# CHARACTERIZATION OF 2-MINIMALLY NONOUTERPLANAR JOIN GRAPHS

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*Abstract.* In this paper, we present characterizations of pairs of graphs whose join graphs are 2-minimally nonouterplanar. In addition, we present a characterization of pairs of graphs whose join graphs are 2-minimally nonouterplanar in terms of forbidden subgraphs.

*Keywords*: join graphs, minimally nonouterplanar, forbidden, homeomorphic, block, cut vertex

*MSC 2000*: 05C50, 05C99

## 1. Introduction

The investigation of various operations on graphs [3] has proved to be a useful approach to the study of the structure of graphs. In particular, the relationship between the groups of two graphs and the group of their product has been considered for several operations. The question of unique factorization has also been studied. In 1971, criteria for planarity of (1) the join of two graphs, (2) their corona, (3) their Cartesian product, (4) their lexicographic product and (5) their strong product, were derived in [2]. Criteria for outerplanarity and minimal nonouterplanarity of the join of two graphs were derived in [5].

In this paper, we present characterizations of graphs whose join graphs are 2-minimally nonouterplanar. In addition, we establish a characterization of graphs whose join graphs are 2-minimally nonouterplanar in terms of forbidden subgraphs.

We need the following definitions.

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A set $S(G)$ of vertices of a planar graph $G$ is called an inner vertex set of $G$, if $G$ can be drawn in the plane in such a way that each vertex of $S(G)$ lies only in the interior region of $G$ and $S(G)$ contains the minimum possible number of vertices of $G$. Each vertex of $S(G)$ is called an inner vertex of $G$. The number of vertices $i(G)$ of $S(G)$ is called the inner vertex number. A graph is said to be $k$-minimally nonouterplanar if $i(G) = k$. A graph is said to be minimally (1-minimally) nonouterplanar if $i(G) = 1$. A graph is said to be 2-minimally nonouterplanar if $i(G) = 2$. This concept has been introduced in [7].

The vertex-semientire and edge-semientire graphs were introduced by Kulli and Akka in [4].

The vertex-semientire graph $e_v(G)$ of a plane graph $G$ is the graph whose vertex set can be put in one-to-one correspondence with the vertices and regions of $G$ in such a way that two vertices of $e_v(G)$ are adjacent if and only if the corresponding elements (vertices and regions) of $G$ are adjacent.

The crossing number $c(G)$ of a graph $G$ is the minimum number of pairwise intersections of its edges when $G$ is drawn in the plane. Obviously $c(G) = 0$ if and only if $G$ is planar.

The complement $\overline{G}$ of a graph $G$ is the graph having the same set of vertices as $G$, in which two vertices are adjacent if and only if they are not adjacent in $G$.

We consider finite non-empty graphs. The vertex set of a graph $G$ is denoted by $V(G)$, its edge set by $E(G)$.

Let $G_i$ denote a graph with vertex set $V_i$ and edge set $E_i$. Let $|V_i| = m_i$ and $|E_i| = n_i$.

The most elementary binary operation on two graphs $G_1$ and $G_2$ is their union $G = G_1 \cup G_2$. The set of vertices of $G_1 \cup G_2$ is simply $V = V_1 \cup V_2$, while the set of edges is $E_1 \cup E_2$.

There is another operation that yields a graph with the vertex set $V_1 \cup V_2$. The join $G_1 + G_2$ of 2 disjoint graphs is obtained from their union $G_1 \cup G_2$ by adding new edges joining each vertex of $V_1$ to every vertex $V_2$. Thus $G_1 + G_2$ has $m_1 + m_2$ vertices and $n_1 + n_2 + m_1 m_2$ edges. This concept was defined by Zykov [8].

The following results are useful.

**Lemma 1** [5]. *Let $P_n$ be a path with $n \geqslant 2$ vertices. Then $i(K_2 + P_n) = n - 1$.*

**Theorem A** [5]. *The join $G + H$ of two connected graphs $G$ and $H$ has crossing number zero if and only if the pair of graphs $(G, H)$ is either $(K_1, H)$ of $(K_2, P)$ where $H$ is outerplanar and $P$ is a path.*

**Theorem B** [6]. *The join $G + H$ of two connected graphs $G$ and $H$ has crossing number 1 if and only if the pair $(G, H)$ satisfies one of the following conditions:*

1. *$G = K_1$ and $H$ is a theta-graph,*
2. *$G = K_2$ and $H$ is a path together with an end edge adjoined to some non-end vertex,*
3. *$G = K_2$ and $H$ is a triangle together with two paths $P_m$ and $P_n$ $(m, n \geqslant 1)$ adjoined at different vertices,*
4. *$G = K_2$ and $H$ is a cycle $C_p$ with $p \geqslant 4$ vertices,*
5. *Both $G$ and $H$ are $K_{1,2}$.*

**Theorem C** [5]**.** *The join $G+H$ of two connected graphs $G$ and $H$ is outerplanar if and only if $G$ is $K_1$ and $H$ is a path.*

**Theorem D** [1]**.** *Let $G$ be a tree. Then $e_v(G)$ is 2-minimally nonouterplanar if and only if $G$ satisfies 1 or 2:*

1. *$\Delta(G) = 4$, $G$ has exactly one vertex which lies on four blocks in which at least two blocks contain end vertices of $G$ and no other vertex of $G$ is of degree 3.*
2. *$\Delta(G) = 3$, and either $G$ has exactly two vertices, each lying in 3 blocks in which at least one block contains an end vertex of $G$, or $G$ has exactly one vertex lying in 3 blocks $e_i$ in which at least one block $e'_i$ adjacent to $e_i$ contains an end vertex of $G$.*

## 2. Main results

In the following theorem we establish a characterization of graphs whose join graphs are 2-minimally nonouterplanar.

**Theorem 1.** *The joint $G + H$ of two connected graphs $G$ and $H$ is 2-minimally nonouterplanar if and only if the pair of graphs $(G, H)$ fulfils one of the following conditions:*

1. *$G = K_1$ and $H$ is a path together with two end edges adjoined at one or two different non-end vertices.*
2. *$G = K_1$ and $H$ is a path $P_m(m \geqslant 5)$ together with a path $P_3$ adjoined to at least 2nd non-end vertex.*
3. *$G = K_1$ and $H$ is a path of length $\geqslant 2$ together with two vertices, each of them adjoined to a pair of adjacent vertices and $H$ has no subgraph homeomorphic to $M_2$ (Fig. 2).*
4. *$G = K_1$ and $H$ is a triangle together with two paths $P_m$, $P_n$ $(m, n \geqslant 2)$ and an end edge adjoined at three different vertices.*
5. *$G = K_1$ and $H$ is a cycle of length 4 together with two paths $P_m$ and $P_n$ $(m, n \geqslant 2)$ adjoined at two consecutive vertices.*

6. $G = K_1$ *and* $H$ *is a cycle of length* $\geqslant 5$ *together with an end edge adjoined at some vertex.*
7. $G = K_2$ *and* $H$ *is a path of length* 2.

P r o o f. Suppose $G + H$ is 2-minimally nonouterplanar. Then by Theorem C, $G = K_1$ and $\deg v \geqslant 3$ for some vertex $v$ of $H$, or $G = K_2$ and $\deg v \leqslant 2$ for every vertex $v$ of $H$. Suppose that $G = K_1$ and $H$ is a tree with $\deg v \geqslant 4$ for some vertex $v$ of $H$. We consider three cases. $\square$

C a s e 1. Assume $\deg v > 4$ for some vertex $v$ of $H$. Suppose $H$ has a vertex of degree 5. Then $K_{1,5}$ is a subgraph of $H$. The vertex $K_1$ of the graph $G$ is adjacent to every vertex of $H$ which gives $(K_2 + \overline{K}_5)$ as a subgraph in $G + H$. Since $i(K_2 + \overline{K}_5) > 2$ (by Theorem D), we have $i(G + H) > 2$, a contradiction.

C a s e 2. Suppose $\deg v = 4$ for some vertex $v$ of $H$. Assume $H$ has at least two vertices $v_1$ and $v_2$ of degree 3 and 4, respectively. Then $H$ has a subgraph homeomorphic to $K_{3,4} - K_{2,3}$ that is $G_1$ (Fig. 1). The vertex $K_1$ of the graph $G$ is adjacent to every vertex of $G_1$. One can easily see that $G + G_1$ is isomorphic to $e_v(G_1)$. By Theorem D, $i(e_v(H)) > 2$. Since $G + G_1 \subset G + H$, hence $i(G + H) > 2$, a contradiction. Thus $H$ has exactly one vertex of degree 4 and no other vertex is of degree 3.

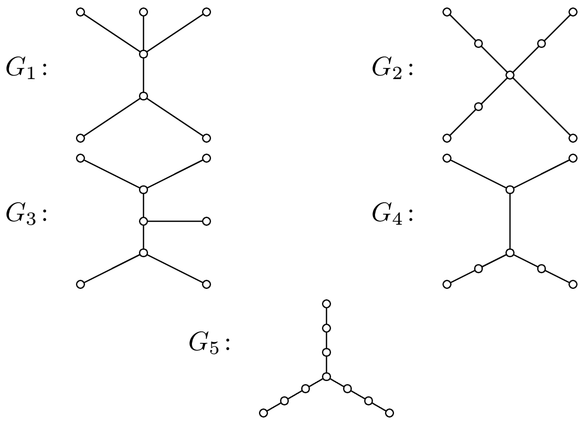


Fig. 1

Next, assume $H$ has exactly one vertex of degree 4 and does not satisfy the condition (1) of the theorem. Then $H$ has a subgraph homeomorphic to $G_2$ (Fig. 1). It is easy to verify that $G + G_2$ is isomorphic to $e_v(G_2)$. By Theorem D, $i(e_v(G_2)) > 2$. Thus $i(G_1 + H) > 2$, a contradiction.

C a s e 3. Suppose $\deg v = 3$ for some vertex $v$ of $H$. Assume $H$ has at least 3 vertices of degree 3. Then $H$ has a subgraph homeomorphic to $G_3$ (Fig. 1). Clearly

$K_1 + G_3 = e_v(G_3)$. Hence by Theorem D, $i(e_v(G_3)) > 2$. Therefore $i(K_1 + H) > 2$, a contradiction.

Suppose $H$ has exactly two vertices of degree 3 and does not satisfy the condition 1 of the theorem. Then $H$ has a subgraph homeomorphic to $G_4$ (Fig. 1). Obviously $K_1+G_4 = e_v(G_4)$. By Theorem $D$, $i(K_1+G_4)) = i(e_v(G_4)) > 2$. Since $K_1+G_4$ is a subgraph of $G+H$, hence $i(G+H) > 2$, a contradiction. This proves the condition 1.

Suppose $H$ has exactly one vertex of degree 3 and does not satisfy the condition 2 of the theorem. Then $H$ contains a subgraph homeomorphic to $G_5$ (Fig. 1). Obviously $K_1 + G_5 = e_v(G_5) \subset (G + H)$. Theorem D implies that $i(e_v(G_5)) > 2$. That is $i(G + H) > 2$, a contradiction. This proves 2.

Assume $H$ is not a tree. We consider the following five cases.

Case 1. Suppose $H$ has at least 3 cycles and each cycle is a block of $H$. Then the vertex $K_1$ of the graph $G$ is adjacent to every vertex of $H$, which gives at least 3 wheels as subgraphs of $G + H$. It is known in [7] that every wheel is minimally nonouterplanar. Thus $i(G + H) > 2$, a contradiction.

Case 2. Suppose $H$ has 3 cycles in which exactly two cycles are in a block. Then there are two subgraphs homeomorphic to $K_4 - x$ and $K_3$. In $G+H$, we have two blocks $(K_4 - x) + K_1 = K_5 - x$ and $K_3 + K_1 = K_4$. It is observed in [7] that $i(K_5 - x) \geqslant 2$ and $i(K_4) \geqslant 1$. Thus $i(G + H) \geqslant 3$, a contradiction.

Case 3. Suppose $H$ has 3 cycles in a block. Then $H$ has a subgraph homeomorphic to $M_1$ (Fig. 2), $P_4 + K_1$, $K_4$ or $K_2 + \overline{K_3}$. The vertex $K_1$ is adjacent to every vertex of $H$, which gives $M_1 + K_1$, $(P_4 + K_1) + K_1$ $(= P_4 + K_2)$, $K_4 + K_1$ $(= K_5)$ or $(K_2 + \overline{K}_3) + K_1$ $(= \overline{K_3 \cup 4K_1})$ as a subgraph of $G + H$. One can easily find out that $M_1 + K_1$ (Fig. 2) and $P_4 + K_3$ (by Lemma 1) are at least 3-minimally nonouterplanar subgraphs of $H$, and $K_5$ and $\overline{K_3 \cup 4K_1}$ are nonplanar subgraphs of $H$, a contradiction.

From cases 1, 2 and 3 we conclude that $H$ has at most two cycles.

Case 4. Suppose $H$ has exactly two cycles in a block. Then $H$ has a subgraph $\overline{M}_1$ or $M_2$ (Fig. 2). In $G + H$ the vertex $K_1$ is adjacent to every vertex of $H$, which gives $\overline{M}_1 + K_1$ $(= \overline{M_1 \cup K_1})$ or $M_2 + K_1$ as a subgraph of $G + H$. It is easy to see that either $\overline{M_1 \cup K_1}$ or $M_2 + K_1$ (Fig. 2) has at least 3 inner vertices. Hence $i(G + H) > 3$, a contradiction. This proves 3.

Case 5. Assume $H$ has exactly one cycle $C$. We consider the following three subcases.

Subcase 5.1. Assume $C$ is a cycle of length $\geqslant 3$ and at least 3 vertices of $C$ are cutvertices. Each cutvertex lies in two blocks in which no block contains an end vertex of $H$. Then there exists a subgraph homeomorphic to $M_3$ (Fig. 2). The vertex $K_1$ of $G$ is adjacent to every vertex of $M_3$. One can see that $i(G + M_3) \geqslant 3$. Since

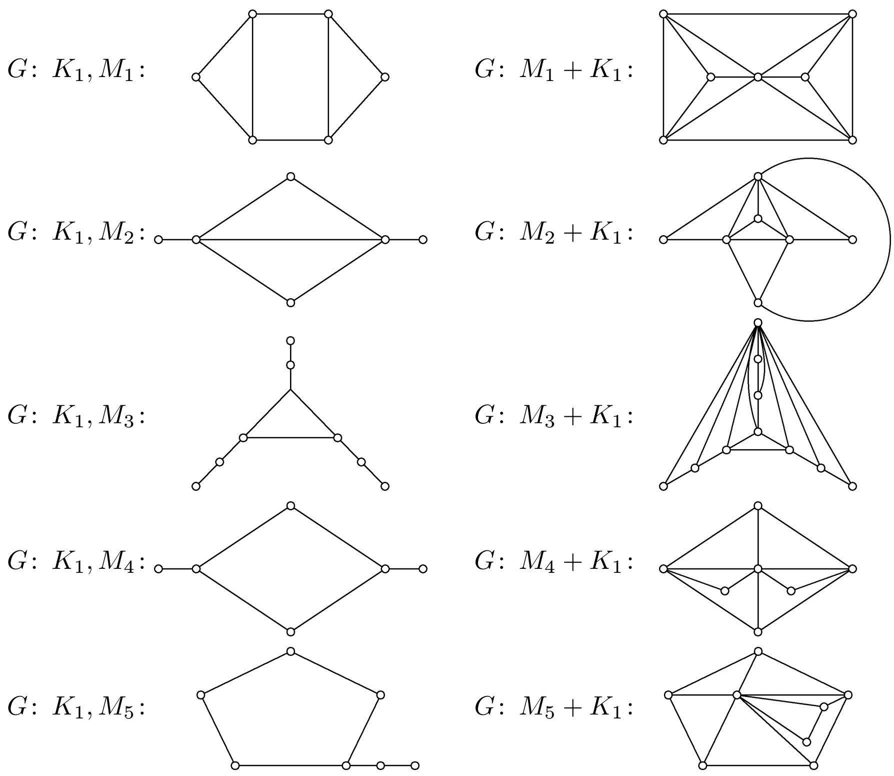


Fig. 2

$G + M_3$ is a subgraph of $G + H$, hence $G + H$ is 3-minimally nonouterplanar. This proves 4.

Subcase 5.2. Suppose $C$ is a cycle of length $\geqslant 4$ which has two alternative cutvertices lying in two blocks of $H$. Then there exists a subgraph homeomorphic to $M_4$ (Fig. 2). It is easy to see that $i(G + M_4) > 2$. Since $G + M_4 \subset G + H$, hence $G + H$ has at least 3 inner vertices, a contradiction. This proves 5.

Subcase 5.3. Suppose $H$ is a cycle of length 5 together with a path at least two, adjoined to some vertex. Then $G = K_1$ is adjacent to every vertex of $C_5 \cdot P_3 = M_5$ (Fig. 2). Obviously $i(G + H) \geqslant 3$, a contradiction. This proves 6.

Next we suppose that $G = K_2$ and $\deg v \leqslant 2$ for every vertex $v$ of $H$. For otherwise, if $\deg v \geqslant 3$, then by condition 2 of Theorem B, $G + H$ is nonplanar. Hence $\deg v \leqslant 2$. This implies that $H$ is either a path or a cycle.

Suppose $H$ is a cycle. Then by condition 4 of Theorem B, $G + H$ is nonplanar. Hence $H$ is a path.

Suppose $H$ is a path of length $\geqslant 3$. Then by Lemma 1, $i(G+H) \geqslant 3$, a contradiction. This proves 7.

Conversely, suppose a pair of graphs $(G, H)$ satisfies 1 and 2. Proof of the converse part of these two conditions is the same as that of the converse part of Theorem D. Thus in each case $i(G+H) = 2$.

Suppose a pair of graphs $(G, H)$ satisfies 3. Then $H$ is a path $\{u_1, u_2, v_1, v_2\}$ of length $\geqslant 3$ together with 2 vertices $u$ and $v$. The vertices $u, v$ are adjacent respectively to two pairs of vertices $(u_1, u_2)$ and $(v_1, v_2)$ or $(u_1, u_2)$ and $(u_2, v_1)$. Clearly $H - \{u, v\}$ is a path. Then $G + (H - \{u, v\})$ is outerplanar by Theorem C. The elements in the respective vertex sets $\{u_1, u_2, K_1\}$ and $\{v_1, v_2, K_1\}$ (or $\{u_1, u_2, K_1\}$ and $\{u_2, v_2, K_1\}$) form two triangles in $G + (H - \{u, v\})$. It is easy to see that $u$ and $v$ are the only 2 vertices in $G + H$ which are not in $G + (H - \{u, v\})$. They are adjacent respectively to elements in the vertex sets $\{u_1, u_2, K_1\}$ and $\{v_1, v_2, K_1\}$ or $\{u_1, u_2, K_1\}$ and $\{u_2, v_1, K_1\}$ in $G + H$. Then $u$ and $v$ are the only inner vertices of $G + H$. Hence $G + H$ is 2-minimally nonouterplanar.

Assume a pair $(G, H)$ satisfies 4. Then $H$ is a triangle $u_1 u_2 u_3$ with three adjacent edges $u_1 v_1$, $u_2 v_2$ and $u_3 v_3$. Clearly $H - u_i$ (say $i = 1$) is a path. Then $G + (H - u_1)$ is outerplanar by Theorem C. The vertices $u_2$, $u_3$ and $K_1$ are mutually adjacent in $G + (H - u_1)$. It is easy to see that $u_1$ and $v_1$ are the only two vertices in $G + H$ $\{$which are not in $G + (H - u_1)\}$ and $u_1$ is adjacent to $u_2$, $u_3$, $v_1$ and $K_1$ and $v_1$ is adjacent to $u_1$ and $K_1$. Then $u_1$ and $v_1$ are the only two inner vertices of $G + H$. Thus $G + H$ is 2-minimally nonouterplanar.

Assume a pair $(G, H)$ satisfies 5. Then $H$ is a cycle, say $C_4 = \{u_1, u_2, u_3, u_4\}$, together with two edges $u_1 w_1$ and $u_2 w_2$. Obviously $H - \{w_1, w_2\}$ is a cycle. Then $G + (H - \{w_1, w_2\})$ is a wheel which is minimally nonouterplanar. By joining a vertex $w_i$ with $u_i$ $(i = 1, 2)$ and $K_1$ we get $G + H$. On the planar embedding of $G + H$ we get $u_3$, $u_4$ as inner vertices.

Suppose a pair $(G, H)$ satisfies 6. Then $H$ is a cycle, say $C$ of length $\geqslant 5$, together with an end edge $uv$ adjoined at some vertex $v$ where $v \in C$. Clearly $H - u$ is a cycle. Then $G + (H - u)$ is a wheel which is minimally nonouterplanar. By joining a vertex $u$ with $v$ and $K_1$, we get $G + H$ which is clearly 2-minimally nonouterplanar.

Finally, suppose a pair $(G, H)$ satisfies 7. Then the join $G + H$ gives a graph $K_5 - x$. But it is observed in 7 that $i(K_5 - x) = 2$. Hence $i(G + H) = 2$. This completes the proof of the theorem.

We now give a characterization of connected and disconnected graphs whose join graphs are 2-minimally nonouterplanar.

**Theorem 2.** *A graph $G_1$ is connected and $G_2$ is disconnected and their join $G_1 + G_2$ is 2-minimally nonouterplanar if and only if the pair $(G_1, G_2)$ is one of the following pairs of graphs.*

1. *to* 6. *$G_1 = K_1$ and $G_2 = H$ where $H$ is $H_1'$, $H_2'$, $H_3'$, $H_4'$, $H_5'$, $H_6'$, $H_7'$ or $H_8'$ (Fig. 3), every other component of $H$ is a path.*

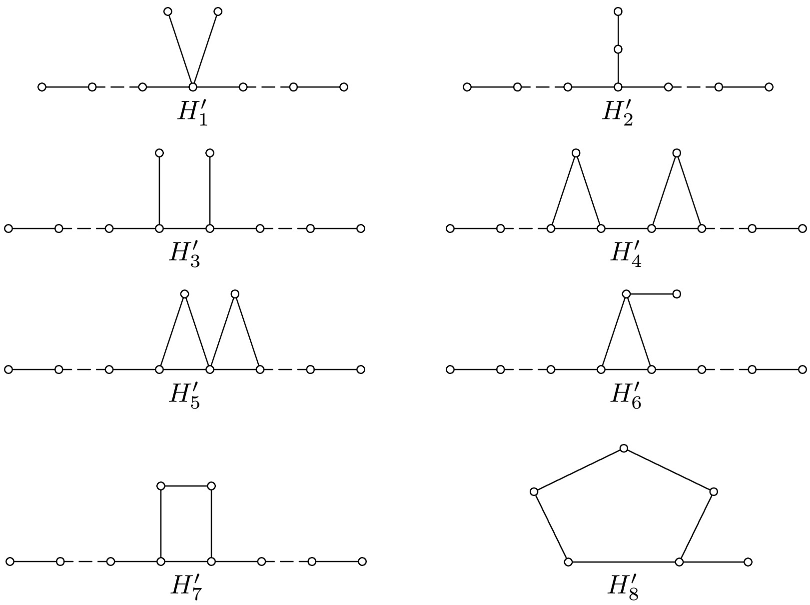


Fig. 3

7. *$G_1 = K_2$ and $G_2 = \overline{K}_4$, $\overline{K_4 - x}$, $\overline{C_4}$ or $\overline{K_3 \cdot K_2}$ ($\overline{K_4 - x}$ is the complement of $K_4 - x$ i.e. $2K_1 \cup K_2$).*
8. *$G_1 = P_4$ or $K_3$ and $G_2 = \overline{K}_2$.*

P r o o f. The proof is similar to that of Theorem 1, hence we omit it. $\square$

We also obtain a characterization of two disconnected graphs whose join graph is 2-minimally nonouterplanar.

**Theorem 3.** *The join $G + H$ of two disconnected graphs $G$ and $H$ is 2-minimally nonouterplanar if and only if the pair $(G, H)$ is either $(\overline{K}_2, \overline{K}_4)$, $(\overline{K}_2, \overline{K_4 - x})$, $(\overline{K}_2, \overline{K_3 \cdot K_2})$ or $(\overline{K}_2, \overline{C}_4)$.*

P r o o f. The proof is similar to that of Theorem 1, hence we omit it. $\square$

## 3. Forbidden subgraphs

In this section we establish a characterization of two connected graphs whose join graph is 2-minimally nonouterplanar in terms of forbidden subgraphs.

**Theorem 4.** *The join $G + H$ of two connected graphs $G$ and $H$ is* 2*-minimally nonouterplanar if and only if it has no subgraph homeomorphic to any one of the pairs of graphs in Fig.* 4.

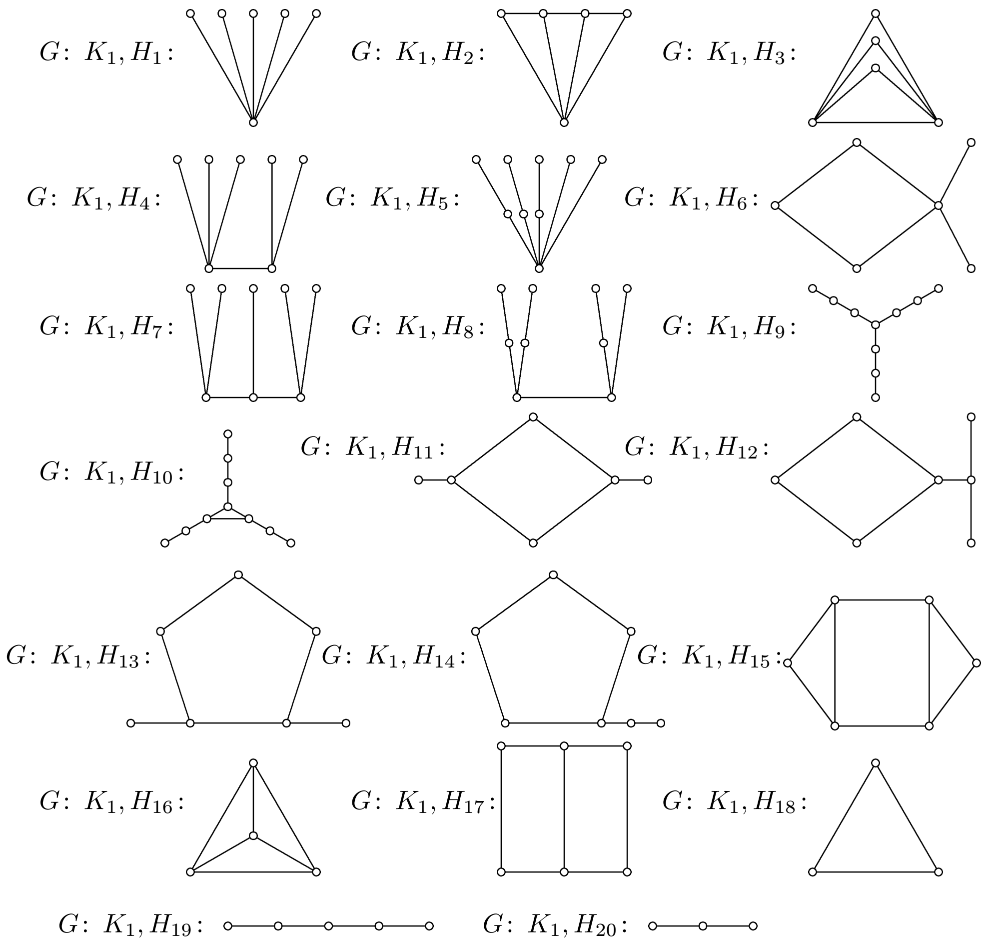


Fig. 4

P r o o f. Suppose the join $G + H$ of two connected graphs $G$ and $H$ is 2-minimally nonouterplanar. Then the pair $(G, H)$ has no subgraph homeomorphic to any one of the pairs of graphs of Fig. 4. This follows from Theorem 1, since graphs homeomorphic to a pair $(G_1, H_1)$ have $G = K_1$ and $H_1$ has a vertex of degree $> 4$, graphs homeomorphic to a pair $(G_i, H_i)$, $i = 2$ or $3$ have $G_i = K_1$ and $H_i$ have a noncutvertex of degree 4, graphs homeomorphic to a pair $(G_4, H_4)$ have $G_4 = K_1$ and $H_4$ has two cutvertices of degrees 3 and 4, graphs homeomorphic to a pair $(G_5, H_5)$ have $G_5 = K_1$ and $H_5$ has a vertex of degree 4 and it lies in 4 blocks in which at least 3 blocks contain no end vertices of $H$. Graphs homeomorphic to a pair $(G_6, H_6)$ have $G_6 = K_1$ and $H_6$ has a unique cutvertex of degree 4 and it lies in 3 blocks in which one block is a cycle of length $\geqslant 4$, graphs homeomorphic to a pair $(G_7, H_7)$ have $G_7 = K_1$ and $H_7$ has at least 3 cutvertices of degree 3, each lying in 3 blocks, graphs homeomorphic to a pair $(G_8, H_8)$ have $G_8 = K_1$ and $H_8$ has 2 cutvertices of degree 3, each lying in 3 blocks in which one cutvertex contains no end block of $H$, graphs homeomorphic to $(G_9, H_9)$ have $G_9 = K_1$ and $H_9$ has a cutvertex of degree 3, lying on 3 paths of length at least three, graphs homeomorphic to a pair $(G_{10}, H_{10})$ have $G_{10} = K_1$ and $H_{10}$ has 3 mutually adjacent cutvertices of degree 3, each lying in two blocks in which no block contains an end vertex of $H$, graphs homeomorphic to $(G_{11}, H_{11})$ have $G_{11} = K_1$ and $H_{11}$ is a cycle of length 4 which has two nonadjacent cutvertices of degree 3, graphs homeomorphic to $(G_{12}, H_{12})$ have $G_{12} = K_1$ and $H_{12}$ has two adjacent cutvertices $v_1$ and $v_2$ of degree 3 and $v_1$ lies in two blocks in which one block is a cycle of length 4 while $v_2$ lies in 3 blocks, graphs homeomorphic to a pair $(G_{13}, H_{13})$ have $G_{13} = K_1$ and $H_{13}$ has a cycle of length 5 containing two adjacent cutvertices of degree 3, graphs homeomorphic to $(G_{14}, H_{14})$ have $G_{14} = K_1$ and $H_{14}$ has a cutvertex of degree 3, it lies in two blocks of which one block is a cycle of length 5 and the other is a non-end edge, graphs homeomorphic to a pair $(G_i, H_i)$, $i = 15$ or $16$ have $G_i = K_1$ and $H_i$ has at least 3 cycles in a block, graphs homeomorphic to $(G_{17}, H_{17})$ have $G_{17} = K_1$ and $H_{17}$ has at least 2 cycles in a block and each cycle has length 4, graphs homeomorphic to $(G_i, H_i)$, $i = 18$ or $19$, have $G_i = K_2$ and $H_i$ is a cycle or a path of length at least 4, and graphs homeomorphic to $(G_{20}, H_{20})$ have $G_{20} = H_{20} = P_3$.

Conversely, suppose a pair $(G, H)$ contains no subgraph homeomorphic to any one of the pairs of graphs in Fig. 4. Then we show that a pair $(G, H)$ satisfies the conditions of Theorem 1 and hence the join $G + H$ is 2-minimally nonouterplanar. Assume $G = K_1$ and $\deg v = 5$ for every vertex $v$ of $H$. Then the pair $(G, H)$ contains a subgraph homeomorphic to $(G_1, H_1)$, a contradiction. Hence $\deg v \leqslant 4$ for every vertex $v$ of $H$.

Assume $H$ has a vertex $v$ of degree 4. We show that $v$ is a cutvertex. If not, let $v_1$, $v_2$, $v_3$ and $v_4$ be the vertices adjacent to $v$. Since $v$ is not a cutvertex, there exist paths between every pair of vertices of $v_1$, $v_2$, $v_3$ and $v_4$ not containing $v$. Then the pair $(G, H)$ has a subgraph homeomorphic to $(G_2, H_2)$ or $(G_3, H_3)$, a contradiction. Thus $v$ is a cutvertex and every vertex of degree 4 is a cutvertex of $H$.

Suppose $H$ has two cutvertices of degrees 3 and 4 respectively lying in 3 and 4 blocks. Then the pair $(G, H)$ has a subgraph homeomorphic to $(G_4, H_4)$, a contradiction. Thus $H$ has exactly one cutvertex $v$ of degree 4 which lies in four blocks and no other vertex is of degree 3.

We consider two cases, depending on the number of blocks of $H$, the cutvertex $v$ lies in. □

Case 1. Suppose $v$ lies in four blocks of $H$ in which at least 3 edge blocks $e_i = vv_i$, $i = 1, 2, 3$ have no end vertices. Let $e'_i = v_i u_i$, $i = 1, 2, 3$ be the edges adjacent to $e_i$. We consider two subcases.

Subcase 1.1. If every pair of $u_i$ are not adjacent then $(G, H)$ has a subgraph homeomorphic to $(G_5, H_5)$, a contradiction.

Subcase 1.2. If any pair of $u_i$ are adjacent, then $(G, H)$ has a subgraph homeomorphic to $(G_6, H_6)$, a contradiction.

From Subcases 1.1 and 1.2, we conclude that $H$ has exactly one cutvertex of degree 4, it lies in 4 blocks in which at most two blocks contain no end vertices of $H$. This proves 1.

Assume $H$ is a tree. Then we consider three cases.

Case 1. Suppose $H$ has three cutvertices of degree 3, each lying in three blocks. Then $(G, H)$ has a subgraph homeomorphic to $(G_7, H_7)$, a contradiction. Thus $H$ has at most two cutvertices of degree 3, each lying in three blocks.

Case 2. Assume $H$ has 2 vertices $v_1$ and $v_2$ of degree 3. Each $v_i$ lies in three blocks in which blocks of one $v_i$ contain no end vertices of $H$. Then the pair $(G, H)$ has a subgraph homeomorphic to $(G_8, H_8)$, a contradiction. This proves 1.

Case 3. Assume $H$ has one cut vertex of degree 3. If $v$ lies on three disjoint paths in which each path has length at least 3, then $(G, H)$ has a subgraph homeomorphic to $(G_9, H_9)$, a contradiction. Thus $v$ lies on 3 disjoint paths from which at least one path has length 2. This proves 2.

Suppose $H$ is not a tree and has at least three cutvertices of degree 3. We consider three cases.

Case 1. Assume $H$ has three mutually adjacent cutvertices of degree 3, each lying in two blocks in which no block contains an end vertex of $H$. Then $(G, H)$ has a subgraph homeomorphic to $(G_{10}, H_{10})$, a contradiction. This proves 4.

Case 2. Assume $H$ has two cutvertices $v_1$ and $v_2$ of degree 3. Each $v_i$ lies in two blocks of $H$. We consider three subcases.

Subcase 2.1. If $v_1$ and $v_2$ lie on a cycle of length 4 and they are not adjacent, then the pair $(G, H)$ contains a subgraph homeomorphic to $(G_{11}, H_{11})$, a contradiction.

S u b c a s e 2.2. If $v_1$ lies on a cycle of length 4 and $v_2$ lies in three blocks, then a pair $(G, H)$ has a subgraph homeomorphic to $(G_{12}, H_{12})$, a contradiction. From cases 2.1 & 2.2. We have proved the condition 5.

S u b c a s e 2.3. If $v_1$ and $v_2$ lie on a cycle of length at least 5 and they are adjacent vertices of a cycle, then $(G, H)$ has a subgraph homeomorphic to $(G_{13}, H_{13})$ a contradiction. This proves 6.

Suppose $H$ has at least 3 cycles, each cycle being a block of $H$. Then $(G, H)$ has a subgraph homeomorphic to $(G_7, H_7)$, a contradiction.

Suppose $H$ has at least 3 cycles in which two cycles are in a block. Then $(G, H)$ has a subgraph homeomorphic to $(G_4, H_4)$, a contradiction.

Suppose $H$ has at least three cycles in a block. Then $(G, H)$ has a subgraph homeomorphic to $(G_i, H_i)$, $i = 2, 3$, 15 or 16, a contradiction.

Thus $H$ has exactly 2 cycles.

If $H$ has two cycles in which at least one cycle has length greater than or equal to 4, then $G$ has a subgraph homeomorphic to $(G_6, H_6)$, $(G_{11}, H_{11}), \ldots, (G_{15}, H_{15})$ or $(G_{17}, H_{17})$, a contradiction.

Thus $H$ has exactly two cycles, each of length 3. This proves 3.

Assume $G = K_2$ and $\deg v \geqslant 3$ for every vertex $v$ of $H$ and let every vertex of degree 3 be a cutvertex. Then $H$ has a subgraph homeomorphic to $K_{1,3}$. By Theorem A, $G + H$ contains a subgraph $K_{3,3}$ which is nonplanar, a contradiction. Thus $\deg v \leqslant 2$ for every vertex $v$ of $H$. There are two cases to be considered.

C a s e 1. Suppose each vertex of degree 2 is a noncutvertex in $H$ and $H$ is a cycle of length at least 3. Clearly, $(G, H)$ has a subgraph homeomorphic to $(G_{18}, H_{18})$, a contradiction.

C a s e 2. Suppose each vertex of degree 2 is a cutvertex in $H$ and $H$ is a path of length $> 3$. Obviously $(G, H)$ has a subgraph homeomorphic to $(G_{19}, H_{19})$, a contradiction.

From cases 1 and 2, we conclude that $H$ has a path of length exactly 3.

Suppose $G = P_3$ and $H = P_n$, $n \geqslant 3$. Then $(G, H)$ contains a subgraph homeomorphic to $(G_{20}, H_{20})$, a contradiction. Thus $(G, H) = (K_2, P_3)$. This proves 7.

In view of Theorem 1, $G + H$ is 2-minimally nonouterplanar. This completes the proof of theorem.

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