ON SUBALGEBRA LATTICES OF A FINITE UNARY ALGEBRA II

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(Received April 15, 1999)

Abstract. We use graph-algebraic results proved in [8] and some results of the graph theory to characterize all pairs $\langle\mathcal{L}_1,\mathcal{L}_2\rangle$ of lattices for which there is a finite partial unary algebra such that its weak and strong subalgebra lattices are isomorphic to $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Next, we describe other pairs of subalgebra lattices (weak and relative, etc.) of a finite unary algebra. Finally, necessary and sufficient conditions are found for quadruples $\langle\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4\rangle$ of lattices for which there is a finite unary algebra having its weak, relative, strong subalgebra and initial segment lattices isomorphic to $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_4$, respectively.

Keywords: graph, finite unary algebra, partial algebra, subalgebras, subalgebra lattices

MSC 2000: 05C20, 05C40, 05C99, 08A30, 08A55, 08A60, 05C90, 06B15, 06D05

Investigations of relationships between properties of algebras or properties of varieties of algebras and those of their subalgebra lattices are an important part of universal algebra (see e.g. [6], and also the introduction to the previous part [8]). The theory of partial algebras provides additional tools for such investigations, because at least four different structures may be considered in this case (see [2] or [4]). Note that these different partial subalgebra lattices yield more information on an algebra (also a total one) than the (total) subalgebra lattice alone. For instance, in [9] we show that for a total and locally finite unary algebra of finite type, its weak subalgebra lattice uniquely determines its subalgebra lattice. Moreover, the theory of graphs (its language and results) is very useful (see e.g. [7], [9], and also [8]) in investigations of subalgebra lattices of partial unary algebras and connections between algebras and their subalgebra lattices. For example, in [8] we characterized, in terms of isomorphisms of some directed graphs, pairs $\langle\mathcal{A},\mathcal{L}\rangle$, where $\mathcal{A}$ is a finite partial unary algebra and $\mathcal{L}$ a lattice such that the strong subalgebra lattice $\mathcal{S}_s(\mathcal{A})$ of $\mathcal{A}$ is isomorphic to $\mathcal{L}$. Recall that the usual subalgebra is called strong to stress that this notion of subalgebra is one of several considered in the theory of partial algebras. Now we give new examples of applications of graphs in such investigations. More precisely, we apply this result and other facts of [8], and several results of the graph theory as well as some of the universal algebra to describe all pairs of lattices $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ for which there is a finite partial unary algebra such that its weak and strong subalgebra lattices are isomorphic to $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We also describe other pairs of subalgebra lattices (weak and relative, relative and strong, etc.) of a finite partial unary algebra. Finally, we characterize the quadruple of subalgebra lattices of a finite partial unary algebra.

We use the notation from the previous part [8]. For any digraph (graph) $\mathcal{D}$, by $V^{\mathcal{D}}$ and $E^{\mathcal{D}}$ we denote its sets of vertices and edges, respectively. Obviously with every digraph $\mathcal{D}$ we can associate a graph $\mathcal{D}^*$ by omitting the orientation of all edges (but not edges themselves). Each partial unary algebra $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k\in K}\rangle$ can be represented (see [8]) by the digraph $\mathcal{D}(\mathcal{A})$ obtained from $\mathcal{A}$ by omitting the names of all operations. Thus we can also associate with $\mathcal{A}$ the graph $\mathcal{D}^*(\mathcal{A}) = \big(\mathcal{D}(\mathcal{A})\big)^*$. Recall also (see [2] or [4]) that a partial unary algebra $\mathcal{B} = \langle B, (k^{\mathcal{B}})_{k\in K}\rangle$ of type K is a weak subalgebra of a similar partial unary algebra $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k\in K}\rangle$ iff $B \subseteq A$ and $k^{\mathcal{B}} \subseteq k^{\mathcal{A}}$ for $k \in K$. The set of all weak subalgebras of $\mathcal{A}$ forms (see also [2] or [4]) a complete and algebraic lattice $\mathcal{S}_w(\mathcal{A})$ under the (weak subalgebra) inclusion $\leqslant_w$.

For a complete lattice $\mathcal{L} = \langle L, \leqslant_{\mathcal{L}}\rangle$, $\mathrm{At}(\mathcal{L})$ is the set of its atoms; $\mathrm{Ir}(\mathcal{L})$ is the set of its non-zero and non-atomic join-irreducible elements; and for each $l \in L$, $\mathrm{At}(l) = \{a \in \mathrm{At}(\mathcal{L})\colon a \leqslant_{\mathcal{L}} l\}$. Recall that $i \in L$ is *join-irreducible* iff for any $k_1, k_2 \in L$, $i = k_1 \vee k_2$ implies $i = k_1$ or $i = k_2$.

By [1] we obtain

Theorem 1. *A lattice $\mathcal{L} = \langle L, \leqslant_{\mathcal{L}}\rangle$ is isomorphic to the weak subalgebra lattice $\mathcal{S}_w(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff $\mathcal{L}$ is finite and distributive and*

(1) *every element of $\mathcal{L}$ is the join of join-irreducible elements,*
(2) *for each $i \in \mathrm{Ir}(\mathcal{L})$, $\mathrm{At}(i)$ has exactly one or exactly two elements,*
(3) *$\langle \mathrm{Ir}(\mathcal{L}), \leqslant_L\rangle$ is an antichain.*

More precisely, [1] shows that $\mathcal{L} \simeq \mathcal{S}_w(\mathcal{B})$ for some partial unary algebra $\mathcal{B}$ iff $\mathcal{L}$ is algebraic, distributive and satisfies (1)–(3). Moreover, such an isomorphism forces existence of a bijection between $\mathrm{At}(\mathcal{L})$ and the carrier of $\mathcal{B}$, because it is known by [1] that one-element subalgebras of $\mathcal{B}$ with empty operations form the set of all atoms of $\mathcal{S}_w(\mathcal{B})$. Hence, $\mathcal{L}$ is finite iff $\mathcal{B}$ is finite.

The above theorem implies (see [7]) that each algebraic and distributive lattice $\mathcal{L}$ satisfying (1)–(3) can be represented by a graph $\mathcal{G}(\mathcal{L})$ defined as follows: $\mathrm{At}(\mathcal{L})$ is its set of vertices, $\mathrm{Ir}(\mathcal{L})$ is its set of (undirected) edges, and for every edge e, $\mathrm{At}(e)$ is the set of endpoints of e.

Having this correspondence between lattices (algebraic, distributive and satisfying (1)–(3) of Theorem 1) and graphs, and also the representations of partial unary algebras by digraphs and graphs, we have proved in [7] the following characterization of partial unary algebras (also infinite ones) with a given lattice of weak subalgebras:

Theorem 2. *Let $\mathcal{L}$ be an algebraic and distributive lattice satisfying* (1)–(3) *of Theorem* 1 *and let $\mathcal{A}$ be a partial unary algebra. Then*

$$\mathcal{L} \simeq \mathcal{S}_w(\mathcal{A}) \text{ iff } \mathcal{G}(\mathcal{L}) \simeq \mathcal{D}^*(\mathcal{A}).$$

In the previous part [8] we recall (see [3], chapter 3.2) the operation of the *contraction* of a vertex set in a digraph or graph. Moreover, if $W_1, \ldots, W_n$ are pairwise disjoint subsets of the vertex set of a graph (digraph) $\mathcal{G}$, then in turn we can contract all these sets. It is easy to see that the order of the contraction of these sets is not important. Therefore the graph (digraph) so obtained can be denoted by $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$. If this family has one element, i.e. $n = 1$, then we write $\mathcal{G}/W_1$.

Observe that every finite digraph $\mathcal{D}$ (i.e. whose vertex and edge sets are finite) can be reduced to a simple digraph, which will be sometimes denoted by $\textsc{sim}(\mathcal{D})$, in the following way: First, we remove all loops. Secondly, for any two distinct vertices v, w, if there are at least two edges going from v to w, then we replace all edges from v to w by a single edge with the same endpoints.

Recall (see [3]) that *a strongly connected component* of $\mathcal{D}$ is a maximal strongly connected subdigraph. A digraph is *strongly connected* iff for any two distinct vertices v and w, there is a path going from v to w. An edge e is an *isthmus* (see also [3]) iff e is regular (i.e. not a loop) and e is the only path from its initial vertex to its final vertex. We assume that no path encounters the same vertex twice.

Now let $\mathcal{D}$ be a finite digraph. Then we define the digraph $\textsc{Tis}(\mathcal{D})$ as follows: First, we take the digraph $\mathcal{T}(\mathcal{D}) = \mathcal{D}/\{V_i\}_{i=1}^{i=n}$, where $V_1, \ldots, V_n$ are the vertex sets of all non-trivial, i.e. with at least two vertices, strongly connected components of $\mathcal{D}$. Secondly, we can reduce $\mathcal{T}(\mathcal{D})$ to the simple digraph $\textsc{sim}(\mathcal{T}(\mathcal{D}))$. Thirdly, we take the subdigraph of the last digraph consisting of all its vertices and all its isthmi. Note that $\textsc{Tis}(\mathcal{D})$ is a simple digraph without directed cycles and each of its edges is an isthmus. Moreover, for every finite partial unary algebra $\mathcal{A}$, let $\textsc{Tis}(\mathcal{A}) := \textsc{Tis}(\mathcal{D}(\mathcal{A}))$.

Recall (see e.g. [8]) that a lattice $\mathcal{L}$ is isomorphic to the strong subalgebra lattice for some finite partial unary algebra iff $\mathcal{L}$ is finite and distributive. Moreover, we showed in [8] that each finite and distributive lattice $\mathcal{L}$ can be also represented by a digraph $\mathcal{D}(\mathcal{L})$ defined as follows: $\mathrm{Ir}(\mathcal{L}) \cup \mathrm{At}(\mathcal{L})$ is its set of vertices, $\{\langle p, q\rangle :\ p, q \in \mathrm{Ir}(\mathcal{L}) \cup \mathrm{At}(\mathcal{L}), q \prec p\}$ is its set of (directed) edges, where $\prec$ is the covering relation on

$\mathrm{Ir}(\mathcal{L}) \cup \mathrm{At}(\mathcal{L})$, i.e. $q \prec p$ iff $q \leqslant_L p$ and there is no join-irreducible element between q and p. Note that $\mathcal{D}(\mathcal{L})$ is a simple digraph without directed cycles and each of its edges is an isthmus.

Using these two constructions we have proved in [8]

Theorem 3. *Let $\mathcal{L}$ be a finite and distributive lattice and let $\mathcal{A}$ be a finite partial unary algebra. Then $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L}$ iff $\mathrm{T{\scriptsize IS}}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{L})$.*

Observe that Theorems 2 and 4 reduce our algebraic problem to the following one: Let $\mathcal{G}$ be a finite graph and let $\mathcal{D}$ be a finite and simple digraph without directed cycles such that each of its edges is an isthmus. When is there a finite digraph $\mathcal{H}$ such that $\mathcal{H}^* \simeq \mathcal{G}$ and $\mathrm{T{\scriptsize IS}}(\mathcal{H}) \simeq \mathcal{D}$? In our solution of this graph problem we need the following technical fact:

Lemma 4. *Let a finite graph $\mathcal{G}$, a family of digraphs $\{\mathcal{K}_i\}_{i=1}^{i=n}$ and a digraph $\mathcal{H}$ satisfy the conditions*

(1) *$\mathcal{K}_1^*, \ldots, \mathcal{K}_n^*$ are pairwise disjoint subgraphs of $\mathcal{G}$,*

(2) *$\mathcal{H}^* \simeq \mathcal{G}/\{V^{\mathcal{K}_i}\}_{i=1}^{i=n}$.*

Then $\mathcal{G}$ can be directed to form a digraph $\mathcal{D}$, i.e. $\mathcal{D}^* = \mathcal{G}$, such that

(a) *$\mathcal{K}_1, \ldots, \mathcal{K}_n$ are subdigraphs of $\mathcal{D}$,*

(b) *$\mathcal{D}/\{V^{\mathcal{K}_i}\}_{i=1}^{i=n} \simeq \mathcal{H}$.*

P r o o f. First, all edges in $\bigcup_{i=1}^{i=n} E^{\mathcal{K}_i}$ are already directed, and each of these turns into a loop in $\mathcal{H}$.

Secondly, if e is an edge of $\mathcal{G}$ with endpoints in $\mathcal{K}_i$ for some $1 \leqslant i \leqslant n$, then e can be arbitrarily directed, because in this case the image of e in $\mathcal{H}$ is a loop.

Thirdly, let e be an edge of $\mathcal{G}$ such that its endpoints belong to two distinct graphs $\mathcal{K}_i$ and $\mathcal{K}_j$, i.e. $i \neq j$. Then it is sufficient to import the orientation of e from the digraph $\mathcal{H}$. More precisely, let $\overline{v_i}$ and $\overline{v_j}$ be vertices of $\mathcal{H}$ corresponding to the sets $V^{\mathcal{K}_i}$ and $V^{\mathcal{K}_j}$, respectively. Moreover, assume that $\overline{v_i}$ is the initial vertex of e in $\mathcal{H}$. Of course, the inverse case is analogous. Then we direct e so that the terminal vertex of e (in $\mathcal{G}$) belonging to $V^{\mathcal{K}_i}$ becomes the initial vertex, and the other endpoint (which belongs to $V^{\mathcal{K}_j}$) becomes the final vertex.

It is easy to verify that in this way $\mathcal{G}$ is directed to form a digraph $\mathcal{D}$ satisfying our conditions. □

Recall that H. E. Robbins proved in [10] the following simple characterization of finite graphs that can be directed to form strongly connected digraphs (this result and its proof is given also in [3], chapter 9.3, Theorem 10, p. 182):

Theorem 5. *A finite graph $\mathcal{G}$ can be directed to form a strongly connected digraph iff $\mathcal{G}$ is connected and each edge lies on an undirected cycle.*

Obviously any loop of a graph (digraph) forms a undirected (directed) cycle, called trivial.

We also need the following technical fact from the previous part [8]:

Lemma 6. *Let $\mathcal{D}$ be a finite and simple digraph without directed cycles. Then for each edge e, there is a path $(f_1, \dots, f_n)$ going from the initial vertex of e to the final vertex of e, and $f_1, \dots, f_n$ are isthmi.*

Now we can formulate and prove our main graph result. For any graph (digraph) $\mathcal{G}$ and its subset $W \subseteq V^{\mathcal{G}}$, we denote by $[W]_{\mathcal{G}}$ the subgraph spanned on W, i.e. W is its set of vertices, and all the edges of $\mathcal{G}$ with endpoints in W form its set of edges.

Theorem 7. *Let $\mathcal{G}$ be a finite graph and let $\mathcal{H}$ be a finite and simple digraph without directed cycles such that each of its edges is an isthmus. Then the following conditions are equivalent:*

(a) *There is a finite digraph $\mathcal{D}$ such that $\mathcal{D}^* \simeq \mathcal{G}$ and $\mathrm{T{\scriptsize IS}}(\mathcal{D}) \simeq \mathcal{H}$.*

(b) *There is a family $\{W_i\}_{i=1}^{i=n}$ of pairwise disjoint subsets of $V^{\mathcal{G}}$ such that each of them has at least two vertices and*

(b.1) *$[W_i]_{\mathcal{G}}$ is connected and each of its edges lies on an undirected cycle (of this subgraph), for every $i = 1, \dots, n$.*

(b.2) *$\mathcal{H}^*$ is a subgraph (up to isomorphism) of $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ containing all its vertices and for each regular edge e of $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$, there is a directed path in $\mathcal{H}$ going from one endpoint of e to the other endpoint of e.*

Proof. (a) $\Rightarrow$ (b): Take a digraph $\mathcal{D}$ such that $\mathcal{D}^* \simeq \mathcal{G}$ and $\mathrm{T{\scriptsize IS}}(\mathcal{D}) \simeq \mathcal{H}$, and let $\mathcal{K}_1, \dots, \mathcal{K}_n$ be all the non-trivial strongly connected components of $\mathcal{D}$ and $W_1, \dots, W_n$ their sets of vertices, respectively. Then first,

$$\mathcal{K}_i^* = \big([W]_{\mathcal{D}}\big)^* = [W]_{\mathcal{D}^*} = [W]_{\mathcal{G}} \quad \text{for } i = 1, \dots, n.$$

Hence and by Theorem 6, $[W_1]_{\mathcal{G}}, \dots, [W_n]_{\mathcal{G}}$ satisfy (b.1).

Secondly, observe that

$$\big(\mathcal{D}/\{W_i\}_{i=1}^{i=n}\big)^* \simeq \mathcal{D}^*/\{W_i\}_{i=1}^{i=n} \simeq \mathcal{G}/\{W_i\}_{i=1}^{i=n}.$$

Thirdly, by the definition of $\mathrm{T{\scriptsize IS}}(\mathcal{D})$ and Lemma 6 we easily obtain that $\mathrm{T{\scriptsize IS}}(\mathcal{D})$ (and thus also $\mathcal{H}$) can be regarded (up to isomorphism) as a subdigraph of $\mathcal{D}/\{W_i\}_{i=1}^{i=n}$ containing all its vertices and for each of its regular edges e there is a directed path in $\mathcal{H}$ going from the initial vertex to the final vertex of e.

(b) $\Rightarrow$ (a): At the beginning we show that $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ can be directed to form a digraph $\mathcal{K}$ such that it has no non-trivial directed cycles and $\mathcal{H}$ is the subdigraph (up to isomorphism) of $\textsc{sim}(\mathcal{K})$ consisting of all its vertices and all its isthmi.

To this purpose observe first that all loops in $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ can be, of course, arbitrarily directed.

Secondly, all edges of $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ which belong to $\mathcal{H}$ (more formally, all edges corresponding to edges of $\mathcal{H}$ by the isomorphism given in (b.2)) can be directed as in $\mathcal{H}$.

Thirdly, all regular edges of $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ outside $\mathcal{H}$ can be directed according to the orientation of $\mathcal{H}$. More precisely, let e be a regular edge of $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ which does not belong to $\mathcal{H}$. Then there is a directed path in $\mathcal{H}$ going from one endpoint v of e to the other endpoint w of e. Thus we can direct e so that v becomes its initial vertex (in $\mathcal{K}$) and w becomes its final vertex. Since $\mathcal{H}$ has no directed cycles, we conclude that every directed path in $\mathcal{H}$ connecting the endpoints of e must go from v to w. Hence, this orientation of e is well-defined.

Obviously in this way we direct all edges of $\mathcal{G}/\{W_i\}_{i=1}^{i=n}$ to form a digraph $\mathcal{K}$ such that $\mathcal{H}$ is (up to the isomorphism given in (b.2)) its subdigraph. Moreover, the above construction easily implies that $\mathcal{K}$ has no non-trivial directed cycles.

Now we show the second property. Since $\mathcal{H}$ is a simple subdigraph of $\mathcal{K}$ containing all its vertices such that for each regular edge e of $\mathcal{K}$, there is a directed path in $\mathcal{H}$ going from the initial vertex of e to the final vertex of e, we obtain that $\mathcal{H}$ is a subdigraph (up to isomorphism) of $\textsc{sim}(\mathcal{K})$ with the same properties. Hence, all the isthmi of $\textsc{sim}(\mathcal{K})$ are contained in $\mathcal{H}$. Moreover, each isthmus of $\mathcal{H}$ is also an isthmus in $\textsc{sim}(\mathcal{K})$. To see this take an isthmus e of $\mathcal{H}$ and assume that $p = (f_1, \ldots, f_k)$ is a directed path in $\textsc{sim}(\mathcal{K})$ going from the initial vertex v of e to the final vertex w of e. Then there are directed paths $p_1, \ldots, p_k$ in $\mathcal{H}$ such that p_i goes from the initial vertex of f_i to the final vertex of f_i for $i = 1, \ldots, k$. Since $\mathcal{H}$ has no directed cycles, these paths form another directed path $\overline{p}$ in $\mathcal{H}$ going from v to w. But e is an isthmus in $\mathcal{H}$, so $\overline{p}$ is equal to e. Thus $k = 1$ and $p_1 = (e)$. Hence $p = (e)$, because $\textsc{sim}(\mathcal{K})$ is simple. This implies that e is indeed an isthmus in $\textsc{sim}(\mathcal{K})$. Thus $\mathcal{H}$ consists of all vertices of $\textsc{sim}(\mathcal{K})$ and all its isthmi, because each edge of $\mathcal{H}$ is an isthmus.

Now by Theorem 5 and (b.1) there are strongly connected digraphs $\mathcal{K}_1, \ldots, \mathcal{K}_n$ such that

$$\mathcal{K}_i^* = [W_i]_{\mathcal{G}} \quad \text{for} \quad i = 1, \ldots, n.$$

Thus by Lemma 4 there is a digraph $\mathcal{D}$ such that $\mathcal{D}^* = \mathcal{G}$ and $\mathcal{D}/\{W_i\}_{i=1}^{i=n} \simeq \mathcal{K}$ and $\mathcal{K}_1, \ldots \mathcal{K}_n$ are subdigraphs of $\mathcal{D}$. Then

$$\big([W_i]_{\mathcal{D}}\big)^* = [W_i]_{\mathcal{D}^*} = [W_i]_{\mathcal{G}} = \mathcal{K}_i^* \quad \text{for} \quad i = 1, \ldots, n,$$

so $[W_i]_{\mathcal{D}}$ and $\mathcal{K}_i$ have the same vertex and edge sets. Hence, $\mathcal{K}_i = [W_i]_{\mathcal{D}}$ for $i = 1, \dots, n$, because they are subdigraphs of $\mathcal{D}$. Thus $[W_1]_{\mathcal{D}}, \dots, [W_n]_{\mathcal{D}}$ are strongly connected subdigraphs of $\mathcal{D}$ and each of them has at least two vertices. Moreover, $\mathcal{D}/\{W_i\}_{i=1}^{i=n}$ has no non-trivial directed cycles. These facts imply that $[W_1]_{\mathcal{D}}, \dots, [W_n]_{\mathcal{D}}$ are all the non-trivial strongly connected components of $\mathcal{D}$. Thus $\mathcal{T}(\mathcal{D}) \simeq \mathcal{K}$. Hence we deduce $\mathrm{T{\small IS}}(\mathcal{D}) \simeq \mathcal{H}$, because $\mathcal{H}$ can be regarded as the subdigraph of $\mathrm{{\small SIM}}(\mathcal{K})$ consisting of all its vertices and all its isthmi. This completes the proof. $\square$

Using the above graph result we obtain the following characterization of the weak and strong subalgebra lattices of one finite partial unary algebra:

Theorem 8. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be arbitrary lattices. Then the following conditions are equivalent:*

(a) *There is a finite partial unary algebra $\mathcal{A}$ such that $\mathcal{S}_w(\mathcal{A}) \simeq \mathcal{L}_1$ and $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L}_2$.*

(b) *$\mathcal{L}_1$ and $\mathcal{L}_2$ are finite and distributive lattices and*

(b.1) *$\mathcal{L}_1$ satisfies* (1)–(3) *of Theorem* 1,

(b.2) *$\mathcal{G}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ satisfy* (b) *of Theorem* 7.

P r o o f. (a) $\Rightarrow$ (b): First, we know that $\mathcal{L}_1$ and $\mathcal{L}_2$ are finite and distributive and $\mathcal{L}_1$ satisfies (1)–(3) of Theorem 1. Secondly, $\mathcal{D}^*(\mathcal{A}) \simeq \mathcal{G}(\mathcal{L}_1)$ and $\mathrm{T{\small IS}}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{L}_2)$, by Theorems 2 and 3. Thirdly, $\mathcal{D}(\mathcal{L}_2)$ is a simple digraph without directed cycles and each of its edges is an isthmus. Thus by Theorem 7 we obtain our implication.

(b) $\Rightarrow$ (a): Since $\mathcal{G}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_2)$ satisfy (b) of Theorem 7 and, of course, the other assumptions of this theorem as well, there is a finite digraph $\mathcal{D}$ such that $\mathcal{D}^* \simeq \mathcal{G}(\mathcal{L}_1)$ and $\mathrm{T{\small IS}}(\mathcal{D}) \simeq \mathcal{D}(\mathcal{L}_2)$. Now it is sufficient to construct (details of this simple construction are given in [7]) a finite partial unary algebra $\mathcal{A}$ such that $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}$. Then by Theorems 2 and 3 (see also Theorem 1) we obtain that $\mathcal{S}_w(\mathcal{A})$ and $\mathcal{S}_s(\mathcal{A})$ are isomorphic to $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. $\square$

Finally, we consider the remaining two kinds of partial subalgebras which also play an important role in the theory of partial algebras (see [2] and [4]): relative subalgebras and initial segments. Recall that a partial unary algebra $\mathcal{B} = \langle B, (k^{\mathcal{B}})_{k \in K} \rangle$ of type K is a relative subalgebra (an initial segment) of a partial unary algebra $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k \in K} \rangle$ of the same type iff $B \subseteq A$ and $k^{\mathcal{B}} = k^{\mathcal{A}}|_{(B \times B)}$ ($k^{\mathcal{B}} = k^{\mathcal{A}}|_{(A \times B)}$) for $k \in K$. The sets of all relative subalgebras and initial segments of $\mathcal{A}$ form complete, and also algebraic, lattices $\mathcal{S}_r(\mathcal{A})$ and $\mathcal{S}_d(\mathcal{A})$, respectively.

First, it is known (see e.g. [2]) that the relative subalgebra lattice of any partial algebra $\mathcal{A}$ (not only a unary one) is isomorphic to the powerset lattice of the carrier of $\mathcal{A}$. Next (see [5]), a lattice $\mathcal{L}$ is isomorphic to the lattice of all subsets of a finite set iff $\mathcal{L}$ is a finite Boolean algebra, and then $\mathcal{L}$ is isomorphic to the powerset lattice of $\mathrm{At}(\mathcal{L})$. Since each set can be viewed as a partial algebra with empty operations, these two facts imply

R e m a r k 1.

(a) A lattice $\mathcal{L}$ is isomorphic to the relative subalgebra lattice for some finite partial algebra iff $\mathcal{L}$ is a finite Boolean algebra.

(b) Let $\mathcal{A}$ be a finite partial algebra and $\mathcal{L}$ a finite Boolean algebra. Then $\mathcal{S}_r(\mathcal{A}) \simeq \mathcal{L}$ iff $|A| = |\mathrm{At}(\mathcal{L})|$.

Because for any partial algebra $\mathcal{A}$, its one-element subalgebras with empty operations form the set of all atoms of its lattice of weak subalgebras (see [1]), we have $|A| = \left|\mathrm{At}\big(\mathcal{S}_w(\mathcal{A})\big)\right|$. Hence and by Theorem 1 and Remark 1 we obtain the following characterization of weak and relative subalgebra lattices:

R e m a r k 2. Lattices $\mathcal{K}$ and $\mathcal{L}$ are isomorphic respectively to $\mathcal{S}_w(\mathcal{A})$ and $\mathcal{S}_r(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff $\mathcal{K}$ is a finite and distributive lattice satisfying (1)–(3) of Theorem 1, $\mathcal{L}$ is a finite Boolean algebra, and $|\mathrm{At}(\mathcal{K})| = |\mathrm{At}(\mathcal{L})|$.

Now we describe the relative and strong subalgebra lattices.

Proposition 9. *Lattices $\mathcal{K}$ and $\mathcal{L}$ are isomorphic respectively to $\mathcal{S}_r(\mathcal{A})$ and $\mathcal{S}_s(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff $\mathcal{K}$ is a finite Boolean algebra, $\mathcal{L}$ is finite and distributive, and $|\mathrm{At}(\mathcal{K})| \geqslant |\mathrm{Ir}(\mathcal{L})|$.*

P r o o f. The implication $\Rightarrow$ follows from Theorem 3 and Remark 1, because for any finite digraph $\mathcal{D}$ (and thus also for finite partial unary algebras), $\mathrm{T\small{IS}}(\mathcal{D})$ has not more vertices than $\mathcal{D}$, i.e. $|V^{\mathrm{T\small{IS}}(\mathcal{D})}| \leqslant |V^{\mathcal{D}}|$.

$\Leftarrow$: Take a vertex v_0 of $\mathcal{D}(\mathcal{L})$ and let $v_1, \dots, v_k$ be pairwise different elements which do not belong to $\mathcal{D}(\mathcal{L})$ and such that $\left|\{v_1, \dots, v_k\} \cup V^{\mathcal{D}(\mathcal{L})}\right| = |\mathrm{At}(\mathcal{K})|$. Let $\mathcal{H}$ be a digraph obtained from $\mathcal{D}(\mathcal{L})$ by adding all vertices $v_1, \dots, v_k$ and directed edges $\langle v_i, v_j\rangle$ for $i, j = 0, \dots, n$ with $i \neq j$. Then, of course, $\mathrm{T\small{IS}}(\mathcal{H}) \simeq \mathcal{D}(\mathcal{L})$ and $|V^{\mathcal{H}}| = |\mathrm{At}(\mathcal{K})|$. Thus we must only take a finite partial unary algebra $\mathcal{A}$ such that $\mathcal{D}(\mathcal{A}) \simeq \mathcal{H}$. $\square$

By Theorem 8 and Remark 1, because $|A| = \left|\mathrm{At}\big(\mathcal{S}_w(\mathcal{A})\big)\right|$, we obtain

R e m a r k 3. Lattices $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ are isomorphic respectively to $\mathcal{S}_w(\mathcal{A})$, $\mathcal{S}_r(\mathcal{A})$, $\mathcal{S}_s(\mathcal{A})$ of a finite partial unary algebra $\mathcal{A}$ iff they are finite and distributive with $\mathcal{L}_1$ satisfying (1)–(3) of Theorem 1, $\mathcal{L}_2$ being a Boolean algebra, $|\mathrm{At}(\mathcal{L}_1)| = |\mathrm{At}(\mathcal{L}_2)|$, and $\mathcal{G}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_3)$ satisfying (b) of Theorem 7.

Now we characterize the initial segment lattice. Take a finite partial unary algebra $\mathcal{A}$ and let $\overline{\mathcal{D}(\mathcal{A})}$ be the digraph obtained from $\mathcal{D}(\mathcal{A})$ by inverting the orientation of all edges, and let $\overline{\mathcal{A}}$ be any finite partial algebra corresponding to this digraph, i.e. $\mathcal{D}(\overline{\mathcal{A}}) \simeq \overline{\mathcal{D}(\mathcal{A})}$. Obviously we can assume that $\mathcal{A}$ and $\overline{\mathcal{A}}$ have the same carrier. It is not difficult to verify that each initial segment of $\mathcal{A}$ is a strong subalgebra of $\overline{\mathcal{A}}$, and conversely, each strong subalgebra of $\overline{\mathcal{A}}$ is an initial segment of $\mathcal{A}$. Thus the identity function on the carrier of $\mathcal{A}$ induces an isomorphism of the lattices $\mathcal{S}_d(\mathcal{A})$ and $\mathcal{S}_s(\overline{\mathcal{A}})$. This fact implies

Proposition 10. *A lattice $\mathcal{L}$ is isomorphic to the initial segment lattice $\mathcal{S}_d(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff $\mathcal{L}$ is finite and distributive.*

Observe also that $\mathrm{TIS}(\overline{\mathcal{A}}) \simeq \overline{\mathrm{TIS}(\mathcal{A})}$, because for any digraph $\mathcal{D}$, each strongly connected component of $\mathcal{D}$ is also a strongly connected component of $\overline{\mathcal{D}}$, and conversely. Moreover, an edge of $\mathcal{D}$ is an isthmus iff it is an isthmus in $\overline{\mathcal{D}}$. Thus Theorem 3 yields

Corollary 11. *Let $\mathcal{A}$ be a finite partial unary algebra and $\mathcal{L}$ a finite and distributive lattice. Then $\mathcal{S}_d(\mathcal{A}) \simeq \mathcal{L}$ iff $\overline{\mathrm{TIS}(\mathcal{A})} \simeq \mathcal{D}(\mathcal{L})$ iff $\mathrm{TIS}(\mathcal{A}) \simeq \overline{\mathcal{D}(\mathcal{L})}$.*

This corollary and Theorem 3 imply

Remark 4. Lattices $\mathcal{K}$ and $\mathcal{L}$ are isomorphic to $\mathcal{S}_s(\mathcal{A})$ and $\mathcal{S}_d(\mathcal{A})$ of one finite partial unary algebra $\mathcal{A}$ iff $\mathcal{K}$ and $\mathcal{L}$ are finite and distributive, and $\mathcal{D}(\mathcal{K}) \simeq \overline{\mathcal{D}(\mathcal{L})}$.

Proposition 12. *Lattices $\mathcal{K}$ and $\mathcal{L}$ are isomorphic respectively to the weak subalgebra and the initial segment lattices $\mathcal{S}_w(\mathcal{A})$ and $\mathcal{S}_d(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff $\mathcal{K}$ and $\mathcal{L}$ are finite and distributive and*
(1) *$\mathcal{K}$ satisfies* (1)–(3) *of Theorem* 1,
(2) *$\mathcal{G}(\mathcal{K})$ and $\overline{\mathcal{D}(\mathcal{L})}$ satisfy* (b) *of Theorem* 7.

Proof. Let $\mathcal{K} \simeq \mathcal{S}_w(\mathcal{A})$ and $\mathcal{L} \simeq \mathcal{S}_d(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$. Then we know that $\mathcal{K}$ and $\mathcal{L}$ must be finite, distributive and, of course, $\mathcal{K}$ must satisfy (1)–(3) of Theorem 1. Moreover, by Theorem 2 and Corollary 11 we have $\mathcal{D}^*(\mathcal{A}) \simeq \mathcal{G}(\mathcal{K})$ and $\mathrm{TIS}(\mathcal{A}) \simeq \overline{\mathcal{D}(\mathcal{L})}$. Hence and by Theorem 7 (the implication (a) $\Rightarrow$ (b)) we infer that $\mathcal{G}(\mathcal{K})$ and $\overline{\mathcal{D}(\mathcal{L})}$ satisfy (b) of Theorem 7.

Assume that $\mathcal{K}$ and $\mathcal{L}$ satisfy (1), (2). Then by Theorem 7 there is a digraph $\mathcal{D}$ such that $\mathcal{D}^* \simeq \mathcal{G}(\mathcal{K})$ and $\mathrm{TIS}(\mathcal{D}) \simeq \overline{\mathcal{D}(\mathcal{L})}$. Let $\mathcal{A}$ be a finite partial unary algebra such that $\mathcal{D}(\mathcal{A}) \simeq \mathcal{D}$. Then by Theorem 2, $\mathcal{S}_w(\mathcal{A}) \simeq \mathcal{K}$. Moreover, $\mathrm{TIS}(\mathcal{A}) = \mathrm{TIS}\big(\mathcal{D}(\mathcal{A})\big) \simeq \mathrm{TIS}(\mathcal{D}) \simeq \overline{\mathcal{D}(\mathcal{L})}$, so by Corollary 11, $\mathcal{S}_d(\mathcal{A}) \simeq \mathcal{L}$. $\square$

Observe that for any partial unary algebra $\mathcal{A}$, its inverse algebra $\overline{\mathcal{A}}$ has the same relative subalgebra lattice, so by Proposition 9 we obtain

R e m a r k 5. Lattices $\mathcal{K}$ and $\mathcal{L}$ are isomorphic respectively to $\mathcal{S}_r(\mathcal{A})$ and $\mathcal{S}_d(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff $\mathcal{K}$ is a finite Boolean algebra, $\mathcal{L}$ is finite and distributive, and $|\operatorname{At}(\mathcal{K})| \geqslant |\operatorname{Ir}(\mathcal{L})|$.

Using Theorem 3, Remark 1, Proposition 9 and Corollary 11 we can characterize the relative, strong subalgebra and the initial segment lattices. More precisely, we have

R e m a r k 6. Lattices $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are isomorphic respectively to $\mathcal{S}_r(\mathcal{A})$, $\mathcal{S}_s(\mathcal{A})$ and $\mathcal{S}_d(\mathcal{A})$ for a finite partial unary algebra $\mathcal{A}$ iff $\mathcal{L}_1$ is a finite Boolean algebra, $\mathcal{L}_2$ and $\mathcal{L}_3$ are finite and distributive, $\mathcal{D}(\mathcal{L}_2) \simeq \overline{\mathcal{D}(\mathcal{L}_3)}$ and $|\operatorname{At}(\mathcal{L}_1)| \geqslant |\operatorname{Ir}(\mathcal{L}_2)|$.

Analogously, using Theorem 8 and Corollary 11 we can characterize the remaining two triplets of subalgebra lattices of one finite partial unary algebra, i.e. the weak, strong subalgebra and the initial segment lattices; and the weak, relative subalgebra and the initial segment lattices.

Finally, we characterize the quadruple of the subalgebra lattices of one finite partial unary algebra.

Theorem 13. *Lattices $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ are isomorphic respectively to the weak, relative, strong subalgebra and the initial segment lattices $\mathcal{S}_w(\mathcal{A})$, $\mathcal{S}_r(\mathcal{A})$, $\mathcal{S}_s(\mathcal{A})$ and $\mathcal{S}_d(\mathcal{A})$ for some finite partial unary algebra $\mathcal{A}$ iff they are finite, distributive and*

(1) *$\mathcal{L}_1$ satisfies* (1)–(3) *of Theorem* 1,

(2) *$\mathcal{L}_2$ is a Boolean algebra,*

(3) $|\operatorname{At}(\mathcal{L}_1)| = |\operatorname{At}(\mathcal{L}_2)|$,

(4) $\mathcal{D}(\mathcal{L}_3) \simeq \overline{\mathcal{D}(\mathcal{L}_4)}$,

(5) *$\mathcal{G}(\mathcal{L}_1)$ and $\mathcal{D}(\mathcal{L}_3)$ satisfy* (b) *of Theorem* 8.

P r o o f. $\Rightarrow$: $(1), \ldots, (5)$ are obtained directly by Theorem 1, Remarks 1 (a), 2, 4 and Theorem 8, respectively.

$\Leftarrow$: By (1), (5) and Theorem 8, there is a finite partial unary algebra $\mathcal{A}$ such that $\mathcal{S}_w(\mathcal{A}) \simeq \mathcal{L}_1$ and $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L}_3$. Hence, in particular, $\textsc{Tis}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{L}_3) \simeq \overline{\mathcal{D}(\mathcal{L}_4)}$, by Theorem 3 and (4). Thus $\mathcal{S}_d(\mathcal{A}) \simeq \mathcal{L}_4$, by Corollary 11. Moreover, since $\mathcal{S}_w(\mathcal{A}) \simeq \mathcal{L}_1$, we have by (3) that $|A| = \big|\operatorname{At}\big(\mathcal{S}_w(\mathcal{A})\big)\big| = \big|\operatorname{At}(\mathcal{L}_1)\big| = \big|\operatorname{At}(\mathcal{L}_2)\big|$, because lattice isomorphisms preserve atoms. Hence and by (2) and Remark 1, $\mathcal{S}_r(\mathcal{A}) \simeq \mathcal{L}_2$. $\square$

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