MONOTONE ITERATIVE TECHNIQUE AND CONNECTEDNESS OF THE SET OF SOLUTIONS

Boris Rudolf, Bratislava

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Abstract. The paper deals with the properties of a monotone operator defined on a subset of an ordered Banach space. The structure of the set of fixed points between the minimal and maximal ones is described.

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The purpose of this paper is to describe the structure of the set of fixed points of a monotone operator defined on a subset of an ordered Banach space.

The well known result of Amman [1] guarantees the existence of the minimal and maximal fixed points between the sub- and superequilibria. We depict the situation in the case when the minimal and maximal fixed points are different.

Our result extends the results of Krasnosel'skij, Lusnikov [6] and Hess [5] to the situation when the operator is strictly order preserving.

We apply our result to the periodic boundary value problem for the second order ordinary differential equation. In this case the minimal and maximal solutions are obtained by the monotone iterative technique developed by Lakshmikantham et al. [7, 8]. In the papers [4, 9, 10] the structure of the set of solutions of certain boundary value problems is discussed and examples of problems with a convex set of solutions are given.

We suggest another possibility how to prove that the set of solutions is connected.

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Let B be an ordered Banach space with an order cone P, $U \subset B$ and let $A\colon U \to U$ be a continuous operator.

The operator A is *order preserving* if $x \leqslant y \Rightarrow A(x) \leqslant A(y)$, *strictly order preserving* if $x < y \Rightarrow A(x) < A(y)$, *strongly order preserving* if $x < y \Rightarrow A(x) \ll A(y)$. ($x \ll y$ means $y - x \in \operatorname{int} P$.)

An element $x \in U$ is called a *subequilibrium (superequilibrium)* provided $x \leqslant Ax$ ($x \geqslant Ax$). A sub- or superequilibrium is called *strict* if the strict inequality holds.

We denote $[v, w] = \{u \in B;\ v \leqslant u \leqslant w\}$.

Theorem 1. [1] *Assume that v, w, $v < w$ are a sub- and superequilibrium, respectively, of an order preserving operator A, the interval $V = [v, w] \subset U$ and $A(V)$ is a relatively compact set.*

Then $A\colon V \to V$ and there are a minimal and maximal fixed points, say ϱ, r, such that the set F of all fixed points of A in V is a subset of the interval $[\varrho, r]$.

A sequence $(x_n)_{n\in Z}$ in U such that $x_{n+1} = Ax_n$, $x_n \to x^-$ for $n \to -\infty$, $x_n \to x^+$ for $n \to \infty$, $x^-,\ x^+ \in U$ is called an *entire orbit* connecting x^- with x^+.

Theorem 2. [5, Proposition 2.1] *Let $u_1 < u_2$ be fixed points of a strictly order preserving continuous operator $A\colon U \to U$. Let $W = [u_1, u_2] \subset U$ and $A(W)$ be relatively compact.*

Then precisely one of the following three cases occurs:

(a) *there is another fixed point of A in W,*

(b) *there is an entire orbit consisting of strict subequilibria, connecting u_1 with u_2,*

(c) *there is an entire orbit consisting of strict superequilibria, connecting u_2 with u_1.*

Let us consider again the situation when $A\colon V \to V$, $V = [v, w]$, $v < w$ are sub- and superequilibria.

The fixed point $u \in V$ is called *stable* with respect to V if for each $\varepsilon > 0$ there is $\delta > 0$ such that $A^n(x) \in O(u, \varepsilon)$ for each $x \in O(u, \delta) \cap V$ and each $n \in N$.

Under the assumptions of stability of each fixed point $u \in V$ and the relative compactness of the set $A(V)$ it is proved that the set of fixed points of a strongly order preserving operator $A\colon V \to V$ is a continuous totally ordered curve [5, Theorem 3.3].

Under weaker assumptions we obtain the following result.

Theorem 3. *Let $A\colon V \to V$ be a strictly order preserving continuous mapping, $A(V)$ be a relatively compact set. Assume that all fixed points of A are stable with respect to V.*

Then the set $F \subset V$ of fixed points of A is connected.

The proof is based on the following lemma.

Lemma. *Let the assumptions of Theorem 3 be satisfied and let $y_1, y_2 \in V$, $y_1 < y_2$ be fixed points of A. Then there is a continuous totally ordered curve of fixed points of A connecting y_1 with y_2.*

Proof. As the set $A(V)$ is relatively compact, the set F of fixed points of A is compact and there exists a countable dense subset F_0, $\overline{F_0} = F$. We denote by $\operatorname{span}(F_0)$ the spanning set of F_0 and $E_1 = \overline{\operatorname{span}(F_0)}$. Obviously E_1 is a separable closed subspace of E. We denote by $P_1 = P \cap E_1$ the positive cone in E_1. As $y_1, y_2 \in F$, $y_1 < y_2$ we have $y_2 - y_1 \in P \cap E_1$. Moreover, the cones P, P_1 induce the same ordering on the set F.

As E_1 is a separable Banach space, there is a strictly positive linear functional $x^* \in {P_1}^*$ [2]. Obviously for each $u_1, u_2 \in F$ we have $u_1 < u_2 \Rightarrow x^*(u_1) < x^*(u_2)$. Let F_1 be the set of fixed points of A in $[y_1, y_2]$.

Denote $Z = \{U \subset F_1,\ U \text{ is a totally ordered set}, y_1 \in U, y_2 \in U\}$. The set Z is inductively ordered by the set inclusion. Denote by U^+ the maximal element of Z. As U^+ is a totally ordered set, x^* is a homeomorphism of U^+ onto a closed set $x^*(U^+)$.

We claim $x^*(U^+)$ is connected. Supposing the contrary there are $u_\alpha, u_\beta \in U^+$ such that $x^*(u_\alpha) = \alpha$, $x^*(u_\beta) = \beta$, $\alpha, \beta \in x^*(U^+)$ and $(\alpha, \beta) \subset R \setminus x^*(U^+)$. That means $[u_\alpha, u_\beta]$ contains no fixed point.

The assumption of stability of fixed points implies that there is no strict superequilibrium in $O(u_\beta, \delta) \cap [u_\alpha, u_\beta]$ and no strict subequilibrium in $O(u_\alpha, \delta) \cap [u_\alpha, u_\beta]$ for δ sufficiently small.

Thus we have obtained a contradiction with Theorem 2 as neither case (a), nor cases (b), (c) occur.

$\square$

Proof of Theorem 3. Theorem 1 implies there are a minimal and a maximal fixed point ϱ, r and that $F \subset [\varrho, r]$.

If $r = \varrho$, the set F is a singleton.

If $r > \varrho$ then for each $u \in F$ there are continuous totally ordered curves of fixed points of A connecting ϱ with u and u with r. That means F is connected.

$\square$

Corollary. *Assuming $\varrho \neq r$ the set $F \subset V$ is a union of continuous totally ordered curves of fixed points of A connecting ϱ with r.*

The above Corollary completes the cascade of results of Krasnosel'skij and Lusnikov [6] concerning the relations between the type of monotony and the structure of the set of fixed points. The authors in [6] use another assumption instead of the stability of each fixed point. They assume the interval $[\varrho, r]$ is *degenerated*, i.e. there is no strict sub- or superequilibrium inside.

The following theorem presents a summary of results of Krasnosel'skij and Lusnikov (parts (a),(b) and (d)) and ours (part (c)).

Theorem 4. *Let an interval $V = [\varrho, r]$ be degenerated for a completely continuous operator $A\colon V \to V$. Then*

(a) *the set F of fixed points of A forms a continuous branch in V (i.e. F has a nonzero intersection with the boundary $\partial\Omega$ of each bounded open set Ω such that $r \in \Omega$ and $\varrho \notin \Omega$);*
(b) *if the operator A is order preserving then F contains a continuous curve;*
(c) *if the operator A is strictly order preserving then F is a union of continuous curves;*
(d) *if the operator A is strongly order preserving then F is a continuous curve.*

Remark. The assumption of degeneracy of the interval $[\varrho, r]$ can be slightly weakened by assuming that for each fixed point $x \in [\varrho, r]$ there is $0 << \delta$ such that there is no strict sub- or superequilibrium in the interval $[x, x+\delta] \cap [\varrho, r]$.

Application

We are interested in the structure of the set of solutions of the periodic boundary value problem

$$u'' + f(t,u) = 0, \tag{1}$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi), \tag{2}$$

where $f\colon I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We assume that there are constants a, $b \in \mathbb{R}$ such that

(i) there is a lower solution α and an upper solution β, $\alpha, \beta \in C^2(I)$, of the problem (1), (2) such that $a \leqslant \alpha(t) \leqslant \beta(t) \leqslant b$,
(ii) there is a constant $M > 0$ such that for each $u, v \in [a, b]$, $t \in I$, if $u \leqslant v$ then $f(t,v) - f(t,u) \geqslant -M^2(v-u)$,

(iii) the function $f(t,.)$ is nonincreasing in the variable x for $a \leqslant x \leqslant b$.

Using the existence result and the method of Lakshmikantham and Leela [8] we obtain that under the assumptions (i), (ii) there are maximal and minimal solutions $r(t)$, $\varrho(t)$ of the boundary value problem (1), (2), and that for each $\eta \in [a,b] \subset C(I)$ the linear problem

(3)
$$-u'' + M^2 u = f(t,\eta) + M^2\eta,$$
$$u(0) = u(2\pi),\ u'(0) = u'(2\pi),$$

has the unique solution

$$u(t) = c_1 e^{Mt} - c_2 e^{-Mt} - \frac{e^{Mt}}{2M}\int_0^t \sigma(s) e^{-Ms}\,ds + \frac{e^{-Mt}}{2M}\int_0^t \sigma(s) e^{Ms}\,ds,$$

where

$$c_1 = \frac{e^{2M\pi}}{2M(e^{2M\pi}-1)}\int_0^{2\pi} \sigma(s) e^{-Ms}\,ds,$$
$$c_2 = \frac{1}{2M(e^{2M\pi}-1)}\int_0^{2\pi} \sigma(s) e^{Ms}\,ds,$$

and

$$\sigma(t) = f(t,\eta) + M^2\eta.$$

The operator $A = V \to V$, $V = [a,b] \subset C(I)$ defined by $A(\eta) = u$, u being a solution of (3), (2) is relatively compact and strictly monotone [8].

Let $x(t)$ be a fixed point of A and let δ be a constant. We denote

$$A\left(x(t)+\delta\right) = x(t) + \varepsilon(t).$$

From the definition of A we obtain that $\varepsilon(t)$ is a solution of the boundary value problem

$$-\varepsilon(t)'' + M^2\varepsilon(t) = F(t),$$
$$\varepsilon(0) = \varepsilon(2\pi),\ \varepsilon'(0) = \varepsilon'(2\pi),$$

where $F(t) = f(t, x+\delta) - f(t,x) + M^2\delta$.

The assumption (iii) implies that $|\varepsilon(t)| \leqslant |\delta|$ and that each fixed point of the operator A is stable. Theorem 3 implies that the set of solutions of the boundary value problem (1), (2) is connected.

R e m a r k . Our example is only an illustrative one. The direct computation yields that $r(t)-\varrho(t)=c_0$, where c_0 is a nonnegative constant and the solution set has the form $S=\{\varrho(t)+c;\ c\in[0,c_0]\}$. See [9, Theorem].

As our second application we give the result concerning the structure of the set of solutions $x(t)\in C(I)$, $I=[0,1]$) of the integral equation

$$x(t)=\int_I K(t,s)f(s,x(s))\,\mathrm{d}s. \tag{4}$$

We assume that

(i) the function $K(t,s)\colon I\times I\to\mathbb{R}$ is continuous and $0\leqslant K(t,s)$,

(ii) the function $f(t,.)$ is increasing in the variable x,

(iii) there is a lower solution α and an upper solution β, $\alpha,\beta\in C(I)$, $\alpha(t)\leqslant\beta(t)$,

(iv) the function f is continuous and there is $\delta>0$ such that

$$f(t,u+\delta)-f(t,u)<\tfrac{\delta}{m}$$

for each $t\in I$ and $u\in[\alpha(t),\beta(t)]$, where $m\in\mathbb{R}$ and

$$m=\int_0^1 K(t_1,s)\,\mathrm{d}s=\max_{t\in I}\int_0^1 K(t,s)\,\mathrm{d}s.$$

(v) $K(t,s)$ is not identically zero in any subset $I\times[s_1,s_2]$, $s_1,s_2\in I$.

The operator $A\colon V\to V$, $V=[\alpha,\beta]\subset C(I)$ defined by $A(\eta)=u$, where

$$u(t)=\int_I K(t,s)f(s,\eta(s))\,\mathrm{d}s,$$

is relatively compact and strictly monotone.

Let $x(t)$ be a fixed point of A, and let $\delta>0$ be a constant.

Then

$$A\,(x(t)+\delta)=x(t)+\varepsilon(t).$$

where $\varepsilon(t)$ is given by

$$\varepsilon(t)=\int_I K(t,s)F(s)\,\mathrm{d}s,$$

$F(t)=f(t,x(t)+\delta)-f(t,x(t))$.

The assumption (iv) implies $\varepsilon(t)\leqslant\delta$.

Thus the solution $x(t)$ is stable. Theorem 3 implies that the set of solutions of the integral equation (4) bounded between $\alpha(t)$ and $\beta(t)$ is connected.

In the paper [3] it is proved that under assumptions somewhat weaker then (i)–(iii) the set of solutions is a complete lattice. Adding the assumptions (iv), (v) we obtain that this lattice is connected (in topology of $C(I)$) and is either a singleton or a union of totally ordered continuous curves.

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Author's address: *Boris Rudolf*, Katedra matematiky FEI STU, Mlynská dolina, 812 19 Bratislava, Slovak Republic, e-mail: `rudolf@kmat.elf.stuba.sk`.