ON A PROBLEM OF E. PRISNER CONCERNING THE BICLIQUE OPERATOR

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Abstract. The symbol $K(B,C)$ denotes a directed graph with the vertex set $B \cup C$ for two (not necessarily disjoint) vertex sets B, C in which an arc goes from each vertex of B into each vertex of C. A subdigraph of a digraph D which has this form is called a bisimplex in D. A biclique in D is a bisimplex in D which is not a proper subgraph of any other and in which $B \neq \emptyset$ and $C \neq \emptyset$. The biclique digraph $\vec{C}(D)$ of D is the digraph whose vertex set is the set of all bicliques in D and in which there is an arc from $K(B_1, C_1)$ into $K(B_2, C_2)$ if and only if $C_1 \cap B_2 \neq \emptyset$. The operator which assigns $\vec{C}(D)$ to D is the biclique operator $\vec{C}$. The paper solves a problem of E. Prisner concerning the periodicity of $\vec{C}$.

Keywords: digraph, bisimplex, biclique, biclique digraph, biclique operator, periodicity of an operator

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Let φ be a graph operator, let φ^n denote the n-th iteration of φ for a positive integer n. Let G be a graph (directed or undirected) for which $\varphi^n(G) \cong G$. Then we say that G is periodic in φ with periodicity n. If $n = 1$, then G is called fixed in φ.

We shall consider directed graphs (digraphs) without loops and without arcs having the same initial vertex and the same terminal one.

Let B, C be two (not necessarily disjoint) sets of vertices. By $K(B,C)$ we denote the digraph with the vertex set $B \cup C$ in which an arc goes from each vertex of B into each vertex of C. If we consider such a digraph as a subdigraph of a digraph D, we call it a bisimplex in D. A bisimplex in D which is not a proper subdigraph of any other and in which $B \neq \emptyset$ and $C \neq \emptyset$ is called a biclique in D.

A biclique digraph $\vec{C}(D)$ of D is the digraph whose vertex set is the set of all bicliques in D and in which there is an arc from a biclique $K(B_1, C_1)$ into a biclique

$K(B_2, C_2)$ if and only if $C_1 \cap B_2 \neq \emptyset$. The operator $\vec{C}$ which assigns $\vec{C}(D)$ to D is called the biclique operator.

In [1], p. 207, E. Prisner suggests the following problem:

Are there, besides the dicycles, any other $\vec{C}$-periodic digraphs in the $\vec{C}$-semibasin of finite strongly connected digraphs?

We shall not reproduce the definition of a semibasin from [1]; it suffices to say that in this problem we might say "in the class of finite strongly connected digraphs".

Before solving this problem we do a consideration concerning bicliques with $B \cap C \neq \emptyset$. In the definition of $K(B, C)$ it was noted that B, C are not necessarily disjoint. Thus consider $B = \{x, z\}$, $C = \{y, z\}$. We consider no loops, therefore $K(B, C)$ has three arcs xy, xz, zy.

The solution of the problem is the following theorem.

Theorem. *There exists a finite strongly connected digraph D which is not a directed cycle and which is fixed in the biclique operator $\vec{C}$.*

Proof. The vertex set of D is $V(D) = \{u, v, w, u', v', w'\}$ and the arc set is $A(D) = \{uv, vw, wu, u'v', v'w', w'u', uu', vv', ww', u'v, v'w, w'u\}$ (Fig. 1). This digraph is evidently finite and strongly connected and is not a directed cycle (dicycle).

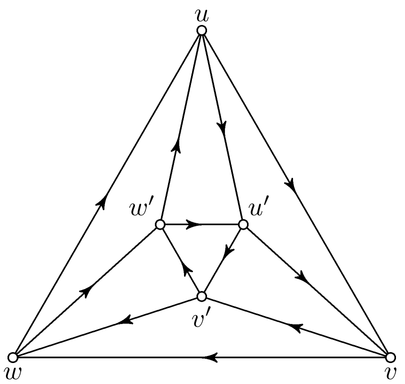

Fig. 1

Put $B_1 = C_3' = \{u, u'\}$, $B_2 = C_1' = \{v, v'\}$, $B_3 = C_2' = \{w, w'\}$, $C_1 = B_1' = \{u', v\}$, $C_2 = B_2' = \{v', w\}$, $C_3 = B_3' = \{w', u\}$. The digraph D has exactly six bicliques, namely $C_i = K(B_i, C_i)$ and $C_i' = K(B_i', C_i')$ for $i \in \{1, 2, 3\}$. The reader may verify himself that there exists a homomorphic mapping $\varphi\colon V(D) \to V(C(D))$ such that $\varphi(u) = C_1$, $\varphi(v) = C_2$, $\varphi(w) = C_3$, $\varphi(u') = C_1'$, $\varphi(v') = C_2'$, $\varphi(w') = C_3'$. $\square$

Note that the digraph D is obtained from the graph of the regular octahedron by directing its edges in such a way that the indegrees and the outdegrees of all vertices become equal to 2.

References

[1] *E. Prisner*: Graph Dynamics. Longman House, Burnt Mill, Harlow, 1995.

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