ON POINTWISE INTERPOLATION INEQUALITIES FOR DERIVATIVES

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. Pointwise interpolation inequalities, in particular,

$$|\nabla_k u(x)| \leqslant c\,(\mathcal{M}u(x))^{1-k/m}\,(\mathcal{M}\nabla_m u(x))^{k/m}\,, \quad k<m,$$

and

$$|I_z f(x)| \leqslant c(\mathcal{M}I_\zeta f(x))^{\operatorname{Re} z/\operatorname{Re}\zeta}(\mathcal{M}f(x))^{1-\operatorname{Re} z/\operatorname{Re}\zeta},\ 0<\operatorname{Re} z<\operatorname{Re}\zeta<n,$$

where ∇_k is the gradient of order k, $\mathcal{M}$ is the Hardy-Littlewood maximal operator, and I_z is the Riesz potential of order z, are proved. Applications to the theory of multipliers in pairs of Sobolev spaces are given. In particular, the maximal algebra in the multiplier space $M(W_p^m(\mathbb{R}^n)\to W_p^l(\mathbb{R}^n))$ is described.

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1. Introduction

It is well known that an arbitrary function on the real line $\mathbb{R}$ with Lipschitz derivative u' satisfies

$$|u'(x)|^2 \leqslant 2\sup|u|\sup|u''|\,, \tag{1}$$

where 2 is the best possible constant (Landau [1]). For the history of this estimate as well as of its analogs and generalizations one can consult, for example, Section 13.5 of our book [2].

In the present paper we are interested in purely pointwise modifications of (1), where both factors in the right-hand side are functions of x. It is trivial, of course, that the second sign of supremum cannot be removed. Moreover, (1) is no longer valid without the first supremum if f has a simple zero at a point.

However, under the additional assumption of non-negativity of u one obtains the following stronger variant of (1)

$$|u'(x)|^2 \leqslant 2u(x) \sup |u''| \tag{2}$$

for all $x \in \mathbb{R}$ (early applications and generalizations of this inequality can be found in [3] and [4]). Indeed, for any $t \in \mathbb{R}$ we have

$$0 \leqslant u(x+t) = u(x) + tu'(x) + \int_0^t (u'(x+\tau) - u'(x)) \, \mathrm{d}\tau.$$

Therefore, the trinomial $u(x) + tu'(x) + \frac{1}{2}t^2 \sup |u''|$ is non-negative and the non-positivity of its discriminant is equivalent to (2).

Kufner and Maz'ya [5] gave generalizations of (2), and showed, in particular, that it can be easily improved as

$$|u'(x)|^2 \leqslant 2u(x) \mathcal{M}_{\pm} u''(x), \tag{3}$$

where the sign + or − is taken if $u'(x) \leqslant 0$ or $u'(x) \geqslant 0$, respectively, and $\mathcal{M}_+$, $\mathcal{M}_-$ are the Hardy-Littlewood right and left maximal operators:

$$\mathcal{M}_+ \varphi(x) = \sup_{\tau > 0} \frac{1}{\tau} \int_x^{x+\tau} |\varphi(y)| \, \mathrm{d}y,$$
$$\mathcal{M}_- \varphi(x) = \sup_{\tau > 0} \frac{1}{\tau} \int_{x-\tau}^{x} |\varphi(y)| \, \mathrm{d}y.$$

We claim that any complex valued function on $\mathbb{R}$ with absolutely continuous first derivative is subject to the inequality

$$|u'(x)|^2 \leqslant 8 \mathcal{M} u(x) \mathcal{M} u''(x), \tag{4}$$

where $\mathcal{M}$ is the Hardy-Littlewood operator

$$\mathcal{M} \varphi(x) = \sup_{\tau > 0} \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} |\varphi(y)| \, \mathrm{d}y.$$

Moreover, (4) can be improved as

$$|u'(x)|^2 \leqslant 8 \mathcal{M}^{\diamond} u(x) \mathcal{M} u''(x), \tag{5}$$

where

$$\mathcal{M}^{\diamond}u(x) = \sup_{\tau>0} \frac{1}{2\tau} \left| \int_{x-\tau}^{x+\tau} \text{sign } (x-y)u(y)\,\mathrm{d}y \right| . \tag{6}$$

In fact, the following identity is readily checked by integration by parts

$$u'(0) = \frac{1}{t^2} \int_{-t}^{t} \text{sign}\, y\, u(y)\,\mathrm{d}y - \frac{1}{2} \int_{-t}^{t} \Big(1 - \frac{|y|}{t}\Big)^2 \text{sign}\, y\, u''(y)\,\mathrm{d}y \tag{7}$$

where $t > 0$. (We replace x by 0 to simplify the notation.) Hence

$$|u'(0)| \leqslant \tfrac{2}{t}\mathcal{M}^{\diamond}u(0) + t\mathcal{M}u''(0),$$

which is equivalent to (5).

Another direct consequence of (7) is the inequality

$$|u'(x)|^2 \leqslant \tfrac{8}{3}\mathcal{M}^{\diamond}u(x) \sup |u''| .$$

The constant $\frac{8}{3}$ in this inequality (and even in the weaker one with $M^{\diamond}$ replaced by M) is best possible. In fact, one can easily check that the odd function u_0 given by

$$u_0(x) = \begin{cases} x(2-x) & \text{for} \quad 0 \leqslant x < \frac{3}{2}, \\ \dfrac{(x-3)^2}{3} & \text{for} \quad \frac{3}{2} \leqslant x < 3, \\ 0 & \text{for} \quad x \geqslant 3 \end{cases}$$

satisfies

$$u_0'(0) = 2, \quad \mathcal{M}^{\diamond}u_0(0) = \mathcal{M}u_0(0) = \tfrac{3}{4}, \quad \sup|u_0''| = 2.$$

As Kufner and Maz'ya noticed in [5], a simple change in the above argument leading to (2) results in the following generalization of (2) with the best possible constant

$$|u'(x)|^{\alpha+1} \leqslant \Big(\frac{\alpha+1}{\alpha}\Big)^{\alpha} u(x)^{\alpha} \sup_y \frac{|u'(y) - u'(x)|}{|y-x|^{\alpha}}, \tag{8}$$

where $u(x) \geqslant 0$ and $0 < \alpha < 1$.

One can arrive at an analogue of (8) for arbitrary complex valued functions, where the role of Hölder's seminorm is played by the function

$$\mathcal{D}_{p,m}u(x) = \left(\int_{\mathbb{R}} \frac{|u'(x) - u'(y)|^p}{|x-y|^{p(m-1)+1}}\,\mathrm{d}y \right)^{1/p}, \quad m \in (1,2),\ p \in [1,\infty] .$$

This function is important because its L_p-norm is a seminorm in the fractional Sobolev space $W_p^m(\mathbb{R})$.

We note that

$$(9) \qquad u'(0) = \frac{1}{t^2}\int_{-t}^{t} \operatorname{sign} y\, u(y)\, \mathrm{d}y + \frac{1}{t^2}\int_{-t}^{t} (t - |y|)\,(u'(0) - u'(y))\, \mathrm{d}y.$$

By Hölder's inequality, the absolute value of the second term on the right-hand side is dominated by

$$t^{m-1}\,(2\mathbf{B}\,(qm, q+1))^{1/q}\,\mathcal{D}_{p,m}u(0),$$

where $m \in (1,2)$, $p^{-1} + q^{-1} = 1$ and $\mathbf{B}$ is Euler's Beta-function. This along with (9) implies

$$|u'(0)| \leqslant 2t^{-1}\mathcal{M}^{\diamond}u(0) + t^{m-1}\,(2\mathbf{B}(qm, q+1))^{1/q}\,\mathcal{D}_{p,m}u(0).$$

Minimizing the right-hand side we conclude that the inequality

$$(10) \qquad |u'(x)|^m \leqslant m\Big(\frac{2m}{m-1}\Big)^{m-1}(2\mathbf{B}(qm, q+1))^{1/q}(\mathcal{M}^{\diamond}u(x))^{m-1}\,\mathcal{D}_{p,m}u(x)$$

is valid for almost all $x \in \mathbb{R}$. In particular, for $p = \infty$ and $m = \alpha + 1$, $0 < \alpha < 1$, we have the following analogue of (8):

$$(11) \qquad |u'(x)|^{\alpha+1} \leqslant \frac{2^{\alpha+1}}{\alpha+2}\Big(\frac{\alpha+1}{\alpha}\Big)^{\alpha}(\mathcal{M}^{\diamond}u(x))^{\alpha}\sup_{y}\frac{|u'(y) - u'(x)|}{|y-x|^{\alpha}}.$$

The constant factor in this inequality (and even in the weaker one with $\mathcal{M}^{\diamond}$ replaced by $\mathcal{M}$) is best possible as can be checked by the odd function u_α given for $x \geqslant 0$ by

$$u_\alpha(x) = \begin{cases} (\alpha+1)x - x^{\alpha+1} & \text{for } 0 \leqslant x < (\frac{\alpha+2}{2})^{1/\alpha}, \\ \frac{\alpha}{\alpha+2}(2(\frac{\alpha+2}{2})^{1/\alpha} - x)^{\alpha+1} & \text{for } (\frac{\alpha+2}{2})^{1/\alpha} \leqslant x < 2(\frac{\alpha+2}{2})^{1/\alpha}, \\ 0 & \text{for } x \geqslant 2(\frac{\alpha+2}{2})^{1/\alpha}. \end{cases}$$

In the sequel we prove n-dimensional generalizations of the above interpolation inequalities and give applications to the theory of pointwise multipliers in pairs of Sobolev spaces.

Let u be a function on $\mathbb{R}^n$ such that its distributional derivatives of order m are locally summable. By $\nabla_m u$ we mean the gradient of u of order m, i.e.

$$\nabla_m u = \left\{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u\right\}, \quad \alpha_1 + \dots + \alpha_n = m,$$

where ∂_{x_j} is a partial derivative. The Euclidean length of $\nabla_m u$ will be denoted by $|\nabla_m u|$ and we write ∂ or ∇ instead of ∇_1.

Let $\mathcal{M}$ be the Hardy-Littlewood maximal operator over centered balls defined by

$$\mathcal{M}\varphi(x) = \sup_{r>0} \bigl(\operatorname{meas}_n B_r\bigr)^{-1} \int_{B_r(x)} |\varphi(y)| \, \mathrm{d}y,$$

where φ is a scalar or vector-valued function in $\mathbb{R}^n$, $B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ and $B_r = B_r(0)$.

Our goal is to prove the following generalization of inequality (4).

Theorem 1. *Let k, l and m be integers, $0 \leqslant l \leqslant k \leqslant m$. Then there exists a positive constant $c = c(k,l,m,n)$ such that*

$$|\nabla_k u(x)| \leqslant c \left(\mathcal{M}\nabla_l u(x)\right)^{\frac{m-k}{m-l}} \left(\mathcal{M}\nabla_m u(x)\right)^{\frac{k-l}{m-l}} \tag{12}$$

for almost all $x \in \mathbb{R}^n$.

P r o o f. Clearly, it suffices to prove (12) for $l = 0$ when it becomes

$$|\nabla_k u(x)| \leqslant c \left(\mathcal{M} u(x)\right)^{\frac{m-k}{m}} \left(\mathcal{M}\nabla_m u(x)\right)^{\frac{k}{m}}. \tag{13}$$

Let η be a function in the ball B_1 with Lipschitz derivatives of order $m-2$ which vanishes on ∂B_1 together with all these derivatives. Also let

$$\int_{B_1} \eta(y) \, \mathrm{d}y = 1.$$

We shall use the Sobolev integral representation:

$$\begin{aligned} v(0) = {} & \sum_{|\beta|<m-k} t^{-n} \int_{B_t} \frac{(-y)^\beta}{\beta!} \partial^\beta v(y) \eta\Bigl(\frac{y}{t}\Bigr) \, \mathrm{d}y \\ & + (-1)^{m-k}(m-k) \sum_{|\alpha|=m-k} \int_{B_t} \frac{y^\alpha}{\alpha!} \partial^\alpha v(y) \int_{|y|/t}^{\infty} \eta\Bigl(\varrho \frac{y}{|y|}\Bigr) \varrho^{n-1} \, \mathrm{d}\varrho \frac{\mathrm{d}y}{|y|^n} \end{aligned}$$

(see [7], Section 1.5.1).

By setting here $v = \partial^\gamma u$ with an arbitrary multiindex γ of order k and integrating by parts in the first integral, we arrive at the identity

$$\partial^\gamma u(0) = (-1)^k t^{-n} \int_{B_t} u(y) \sum_{|\beta|<m-k} \frac{1}{\beta!} \partial^{\beta+\gamma}\Big(y^\beta \eta\Big(\frac{y}{t}\Big)\Big)\,\mathrm{d}y$$
$$+ \sum_{|\alpha|=m-k} (-1)^{m-k}(m-k) \int_{B_t} \frac{y^\alpha}{\alpha!} \partial^{\alpha+\gamma} u(y) \int_{|y|/t}^{\infty} \eta\Big(\varrho\frac{y}{|y|}\Big)\varrho^{n-1}\,\mathrm{d}\varrho \frac{\mathrm{d}y}{|y|^n}. \tag{14}$$

Hence

$$|\nabla_k u(0)| \leqslant c_1 t^{-n-k} \int_{B_t} |u(y)|\,\mathrm{d}y + c_2 \int_{B_t} |\nabla_m u(y)| \frac{\mathrm{d}y}{|y|^{n-m+k}}. \tag{15}$$

If $m-k \geqslant n$, the second integral does not exceed

$$t^{m-k-n} \int_{B_t} |\nabla_m u(y)|\,\mathrm{d}y.$$

In the case $m-k<n$ the second integral in (15) equals

$$t^{m-k-n} \int_{B_t} |\nabla_m u(y)|\,\mathrm{d}y + (n-m+k)\int_0^t \frac{\mathrm{d}\tau}{\tau^{n-m+k+1}} \int_{B\tau} |\nabla_m u(y)|\,\mathrm{d}y.$$

Therefore

$$\int_{B_t} |\nabla_m u(y)| \frac{\mathrm{d}y}{|y|^{n-m+k}} \leqslant \frac{n}{m-k} t^{m-k} \sup_{\tau\leqslant t} \tau^{-n} \int_{B_\tau} |\nabla_m u(y)|\,\mathrm{d}y. \tag{16}$$

Thus, for any $t>0$,

$$|\nabla_k u(0)| \leqslant c_3 t^{-k}\mathcal{M}u(0) + c_4 t^{m-k}\mathcal{M}\nabla_m u(0) \tag{17}$$

which implies (13). The result follows. □

R e m a r k 1. The above proof enables one to improve (13) replacing $\mathcal{M}$ by the maximal operator

$$\sup_{t>0} \frac{1}{\operatorname{meas}_n B_t} \sum_{|\gamma|=k} \Big| \int_{B_t(x)} u(y) H_\gamma(t^{-1}(y-x))\,\mathrm{d}y \Big|.$$

Here $\{H_\gamma\}_{|\gamma|=k}$ is a collection of bounded measurable functions such that, for all multiindices α of order $|\alpha| \leqslant k$,

$$\int_{B_t} y^\alpha H_\gamma(y)\,\mathrm{d}y = \delta_\alpha^\gamma,$$

where δ_α^γ is Kronecker's symbol.

R e m a r k 2. One can specify the constants c_3 and c_4 in (17) but we do not dwell upon this being unaware if the values obtained are best possible. In the special case $k=1$, $m=2$ a direct generalization of (5) is proved as follows. First, (14) becomes

$$\begin{aligned}\frac{\partial u(0)}{\partial x_i} = & -t^{-n}\int_{B_t} u(y)\frac{\partial}{\partial y_i}\Big(\eta\Big(\frac{y}{t}\Big)\Big)\,\mathrm{d}y \\ & -\sum_{j=1}^{n}\int_{B_t} y_j\frac{\partial^2 u(y)}{\partial y_i\partial y_j}\int_{|y|/t}^{\infty}\eta\Big(\varrho\frac{y}{|y|}\Big)\varrho^{n-1}\,\mathrm{d}\varrho\frac{\mathrm{d}y}{|y|^n}.\end{aligned}$$

We choose

$$\eta(y)=\frac{n+1}{\mathrm{meas}_n B_1}\,(1-|y|)_+\,. \tag{18}$$

Then

$$\begin{aligned}|\nabla u(0)| \leqslant\; & t^{-1}\frac{n+1}{\mathrm{meas}_n B_t}\left|\int_{B_t} u(y)\frac{y}{|y|}\,\mathrm{d}y\right| \\ & +\frac{1}{\mathrm{meas}_{n-1}\partial B_1}\int_{B_t}|\nabla_2 u(y)|\,\frac{\mathrm{d}y}{|y|^{n-1}}.\end{aligned}$$

By (16) the second term on the right is not greater than $t\mathcal{M}\nabla_2 u(0)$. Therefore

$$|\nabla u(0)| \leqslant \tfrac{n+1}{t}\mathcal{M}^{\diamond}u(0)+t\mathcal{M}\nabla_2 u(0),$$

where $\mathcal{M}^{\diamond}$ is a multidimensional generalization of (6):

$$\mathcal{M}^{\diamond}u(x)=\sup_{t>0}\frac{1}{\mathrm{meas}_n B_t}\left|\int_{B_t(x)} u(y)\frac{y-x}{|y-x|}\,\mathrm{d}y\right|. \tag{19}$$

Finally,

$$|\nabla u(x)|^2 \leqslant 4(n+1)\mathcal{M}^{\diamond}u(x)\;\mathcal{M}\nabla_2 u(x)$$

for almost all $x\in\mathbb{R}^n$.

R e m a r k 3. Suppose that instead of $\mathcal{M}$ we use the modified maximal operator $\mathcal{M}_\delta$ given by

$$\mathcal{M}_\delta\varphi(x)=\sup_{0<r<\delta}(\mathrm{meas}_n B_r)^{-1}\int_{B_r(x)}|\varphi(y)|\,\mathrm{d}y.$$

Then the proof of Theorem 1 (with small changes) leads to the following alternatives: either

$$\mathcal{M}_\delta u(x) < \tfrac{m-k}{m}\delta^m\mathcal{M}_\delta\nabla_m u(x)$$

and

$$|\nabla_k u(x)| \leqslant c\,(\mathcal{M}_\delta u(x))^{1-k/m}\,(\mathcal{M}_\delta\nabla_m u(x))^{k/m}$$

or

$$\mathcal{M}_\delta u(x) \geqslant \tfrac{m-k}{m}\delta^m \mathcal{M}_\delta \nabla_m u(x)$$

and

$$|\nabla_k u(x)| \leqslant c\delta^{-k}\mathcal{M}_\delta u(x).$$

As a consequence, the local variant of (13)

$$|\nabla_k u(x)| \leqslant c\left(\delta^{-m}\mathcal{M}_\delta u(x) + \mathcal{M}_\delta \nabla_m u(x)\right)^{k/m}\left(\mathcal{M}_\delta u(x)\right)^{1-k/m}$$

is valid with c independent of δ. A similar remark can be made concerning Theorems 2–4 in the next sections.

R e m a r k 4. Estimate (13) leads directly to the Gagliardo-Nirenberg ([8], [9]) inequality

$$\left\|\nabla_k u; \mathbb{R}^n\right\|_{L_s} \leqslant c\left\|u; \mathbb{R}^n\right\|_{L_q}^{1-k/m}\left\|\nabla_m u; \mathbb{R}^n\right\|_{L_p}^{k/m}, \tag{20}$$

where $1 < q \leqslant \infty, 1 < p \leqslant \infty$ and

$$\tfrac{1}{s} = \tfrac{k}{m}\tfrac{1}{p} + \left(1 - \tfrac{k}{m}\right)\tfrac{1}{q}.$$

Indeed, by (13) the left-hand side of (20) does not exceed

$$c\left(\int_{\mathbb{R}^n}\left(\mathcal{M}u(x)\right)^{s(1-k/m)}\left(\mathcal{M}\nabla_m u(x)\right)^{sk/m}\,\mathrm{d}x\right)^{1/r}$$

which by Hölder's inequality is majorized by

$$c\left\|\mathcal{M}u; \mathbb{R}^n\right\|_{L_q}^{1-k/m}\left\|\mathcal{M}\nabla_m u; \mathbb{R}^n\right\|_{L_p}^{k/m},$$

and it remains to refer to the boundedness of the operator $\mathcal{M}$ in $L_\sigma\left(\mathbb{R}^n\right)$, $1 < \sigma \leqslant \infty$.

If m is even and u is the Riesz potential of order m with non-negative density, then the estimate

$$|\nabla_k u(x)| \leqslant c\, \mathcal{M}u(x)^{1-k/m} \left(\mathcal{M}\Delta^{m/2}u(x)\right)^{k/m}, \tag{21}$$

which is stronger than (2), follows directly from Hedberg's inequality

$$I_t f(x) \leqslant c\, (I_\tau f(x))^{t/\tau}(\mathcal{M}f(x))^{1-t/\tau}, \tag{22}$$

where $0 < t < \tau < n$, f is a non-negative locally summable function, and the Riesz potential is defined by

$$I_t f(x) = c \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-t}}\, \mathrm{d}y \tag{23}$$

(see [10] or [6], Proposition 3.1.2(b)). The constant c in (23) is chosen in such a way that $I_t = (-\Delta)^{-t/2}$, i.e.

$$I_t f(x) = F^{-1}_{\xi\to x}|\xi|^{-t}F_{x\to\xi}f(x),$$

where F is the Fourier transform in $\mathbb{R}^n$.

However, in our case u is not a potential with non-negative density and we cannot refer to (22). Nevertheless, (21) is a direct corollary of the following assertion which seems to be of independent interest.

Theorem 2. *Let z and ζ be complex numbers subject to*

$$0 < \operatorname{Re} z < \operatorname{Re}\zeta < n$$

and let $I_z f$ denote the Riesz potential of order z defined by (23) where $t = z$ and f is a complex valued function in $L_1(\mathbb{R}^n)$ with compact support. Then there exists a constant c independent of f such that

$$|I_z f(x)| \leqslant c(\mathcal{M}I_\zeta f(x))^{\operatorname{Re} z/\operatorname{Re}\zeta}(\mathcal{M}f(x))^{1-\operatorname{Re} z/\operatorname{Re}\zeta} \tag{24}$$

for almost all $x \in \mathbb{R}^n$.

Proof. Denote by χ a function in the Schwartz space $\mathcal{S}$ such that $F\chi = 1$ in a neighbourhood of the origin. From the identity

$$|\xi|^{-z} = |\xi|^{\zeta-z}F\chi(\xi)|\xi|^{-\zeta} + |\xi|^{-z}(1-F\chi(\xi)), \quad \xi \in \mathbb{R}^n,$$

it follows that

$$I_z f(0) = P * I_\zeta f(0) + Q * f(0), \tag{25}$$

where $*$ stands for the convolution and

$$\begin{aligned} P(x) &= c_1 F^{-1}_{\xi\to x}(|\xi|^{\zeta-z} F\chi(\xi)),\\ Q(x) &= c_2 F^{-1}_{\xi\to x}(|\xi|^{-z}(1 - F\chi(\xi)). \end{aligned}$$

Let m be a positive integer such that

$$0 \leqslant 2m - \operatorname{Re}\zeta + \operatorname{Re} z < 2.$$

In the case $n \geqslant 2$ we have

$$|P(x)| = c\left|\Delta_x^m \int_{\mathbb{R}^n} \frac{\chi(y)\,\mathrm{d}y}{|x-y|^{n-2m+\zeta-z}}\right| \leqslant c(|x|+1)^{-n-\operatorname{Re}\zeta+\operatorname{Re} z}.$$

Analogously, for $n = 1$ we obtain

$$|P(x)| = c\left|\partial_x \int_{\mathbb{R}} \frac{(x-y)\chi(y)\,\mathrm{d}y}{|x-y|^{1+\zeta-z}}\right| \leqslant c(|x|+1)^{-1-\operatorname{Re}\zeta+\operatorname{Re} z}.$$

Hence

$$\begin{aligned} |P * I_\zeta f(0)| &\leqslant c\int_{\mathbb{R}^n} \frac{|I_\zeta f(y)|\,\mathrm{d}y}{(|y|+1)^{n+\operatorname{Re}\zeta-\operatorname{Re} z}}\\ &= c\int_0^\infty \int_{B_\varrho} |I_\zeta f(y)|\,\mathrm{d}y \frac{\mathrm{d}\varrho}{(\varrho+1)^{n+1+\operatorname{Re}\zeta-\operatorname{Re} z}}\\ &\leqslant c\mathcal{M} I_\zeta f(0). \end{aligned} \tag{26}$$

The function $|\xi|^{-z}(1 - F\eta(\xi))$ is smooth which implies

$$|Q(y)| \leqslant c(N)|y|^{-N} \tag{27}$$

for $|y| \geqslant 1$ and for sufficiently large N. If $|y| < 1$ we have

$$|Q(y)| \leqslant c|y|^{-n+\operatorname{Re} z} + |I_z\chi(y)|$$

and since the second term on the right is bounded,

$$|Q(y)| \leqslant c|y|^{-n+\operatorname{Re} z} \tag{28}$$

for $|y| < 1$. Using (27) and (28) we arrive at

$$|Q * f(0)| \leqslant c\Big(\int_{|y|>1} \frac{|f(y)|\,\mathrm{d}y}{|y|^{n+1}} + \int_{|y|<1} \frac{|f(y)|\,\mathrm{d}y}{|y|^{n-\operatorname{Re} z}}\Big) \leqslant c\mathcal{M}f(0).$$

Combining this inequality with (26) we obtain from (25) that

$$|I_z f(0)| \leqslant c(\mathcal{M}I_\zeta f(0) + \mathcal{M}f(0)).$$

Now the dilation $y \to y/r$ with an arbitrary positive r implies

$$|I_z f(0)| \leqslant c(r^{\operatorname{Re} z - \operatorname{Re}\zeta}\mathcal{M}I_\zeta f(0) + r^{\operatorname{Re} z}\mathcal{M}f(0))$$

and it remains to minimize the right-hand side with respect to r. □

The following analogue of the inequality (20) can be easily obtained from (24).

Corollary 1. *Let* $1 < q \leqslant \infty$, $1 < p \leqslant \infty$, *and*

$$\frac{1}{s} = \Big(1 - \frac{\operatorname{Re} z}{\operatorname{Re}\zeta}\Big)\frac{1}{p} + \frac{\operatorname{Re} z}{\operatorname{Re}\zeta}\frac{1}{q}.$$

Then

$$\|I_z f; \mathbb{R}^n\|_{L_s} \leqslant c\|I_\zeta f; \mathbb{R}^n\|_{L_q}^{\operatorname{Re} z/\operatorname{Re}\zeta}\|f; \mathbb{R}^n\|_{L_p}^{1-\operatorname{Re} z/\operatorname{Re}\zeta}. \tag{29}$$

P r o o f. By (24), the left-hand side in (29) does not exceed

$$\int_{\mathbb{R}^n} \big(\mathcal{M}I_\zeta f(x)\big)^{s\operatorname{Re} z/\operatorname{Re}\zeta}\big(\mathcal{M}f(x)\big)^{s(1-\operatorname{Re} z/\operatorname{Re}\zeta)}\,\mathrm{d}x$$

which by Hölder's inequality is majorized by

$$c\|\mathcal{M}I_\zeta f; \mathbb{R}^n\|_{L_q}^{\operatorname{Re} z/\operatorname{Re}\zeta}\|\mathcal{M}f; \mathbb{R}^n\|_{L_p}^{1-\operatorname{Re} z/\operatorname{Re}\zeta}.$$

It remains to refer to the boundednesss of the operator $\mathcal{M}$ in $L_\sigma(\mathbb{R}^n)$ for $1 < \sigma \leqslant \infty$. □

R e m a r k 5. Note that Hedberg's inequality (22) with $f \geqslant 0$ follows from (24) since, obviously,

$$\int_{B_r} I_\tau f(y)\,\mathrm{d}y \leqslant c\int_{\mathbb{R}^n} \frac{r^n f(y)\,\mathrm{d}y}{r^{n-\tau} + |y|^{n-\tau}}$$

and hence

$$\mathcal{M}I_\tau f(x) \leqslant cI_\tau f(x) \quad \text{a.e.} \tag{30}$$

if f is non-negative. Moreover, the proof of Corollary 1, along with (30), gives inequality (29) with real z, ζ and with non-negative f also for $q \in (0, 1]$. (This is an alternative proof of the corresponding inequality in Theorem 3.1.6 of [6].)

4. Multidimensional variants of inequality (10)

Let m be a fractional number with $[m]$ and $\{m\}$ denoting its integer and fractional parts. We introduce the function

$$\Big(\mathcal{D}_{p,m}u\Big)(x) = \left(\int_{\mathbb{R}^n} \left|\nabla_{[m]}u(x) - \nabla_{[m]}u(y)\right|^p |x-y|^{-n-p\{m\}}\,\mathrm{d}y\right)^{1/p}.$$

Theorem 3. *Let k, l be integers and let m be fractional, $0 \leqslant l \leqslant k < m$. Then there exists a positive constant $c = c(k,l,m,n)$ such that*

$$|\nabla_k u(x)| \leqslant c\,[\mathcal{M}\nabla_l u(x)]^{\frac{m-k}{m-l}}\,[\mathcal{D}_{p,m}u(x)]^{\frac{k-l}{m-l}} \tag{31}$$

for almost all $x \in \mathbb{R}^n$.

Proof. It suffices to prove inequality (31) for $l=0$ and $x=0$. By (14) we have

$$|\nabla_k u(0)| \leqslant c\bigg(t^{-k}\mathcal{M}u(0) + t^{[m]-k}\left|\nabla_{[m]}u(0)\right| + \int_{B_t} \frac{\left|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)\right|}{|y|^{n-[m]+k}}\,\mathrm{d}y\bigg). \tag{32}$$

Hölder's inequality implies

$$\int_{B_t} \frac{\left|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)\right|}{|y|^{n-[m]+k}}\,\mathrm{d}y \leqslant ct^{m-k}\mathcal{D}_{p,m}u(0). \tag{33}$$

Using the function η, introduced in the proof of Theorem 1, we obtain for any multiindex γ of order $[m]$

$$\partial^\gamma u(0) = t^{-n}\int_{B_t} \eta\left(\frac{y}{t}\right)\partial^\gamma u(y)\,\mathrm{d}y + t^{-n}\int_{B_t}\eta\left(\frac{y}{t}\right)[\partial^\gamma u(0) - \partial^\gamma(y)]\,\mathrm{d}y. \tag{34}$$

Hence

$$\left|\nabla_{[m]}u(0)\right| \leqslant t^{-n-[m]}\left|\int_{B_t} u(y)\left(\nabla_{[m]}\eta\right)\left(\frac{y}{t}\right)\mathrm{d}y\right| + t^{\{m\}}\left(\int_{B_t}|\eta(y)|^q\,|y|^{\left(\frac{n}{p}+\{m\}\right)q}\,\mathrm{d}y\right)^{1/q}\mathcal{D}_{p,m}u(0), \tag{35}$$

where $p^{-1}+q^{-1}=1$. Combining (32), (33) and (35) we arrive at

$$|\nabla_k u(0)| \leqslant c\left(t^{-k}\mathcal{M}u(0) + t^{m-k}\mathcal{D}_{p,m}u(0)\right).$$

The minimization of the right-hand side in t completes the proof. □

R e m a r k 6. The same argument as in Remark 4 applied to (31) gives the inequality

$$\|\nabla_k u; \mathbb{R}^n\|_{L_s} \leqslant c\,\|u; \mathbb{R}^n\|_{L_q}^{1-k/m}\,\|\mathcal{D}_{p,m}u; \mathbb{R}^n\|_{L_p}^{k/m}$$

where m is fractional, $k < m$, $1 \leqslant p \leqslant \infty$, and q, s are the same as in Remark 4.

R e m a r k 7. By (35), inequality (10) can be easily extended to the n-dimensional case. Indeed, let $m \in (1,2)$. Then, inserting the function (18) into (35) we arrive at

$$|\nabla u(0)| \leqslant t^{-1}\frac{n+1}{\operatorname{meas}_n B_t}\left|\int_{B_t} u(y)\frac{y}{|y|}\,\mathrm{d}y\right| + t^{m-1}\frac{n+1}{\operatorname{meas}_n B_1}\left(\int_{B_1}(1-|y|)^q\,|y|^{\left(\frac{n}{p}+m-1\right)q}\,\mathrm{d}y\right)^{1/q}\mathcal{D}_{p,m}u(0)$$

which implies

$$|\nabla u(0)| \leqslant t^{-1}\,(n+1)\,\mathcal{M}^{\diamond}u(0) + \frac{t^{m-1}(n+1)n^{1/q}}{(\operatorname{meas}_n B_1)^{1/p}}\left(\mathbf{B}\left(q(m+n-1), q+1\right)\right)^{1/q}\mathcal{D}_{p,m}u(0)$$

with $\mathcal{M}^{\diamond}$ given by (19). The minimization of the right-hand side results in the inequality

$$|\nabla u(x)|^m \leqslant \frac{(n+1)^m m^m\,(n\mathbf{B}(q(m+n-1), q+1))^{1/q}}{(m-1)^{m-1}\,(\operatorname{meas}_n B_1)^{1/p}}\,(\mathcal{M}^{\diamond}u(x))^{m-1}\,\mathcal{D}_{p,m}u(x)$$

containing (10) as a special case. In particular for $p = \infty$, $m = \alpha + 1$, $0 < \alpha < 1$, we have

$$|\nabla u(x)|^{\alpha+1} \leqslant \frac{n(n+1)^{\alpha+1}(\alpha+1)^{\alpha+1}}{(n+\alpha)(n+\alpha+1)\alpha^{\alpha}}\,(\mathcal{M}^{\diamond}u(x))^{\alpha}\sup_y\frac{|\nabla u(y)-\nabla u(x)|}{|y-x|^{\alpha}}$$

which is a multidimensional generalization of (11).

We conclude this section with two inequalities of the same nature as (31).

Theorem 4.

(i) *Let k, m be integers, and let l be noninteger, $0 < l < k \leqslant m$. Then there exists a positive constant $c = c(k,l,m,n)$ such that*

$$|\nabla_k u(x)| \leqslant c\,(\mathcal{D}_{p,l}u(x))^{\frac{m-k}{m-l}}\,(\mathcal{M}\nabla_m u(x))^{\frac{k-l}{m-l}} \tag{36}$$

for almost all $x \in \mathbb{R}^n$.

(ii) *Let k be integer and let l, m be noninteger, $0 < l < k < m$. Then there exists a positive constant $c = c(k, l, m, n)$ such that*

$$|\nabla_k u(x)| \leqslant c\,(\mathcal{D}_{p,l}u(x))^{\frac{m-k}{m-l}}\,(\mathcal{D}_{p,m}u(x))^{\frac{k-l}{m-l}} \tag{37}$$

for almost all $x \in \mathbb{R}^n$.

Proof. It is sufficient to take $l \in (0,1)$ and $x = 0$.

(i) Since the function $\partial^{\beta+\gamma}\left(y^\beta \eta(y)\right)$ in (14) is orthogonal to 1 in $L_2(B_1)$, it follows from (14) that

$$|\nabla_k u(0)| \leqslant c\Bigg(t^{-n-k}\int_{B_t} |u(y) - u(0)|\,\mathrm{d}y + \int_{B_t} \frac{|\nabla_m u(y)|}{|y|^{n-m+k}}\,\mathrm{d}y\Bigg). \tag{38}$$

By Hölder's inequality, applied to the first integral and by (16) we have

$$|\nabla_k u(0)| \leqslant c\left(t^{-k}\mathcal{D}_{p,l}u(0) + t^{m-k}\mathcal{M}\nabla_m(x)\right).$$

The result follows.

(ii) By (38) with m replaced by $[m]$

$$|\nabla_k u(0)| \leqslant c\Bigg(t^{l-k}\mathcal{D}_{p,l}u(0) + t^{[m]-k}\left|\nabla_{[m]}u(0)\right| + \int_{B_t}\frac{\left|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)\right|}{|y|^{n-[m]+k}}\,\mathrm{d}y\Bigg).$$

By (33) the third term in the right-hand side does not exceed

$$t^{m-k}\mathcal{D}_{p,m}u(0).$$

Now we note that (35) implies

$$|\nabla_{[m]}u(0)| \leqslant c(t^{l-[m]}\mathcal{D}_{p,l}u(0) + t^{\{m\}}\mathcal{D}_{p,m}u(0)).$$

Hence

$$|\nabla_k u(0)| \leqslant c\left(t^{-k}\mathcal{D}_{p,l}u(0) + t^{m-k}\mathcal{D}_{p,m}u(0)\right).$$

The result follows. □

5.1. The maximal algebra in $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$.

Let m and l be integers, $m \geqslant l$, and let $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ denote the space of pointwise multipliers acting from $W_p^m(\mathbb{R}^n)$ to $W_p^l(\mathbb{R}^n)$ (see [11]). Analytical descriptions of $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ as well as separate necessary and sufficient conditions for the membership in this multiplier space can be found in [11]. We characterize the maximal algebra in $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ by using inequality (13).

Theorem 5. *The maximal Banach algebra in $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$, $m \geqslant l$, $1 < p < \infty$, is isomorphic to the space*

$$M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big) \cap L_\infty(\mathbb{R}^n)\,. \tag{39}$$

Remark 8. In the case $m = l$ the statement of Theorem 5 is trivial since the multiplier space $M\big(W_p^l(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ is an algebra and is embedded into $L_\infty(\mathbb{R}^n)$.

Proof of Theorem 5. Let A be a subalgebra of $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$. Then, for any $N = 1, 2, \dots$ and for any $\gamma \in A$, $u \in W_p^m(\mathbb{R}^n)$,

$$\|\gamma^N u\|_{L_p}^{1/N} \leqslant \|\gamma^N u\|_{W_p^l}^{1/N} \leqslant c^{1/N}\,\|\gamma\|_A\,\|u\|_{W_p^m}^{1/N}\,.$$

(Here and elsewhere in the present section we omit $\mathbb{R}^n$ in the notations of norms.) Passing to the limit as $N \to \infty$ we obtain $\gamma \in L_\infty(\mathbb{R}^n)$. Hence A is a part of the intersection (39).

Let γ_1, γ_2 belong to (39). Then, for any $u \in W_p^m(\mathbb{R}^n)$,

$$\begin{aligned}\|\nabla_l(\gamma_1\gamma_2 u)\|_{L_p} \leqslant c\Big(&\|\gamma_1\|_{L_\infty}\,\|\nabla_l\,(\gamma_2 u)\|_{L_p}\\ &+\|\gamma_2\|_{L_\infty}\sum_{h=1}^{l}\|\,|\nabla_h\gamma_1|\,|\nabla_{l-h}u|\,\|_{L_p}\\ &+\sum_{h=1}^{l-1}\sum_{k=1}^{l-h}\|\,|\nabla_h\gamma_1|\,|\nabla_k\gamma_2|\,|\nabla_{l-h-k}u|\,\|_{L_p}\Big).\end{aligned} \tag{40}$$

The first term in the right-hand side is majorized by

$$c\,\|\gamma_1\|_{L_\infty}\,\|\gamma_2\|_{M\left(W_p^m \to W_p^l\right)}\,\|u\|_{W_p^m}\,.$$

Before estimating the second term we note that if $\Gamma \in M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ then, for any $h = 0, \dots, l$,

$$\nabla_h \Gamma \in M\big(W_p^m(\mathbb{R}^n) \to W_p^{l-h}(\mathbb{R}^n)\big) \subset M\big(W_p^{m-l+h}(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\big)$$

and the estimate

$$\|\nabla_h \Gamma\|_{M\left(W_p^{m-l+h} \to L_p\right)} \leqslant c\, \|\Gamma\|_{M\left(W_p^m \to W_p^l\right)} \tag{41}$$

holds (see [11], Section 1.3). Therefore, the second term in the right-hand side of (30) is not greater than

$$c\, \|\gamma_2\|_{L_\infty} \|\gamma_1\|_{M\left(W_p^m \to W_p^l\right)} \|u\|_{W_p^m}.$$

To estimate the remaining terms in the right-hand side of (40) we need the inequality

$$|\nabla_h \Gamma(x)| \leqslant c\, \|\Gamma\|_{L_\infty}^{\frac{k}{h+k}} \left(\mathcal{M} \nabla_{h+k} \Gamma(x)\right)^{\frac{h}{k+h}}$$

stemming from (13). Hence

$$\begin{aligned}
&\big\| |\nabla_h \gamma_1| |\nabla_k \gamma_2| |\nabla_{l-h-k} u| \big\|_{L_p} \\
&\quad \leqslant c \|\gamma_1\|_{L_\infty}^{\frac{k}{h+k}} \|\gamma_2\|_{L_\infty}^{\frac{h}{h+k}} \big\| (\mathcal{M} \nabla_{h+k} \gamma_1)^{\frac{h}{h+k}} (\mathcal{M} \nabla_{h+k} \gamma_2)^{\frac{k}{h+k}} |\nabla_{l-h-k} u| \big\|_{L_p} \\
&\quad \leqslant c \|\gamma_1\|_{L_\infty}^{\frac{k}{h+k}} \|\gamma_2\|_{L_\infty}^{\frac{h}{h+k}} \big\| (\mathcal{M} \nabla_{h+k} \gamma_1) |\nabla_{l-h-k} u| \big\|_{L_p}^{\frac{h}{h+k}} \big\| (\mathcal{M} \nabla_{h+k} \gamma_2) |\nabla_{l-h-k} u| \big\|_{L_p}^{\frac{k}{h+k}}.
\end{aligned}$$

By Verbitsky's theorem (see [12], Lemma 3.1)

$$\|\mathcal{M} \Gamma\|_{M\left(W_p^s \to L_p\right)} \leqslant c\, \|\Gamma\|_{M\left(W_p^s \to L_p\right)} \tag{42}$$

which along with (41) implies

$$\begin{aligned}
&\big\| |\nabla_h \gamma_1| |\nabla_k \gamma_2| |\nabla_{l-h-k} u| \big\|_{L_p} \\
&\quad \leqslant c \|\gamma_1\|_{L_\infty}^{\frac{k}{h+k}} \|\gamma_2\|_{L_\infty}^{\frac{h}{h+k}} \big\| |\nabla_{h+k} \gamma_1| |\nabla_{l-h-k} u| \big\|_{L_p}^{\frac{h}{h+k}} \big\| |\nabla_{h+k} \gamma_2| |\nabla_{l-h-k} u| \big\|_{L_p}^{\frac{k}{h+k}} \\
&\quad \leqslant c \|\gamma_1\|_{L_\infty}^{\frac{k}{h+k}} \|\gamma_2\|_{L_\infty}^{\frac{h}{h+k}} \|\gamma_1\|_{M(W_p^m \to W_p^l)}^{\frac{h}{h+k}} \|\gamma_2\|_{M(W_p^m \to W_p^l)}^{\frac{k}{h+k}} \|u\|_{W_p^m}.
\end{aligned}$$

The proof is complete. □

From Theorem 5 and the description of $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ obtained in [11, Chapter 1] we arrive at

Corollary 2. *The maximal algebra in* $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$, $m \geqslant l$, *consists of* $\gamma \in W^l_{p,\mathrm{loc}}(\mathbb{R}^n)$ *with the finite norm*

$$\sup_{e \subset \mathbb{R}^n, \mathrm{diam}(e) \leqslant 1} \frac{\|\nabla_l \gamma; e\|_{L_p}}{\big(\mathrm{cap}\,(e, W_p^m)\big)^{1/p}} + \|\gamma\|_{L_\infty}, \tag{43}$$

where e *is a compact set and* $\mathrm{cap}\,\big(e, W_p^m\big)$ *is the capacity of* e *generated by the norm in* $W_p^m(\mathbb{R}^n)$, *i.e.*

$$\mathrm{cap}(e, W_p^m) = \inf\big\{\|u\|^p_{W_p^m} : u \in C_0^\infty(\mathbb{R}^n), \quad u \geqslant 1 \text{ on } e\big\}.$$

In the case $mp > n$ the norm (43) can be simplified as

$$\sup_{x \in \mathbb{R}^n} \|\nabla_l \gamma; B_1(x)\|_{L_p} + \|\gamma\|_{L_\infty},$$

which also follows from the fact that the norm in $M\big(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n)\big)$ is equivalent to the norm

$$\sup_{x \in \mathbb{R}^n} \|\gamma; B_1(x)\|_{W_p^m}$$

in the case $mp > n$ ([11, Chapter 1]).

5.2. Estimate for the norm in $M(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n))$.
According to Theorem 1.3.2/1 [11], the equivalence relation

$$\|\gamma\|_{M(W_p^m \to W_p^l)} \sim \|\nabla_l \gamma\|_{M(W_p^m \to L_p)} + \|\gamma\|_{M(W_p^{m-l} \to L_p)}$$

holds with $mp \leqslant n$ and $m \geqslant l$. The proof of the upper estimate for the norm in $M(W_p^m(\mathbb{R}^n) \to W_p^l(\mathbb{R}^n))$ given in [11] is based on the complex interpolation. The inequality (13) enables one to arrive at the same result in a different way. The argument is as follows.

Let $\gamma \in M(W_p^{m-l}(\mathbb{R}^n) \to L_p(\mathbb{R}^n))$ and $\nabla_l \gamma \in M(W_p^m(\mathbb{R}^n) \to L_p(\mathbb{R}^n))$. For any $u \in C_0^\infty(\mathbb{R}^n)$

$$\|\gamma u\|_{W_p^l} \leqslant c\bigg(\sum_{k=0}^{l} \big\||\nabla_k \gamma \nabla_{l-k} u|\big\|_{L_p} + \|\gamma u\|_{L_p}\bigg).$$

By (13) and the Hölder inequality we have for $k = 1, \dots, l-1$

$$\begin{aligned} \big\||\nabla_k \gamma \nabla_{l-k} u|\big\|_{L_p} &\leqslant c\|(\mathcal{M}\gamma)^{1-k/l}(\mathcal{M}\nabla_l \gamma)^{k/l}(\mathcal{M}u)^{k/l}(\mathcal{M}\nabla_l u)^{1-k/l}\|_{L_p} \\ &\leqslant c\|(\mathcal{M}\gamma)(\mathcal{M}\nabla_l u)\|^{1-k/l}_{L_p}\|(\mathcal{M}\nabla_l \gamma)(\mathcal{M}u)\|^{k/l}_{L_p}. \end{aligned} \tag{44}$$

Clearly,

$$|\nabla_l u| \leqslant cI_{m-l}|\nabla_m u|, \quad |u| \leqslant cI_m|\nabla_m u|.$$

Hence,

$$\mathcal{M}\nabla_l u \leqslant cI_{m-l}\mathcal{M}\nabla_m u, \quad \mathcal{M}u \leqslant cI_m\mathcal{M}\nabla_m u.$$

This along with (42) leads to

$$\begin{aligned}\|(\mathcal{M}\gamma)(\mathcal{M}\nabla_l u)\|_{L_p} &\leqslant c\|\gamma\|_{M(W_p^m \to W_p^l)}\|I_{m-l}\mathcal{M}\nabla_m u\|_{W_p^{m-l}} \\ &\leqslant c\|\gamma\|_{M(W_p^{m-l} \to L_p)}\|\mathcal{M}\nabla_m u\|_{L_p}\end{aligned}$$

and similarly,

$$\begin{aligned}\|(\mathcal{M}\nabla_l\gamma)(\mathcal{M}u)\|_{L_p} &\leqslant c\|\nabla_l\gamma\|_{M(W_p^m \to W_p^l)}\|I_m\mathcal{M}\nabla_m u\|_{W_p^m} \\ &\leqslant c\|\nabla_l\gamma\|_{M(W_p^m \to L_p)}\|\mathcal{M}\nabla_m u\|_{L_p}.\end{aligned}$$

The result follows from (44) and the boundedness of the operator $\mathcal{M}$ in $L_p(\mathbb{R}^n)$.

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