2$. We distinguish two subcases.

Subcase 2.1. Let $d(x,y) = 2$. Then $x \in \{u_0, u_k, u_{2k}\}$ or $y \in \{u_{3k}, u_{4k}, u_{5k}\}$. Without loss of generality we assume that $x = u_0$. Then $y = w_{3k}$ or $y = u_{3k+1}$ or $y = u_{6k-1}$.

First, let $y = w_{3k}$. Consider the following five sequences:

$$u_0, u_{3k}, w_{3k};$$

$$u_0, u_1, \ldots, u_{k-1}, u_k, u_{4k}, u_{4k-1}, \ldots, u_{3k+1}, u_{3k}, w_{3k};$$

$$u_0, u_{3k-1}, u_{3k-2}, \ldots, u_{k+1}, u_k, u_{4k}, u_{4k+1}, \ldots, u_{6k-2}, u_{6k-1}, u_{3k}, w_{3k};$$

$$u_0, w_0, w_1, \ldots, w_{k-1}, w_k, u_k, u_{4k}, w_{4k}, w_{4k-1}, \ldots, w_{3k+1}, w_{3k};$$

$$u_0, w_0, w_{3k-1}, w_{3k-2}, \ldots, w_k, u_k, u_{4k}, w_{4k}, w_{4k+1}, \ldots w_{6k-1}, w_{3k}.$$

Each vertex of F_k belongs to at least one of these sequences. Moreover, each of these sequences is an induced $x - y$ path in F_k. Thus $J(x,y) = V(F_k)$.

Now, let $y \neq w_{3k}$. Without loss of generality we assume that $y = u_{3k+1}$. Consider the following five sequences:

$$u_0, u_{3k}, u_{3k+1};$$

$$u_0, u_1, \ldots, u_{k-1}, u_k, u_{4k}, u_{4k-1}, \ldots, u_{3k+1};$$

$$u_0, u_{3k-1}, \ldots, u_{k+1}, u_k, u_{4k}, u_{4k+1}, \ldots, u_{6k-2}, u_{6k-1}, w_{6k-1}, w_{3k}, w_{3k+1}, u_{3k+1};$$

$$u_0, w_0, w_1, \ldots, w_{k-1}, w_k, u_k, u_{4k}, w_{4k}, w_{4k-1}, \ldots, w_{3k+1}, u_{3k+1};$$

$$u_0, w_0, w_{3k-1}, w_{3k-2}, \ldots, w_k, u_k, u_{4k}, w_{4k}, w_{4k+1}, \ldots, w_{6k-1}, w_{3k}, w_{3k+1}, u_{3k+1}.$$

Again, each vertex of F_k belongs to at least one of these sequences and each of these sequences is an induced $x - y$ path in F_k. Thus $J(x,y) = V(F_k)$.

Subcase 2.2. Let $d(x,y) \geqslant 3$. Then there exist $x_2 \in \{x^L, x^R\}$ and $y_2 \in \{y^L, y^R\}$ such that $d(x_2,y) \geqslant 3$ and $d(x,y_2) \geqslant 3$. Define $x^* = u_{\operatorname{ind}(x_2)+3k}$ and $y^* = u_{\operatorname{ind}(y_2)-3k}$. Obviously, $d(x^*,y) \geqslant 2$ and $d(x,y^*) \geqslant 2$. It is clear that $V_1 \subseteq J(x,y^*)$ and $V_2 \subseteq J(x^*,y)$. This implies that $J(x,y) = V(F_k)$.

The proof is complete. □

Let $k > 2$. By F'_k we denote the graph with vertices

$$u_0, w_0, u_1, w_1, \ldots, u_{6k-1}, w_{6k-1}$$

and with edges

$$u_0u_1, u_1u_2, \ldots, u_{2k-2}u_{2k-1}, u_{2k-1}u_0,$$
$$u_{2k}u_{2k+1}, u_{2k+1}u_{2k+2}, \ldots, u_{4k-2}u_{4k-1}, u_{4k-1}u_{2k},$$
$$u_{4k}u_{4k+1}, u_{4k+1}u_{4k+2}, \ldots, u_{6k-2}u_{6k-1}, u_{6k-1}u_{4k},$$
$$w_0w_1, w_1w_2, \ldots, w_{2k-2}w_{2k-1}, w_{2k-1}w_0,$$
$$w_{2k}w_{2k+1}, w_{2k+1}w_{2k+2}, \ldots, w_{4k-2}w_{4k-1}, w_{4k-1}w_{2k},$$
$$w_{4k}w_{4k+1}, w_{4k+1}w_{4k+2}, \ldots, w_{6k-2}w_{6k-1}, w_{6k-1}w_{4k},$$
$$u_0w_0, u_1w_1, \ldots, u_{6k-1}w_{6k-1},$$
$$u_ku_{2k}, u_{3k}u_{4k}, u_{5k}u_0.$$

A diagram of F'_3 is presented in Fig. 2.

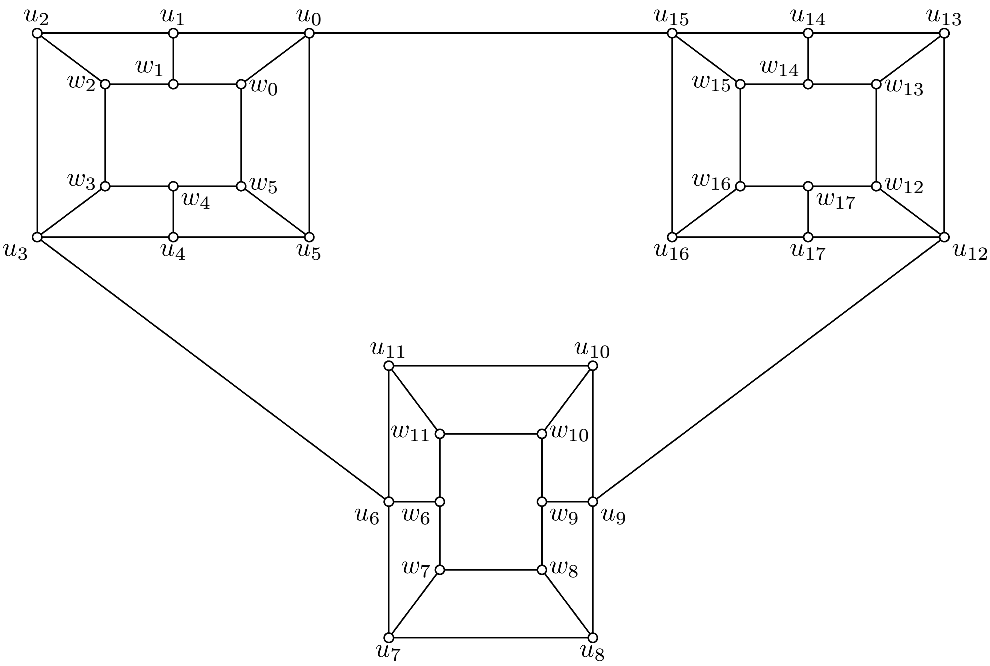

Fig. 2.

Lemma 2. *Let $k \geqslant 3$. Then the B-structure of F'_k is not scant.*

P r o o f. Let J denote the pseudointerval function of the B-structure of F'_k. Since $J(u_{k-1}, u_{k+1}) \neq V(F'_k)$, the result follows. □

Lemma 3. *Let $n \geqslant 1$ and $k > 2^{n+1}$. Assume that (X_1, T_1) and (X_2, T_2) are scant ternary structures such that the underlying graph of (X_1, T_1) is F_k and the underlying graph of (X_2, T_2) is F'_k. Then (X_1, T_1) and (X_2, T_2) satisfy the same sentences σ with $qr(\sigma) \leqslant n$.*

Proof. Put $U = \{u_0, u_1, \dots, u_{6k-1}\}$, $U^\flat = \{u_0, u_k, u_{2k}, u_{3k}, u_{4k}, u_{5k}\}$, $W = \{w_0, w_1, \dots, w_{6k-1}\}$ and $W^\flat = \{w_0, w_k, w_{2k}, w_{3k}, w_{4k}, w_{5k}\}$. Obviously, $X_1 = U \cup W = X_2$.

If $x, y \in U \cup W$, then we will write $x \sim y$ if and only if $x, y \in U$ or $x, y \in W$. We define $u_i^\diamond = w_i$ and $w_i^\diamond = u_i$ for all i, $0 \leqslant i \leqslant 6k-1$. Thus $(x^\diamond)^\diamond = x$ for each $x \in U \cup W$ and $y^\diamond \sim z^\diamond$ if and only if $y \sim z$ for all $y, z \in U \cup W$. We define $[x] = x$ for every $x \in U$ and $[x] = x^\diamond$ for every $x \in W$.

By F^* we mean F_k or F'_k. Let d^* denote the distance function of F^*. Define

$$e^*(x, y) = d^*([x], [y]) \text{ for all } x, y \in U \cup W.$$

Obviously, $e^*(x, y) = 0$ if and only if $x = y$ or $x^\diamond = y$ for all $x, y \in U \cup W$.

Recall that $k > 2^{n+1}$. Consider an arbitrary $x \in U \cup W$ and denote $D(x) = \{y \in U^\flat \cup W^\flat;\ e^*(x, y) \leqslant 2^n\}$; it is easy to see that $|D(x)| \leqslant 4$ and if $D(x) \neq \emptyset$, then the subgraph of F^* induced by $D(x)$ is a path of length either one or three.

Consider arbitrary $x, y \in U \cup W$ such that $e^*(x, y) \leqslant 2^n$. It is easy to see that (i) every $x - y$ geodesic in F^* contains at most two vertices in $U^\flat$; (ii) if at least one $x - y$ geodesic in F^* contains two vertices in $U^\flat$, then every $x - y$ geodesic in F^* contains two vertices in $U^\flat$ and these two vertices are adjacent in F^*. We will write $f^*(x, y) = 1$ if every $x - y$ geodesic in F^* contains at most one vertex in $U^\flat$ and $f^*(x, y) = 2$ otherwise.

For every m, $0 \leqslant m \leqslant n$ and for all $x, y \in U \cup W$ we define

$$e^*_m(x, y) = e^*(x, y) \text{ if } e^*(x, y) \leqslant 2^m,$$
$$e^*_m(x, y) = \infty \text{ if } e^*(x, y) > 2^m.$$

Consider an arbitrary m, $0 \leqslant m < n$. We see that

(1) if $e^*_{m+1}(x, y) = \infty$ and $e^*_m(y, z) < \infty$, then $e^*_m(x, z) = \infty$ for all $x, y, z \in U \cup W$.

We will write e, e_m and f instead of e^*, e^*_m and f^* respectively if F^* is F_k, and e', e'_m and f' instead of e^*, e^*_m and f^* respectively if F^* is F'_k.

Recall that (X_1, T_1) and (X_2, T_2) are scant. We denote by PART the set of all partial isomorphisms p from F_k to F'_k such that $U^\flat \cup W^\flat \subseteq \mathrm{Def}(p)$,

$$p(x) \sim x \text{ for all } x \in \mathrm{Def}(p),$$

and

$$p(u_0) = u_0, p(w_0) = w_0, p(u_k) = u_k, p(w_k) = w_k, p(u_{2k}) = u_{4k}, p(w_{2k}) = w_{4k},$$
$$p(u_{3k}) = u_{5k}, p(w_{3k}) = w_{5k}, p(u_{4k}) = u_{2k}, p(w_{4k}) = w_{2k},$$
$$p(u_{5k}) = u_{3k} \text{ and } p(w_{5k}) = w_{3k}.$$

Obviously, there exists exactly one $p_0 \in \mathrm{PART}$ such that $\mathrm{Def}(p_0) = U^\flat \cup W^\flat$.

For every m, $0 \leqslant m \leqslant n$, we denote by $\mathbf{Q}_m$ the set of all $q \in \mathrm{PART}$ such that $|\mathrm{Def}(q)| \leqslant 12 + n - m$ and that $e'_m(q(x), q(y)) = e_m(x, y)$ for all $x, y \in \mathrm{Def}(q)$.

It is clear that $\mathbf{Q}_n = \{p_0\}$. As follows from the definition, $\mathbf{Q}_n \subseteq \ldots \subseteq \mathbf{Q}_0$.

Consider an arbitrary m, $0 \leqslant m < n$. We need to show that conditions (I) and (II) (of Theorem 1) hold.

Consider an arbitrary $q \in \mathbf{Q}_{m+1}$ and an arbitrary $x \in U \cup W$. If $x \in \mathrm{Def}(q)$, we put $r = q$. Assume that $x \notin \mathrm{Def}(q)$. Then $x \notin U^\flat \cup Z^\flat$. We distinguish two cases.

C a s e 1. Assume that there exists $y \in \mathrm{Def}(q)$ such that $e_m(x, y) < \infty$. Without loss of generality we assume that $e_m(x, y) \leqslant e_m(x, y_0)$ for every $y_0 \in \mathrm{Def}(q)$.

First, let $e_m(x, y) = 0$. Since $x \notin \mathrm{Def}(q)$, we have $y = x^\diamond$. We put $x' = (q(y))^\diamond$.

Now, we assume that $e_m(x, y) > 0$. We distinguish four subcases.

S u b c a s e 1.1. Assume that

$$\text{(2)} \qquad \begin{aligned} &\text{there exists } z \in \mathrm{Def}(q) \text{ such that} \\ &e_m(x, z) < \infty \text{ and } e(y, z) = e_m(y, x) + e_m(x, z). \end{aligned}$$

Without loss of generality we assume that $e_m(x, z) \leqslant e_m(x, z_0)$ for every $z_0 \in \mathrm{Def}(q)$ such that $e_m(x, z_0) < \infty$ and $e(y, z_0) = e_m(y, x) + e_m(x, z_0)$. Since $e_m(x, y) > 0$, it is obvious that $e_m(x, z) > 0$. Since $e_m(x, y) < \infty$ and $e_m(x, z) < \infty$, we get $e_{m+1}(y, z) < \infty$. Since $y, z \in \mathrm{Def}(q)$, we have $e'_{m+1}(q(y), q(z)) = e_{m+1}(y, z)$. There exists exactly one $x' \in (U \cup W) \setminus \mathrm{Im}(q)$ such that $e'(q(y), q(z)) = e'_m(q(y), x') + e_m(x', q(z))$ and $x' \sim x$.

S u b c a s e 1.2. Assume (2) does not hold and

$$\text{(3)} \qquad \begin{aligned} &\text{there exists } z \in \mathrm{Def}(q) \text{ such that} \\ &0 < e_{m+1}(y, z) < \infty, f(y, z) = 1 \text{ and } e(x, z) = e_m(x, y) + e_{m+1}(y, z). \end{aligned}$$

Without loss of generality we assume that $e_{m+1}(y, z) \leqslant e_{m+1}(y, z_0)$ for every $z_0 \in \mathrm{Def}(q)$ such that $0 < e_{m+1}(y, z_0) < \infty, f(y, z_0) = 1$ and $e(x, z_0) = e_m(x, y) + e_{m+1}(y, z_0)$. Since $y, z \in \mathrm{Def}(q)$, we get $e'_{m+1}(q(y), q(z)) = e_{m+1}(y, z)$. There exists exactly one $x' \in (U \cup W) \setminus \mathrm{Im}(q)$ such that $e'(x', q(z)) = e'_m(x', q(y)) + e'_{m+1}(q(y), q(z))$ and $x' \sim x$.

S u b c a s e 1.3. Assume (2) and (3) do not hold and

$$\text{there exists } z \in \operatorname{Def}(q) \text{ such that} \\ 0 < e_{m+1}(y,z) < \infty, f(y,z) = 2 \text{ and } e(x,z) = e_m(x,y) + e_{m+1}(y,z). \tag{4}$$

Without loss of generality we assume that $e_{m+1}(y,z) \leqslant e_{m+1}(y,z_0)$ for every $z_0 \in \operatorname{Def}(q)$ such that $0 < e_{m+1}(y,z_0) < \infty$, $f(y,z_0) = 2$ and $e(x,z_0) = e_m(x,y) + e_{m+1}(y,z_0)$. It is easy to see that $y, z \in U^\flat \cup W^\flat$ and $e(y,z) = 1$. Since $y, z \in \operatorname{Def}(q)$, we get $q(y), q(z) \in U^\flat \cup W^\flat$ and $e'(q(y), q(z)) = 1$. There exist exactly two vertices belonging to $(U \cup W) \setminus \operatorname{Im}(q)$, say vertices v_1 and v_2, such that $e'(v_j, q(z)) = e'_m(v_j, q(y)) + 1$ and $v_j \sim x$ for $j = 1, 2$. Consider an arbitrary $x' \in \{v_1, v_2\}$.

S u b c a s e 1.4. Assume (2), (3) and (4) do not hold. Then there exists no $z \in \operatorname{Def}(q)$ such that $0 < e_{m+1}(y,z) \leqslant e_m(x,y) + 2^m$. Thus there exists no $z \in \operatorname{Def}(q)$ such that $0 < e'_{m+1}(q(y), q(z)) \leqslant e_m(x,y) + 2^m$. This implies that there exist exactly two vertices belonging to $(U \cup W) \setminus \operatorname{Im}(q)$, say vertices v_1 and v_2, such that $e'_m(v_j, q(y)) = e_m(x,y)$ and $v_j \sim x$ for $j = 1, 2$. Consider an arbitrary $x' \in \{v_1, v_2\}$.

C a s e 2. Assume that $e_m(x,y) = \infty$ for every $y \in \operatorname{Def}(q)$. There exists $x' \in (U \cup W) \setminus \operatorname{Im}(q)$ such that $x' \sim x$ and $e'_m(x', q(y)) = \infty$ for every $y \in \operatorname{Def}(q)$.

Define $r = q \cup \{(x, x')\}$. If we take (1) into account, we can see that $r \in \mathbf{Q}_m$. Thus condition (I) holds.

The fact that condition (II) holds can be proved analogously. Applying Theorem 1, we obtain the result of the lemma. □

R e m a r k. The introduction of functions e^*_m in the proof of Lemma 3 is a modification of one of the ideas in Example 1.3.5 of [4].

Theorem 2. *There exists no sentence σ of the first-order logic of vocabulary $\{T\}$ such that a connected ternary structure is a B-structure if and only if it satisfies σ.*

P r o o f. Combining Lemmas 1, 2 and 3, we get the theorem. □

Note that Theorem 2 can be reformulated as follows: There exists no finite set S of sentences of first-order logic of vocabulary $\{T\}$ such that a connected ternary structure is a B-structure if and only if it satisfies each sentence in S.

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