CONGRUENCES IN ORDERED SETS

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Abstract. A concept of congruence preserving upper and lower bounds in a poset P is introduced. If P is a lattice, this concept coincides with the notion of lattice congruence.

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There exist various concepts of a congruence relation in ordered sets. We use the term "ordered set" for the partially ordered set. All of them define a congruence as an equivalence relation whose classes are convex subsets. However, this concept is too weak, namely the factor set by such an equivalence need not be an ordered set. Hence, the definitions are usually amended by additional conditions. As an example we can show the definition by M. Kolibiar [Kol]. A natural condition for a congruence on an ordered set is that if this set is a lattice (w.r.t. the order) then this congruence coincides with the lattice congruence. The aim of our paper is to introduce a concept of congruence in an ordered set satisfying all the foregoing assumptions which, moreover, corresponds to the concept of morphism preserving upper and lower bounds.

Let $A = (A, \leqslant)$ be an ordered set. If there is no danger of misunderstanding, the symbol $\leqslant$ will be omitted. For a subset $B \subseteq A$ (with the induced order) we denote by $L(B)$ or $U(B)$ the set of all lower or upper bounds of B, i. e.

$$L(B) = \{x \in A;\ x \leqslant a \text{ for all } a \in B\},$$
$$U(B) = \{x \in A;\ x \geqslant a \text{ for all } a \in B\}.$$

We adopt the notation $U(B, C) = U(B \cup C)$ and $L(B, C) = L(B \cup C)$. If $B = \{b_1, b_2, \ldots, b_n\}$, we will write briefly $U(B) = U(b_1, b_2, \ldots, b_n)$, dually for $L(B)$. If more than one set is considered, we use subscripts, i. e. we write $U_A(B)$ and $L_A(B)$ to indicate the carrier set.

Indeed, if $B \subseteq C \subseteq A$ then $U(B) \supseteq U(C)$ and $L(B) \supseteq L(C)$.

Let P, Q be ordered sets. A mapping $f\colon P \to Q$ is *order preserving* if $x \leqslant y$ in P implies $f(x) \leqslant f(y)$ in Q. A mapping $f\colon P \to Q$ is called a *strong morphism* if f is order preserving and if $f(a) \leqslant f(b)$ in Q then there exist $c, d \in P$ such that $c \leqslant d$ in P with $f(a) = f(c)$, $f(b) = f(d)$. We will write $P \simeq Q$ if there exists an order preserving bijection of P onto Q. If Θ is an equivalence on a set A and $b \in A$, denote by $[b]_\Theta = \{a \in A;\ a\Theta b\}$. If $f\colon P \to Q$ is a mapping, denote by Θ_f the equivalence on P induced by f, i. e. $a\Theta_f b$ if and only if $f(a) = f(b)$.

Definition 1. Let P, Q be ordered sets. A surjective mapping $f\colon P \to Q$ is an *LU-morphism* if $\operatorname{card} f(P) = 1$ or

$$f(L_P(x, y)) = L_Q(f(x), f(y)))$$

and

$$f(U_P(x, y)) = U_Q(f(x), f(y))$$

for all x, y of P.

Lemma 1. *Every LU- morphism is a strong morphism.*

Proof. Let P, Q be ordered sets and $f\colon P \to Q$ an LU-morphism. Suppose $x, y \in P$ and $x \leqslant y$. Then $L_P(x, y) = L_P(x)$ whence $L_Q(f(x), f(y)) = f(L_P(x, y)) = f(L_P(x)) = L_Q(f(x))$, i. e. $f(x) \leqslant f(y)$. Moreover, if $f(a) \leqslant f(b)$ for $a, b \in P$ then $f(L_P(a, b)) = L_Q(f(a), f(b)) = L_Q(f(a)) = f(L_P(a))$. Analogously, we can show $f(U_P(a, b)) = f(U_P(b))$. Hence, there exist elements $c \in L_P(a, b)$, $d \in U_P(a, b)$ with $f(c) = f(a)$ and $f(d) = f(b)$. Evidently, $c \leqslant d$. $\square$

Theorem 1. *Let P, Q be ordered sets and $f\colon P \to Q$ a surjective mapping. The following conditions are equivalent:*

1. *f is an LU-morphism;*
2. *f is order preserving and for each $x, y \in P$ with $f(x) \leqslant f(y)$ there exist $u, v \in P$ such that $v \leqslant x \leqslant u$, $v \leqslant y \leqslant u$ and $f(u) = f(y), f(v) = f(x)$.*

Proof. (1) $\Rightarrow$ (2) directly by Lemma 1. (2) $\Rightarrow$ (1): Since f is order preserving, we have $f(U_P(x, y)) \subseteq U_Q(f(x), f(y))$. Prove the converse inclusion. Suppose $z \in U_Q(f(x), f(y))$. Then $z = f(w)$ for some $w \in P$ and $f(x) \leqslant f(w)$, $f(y) \leqslant f(w)$. By (2) there exist $c, d \in P$ with $x \leqslant c$, $w \leqslant c$ and $y \leqslant d$, $w \leqslant d$ such that $f(c) = f(w) = f(d)$. By (2) there is $u \in P$ with $c \leqslant u$, $d \leqslant u$ and $f(u) = f(c) = f(w) = z$. Thus also

$x \leqslant u$ and $y \leqslant u$. Since f is order preserving, we have $f(x) \leqslant f(u)$, $f(y) \leqslant f(u)$, i.e. $z = f(u) \in f(U_P(x,y))$.

Dually it can be shown that $f(L_P(x,y)) = L_Q(f(x), f(x))$. $\square$

In what follows we give the definition of a congruence in an ordered set which is simpler than that of M. Kolibiar [Kol]:

Definition 2. An equivalence Θ on an ordered set P is called a *congruence* if either $\Theta = P \times P$ or it satisfies

(i) $[a]_\Theta$ is a convex subset of P for each $a \in P$;

(ii) for each $x, y \in [a]_\Theta$ there exist $c, d \in [a]_\Theta$ such that $d \leqslant x \leqslant c$ and $d \leqslant y \leqslant c$;

(iii) if $u \leqslant a$, $u \leqslant b$ and $u\Theta a$ then there exists $t \in P$ with $a \leqslant t$, $b \leqslant t$ and $b\Theta t$; if $a \leqslant v$, $b \leqslant v$ and $v\Theta b$ then there exists $s \in P$ with $s \leqslant a$, $s \leqslant b$ and $a\Theta s$.

Of course, the identity relation on P is a congruence on P. We are going to show that the factor set by a congruence is an ordered set again:

Theorem 2. *Let P be an ordered set and let Θ be a congruence on P. The factor relation defined on P/Θ by setting $[a]_\Theta \leqslant_{/\Theta} [b]_\Theta$ iff there exist $x \in [a]_\Theta, y \in [b]_\Theta$ with $x \leqslant y$ is an order on P/Θ.*

Proof. Of course, $\leqslant_{/\Theta}$ is reflexive.

Suppose $[a]_\Theta \leqslant_{/\Theta} [b]_\Theta$ and $[b]_\Theta \leqslant_{/\Theta} [a]_\Theta$. Then there are $x, x' \in [a]_\Theta$ and $y, y' \in [b]_\Theta$ such that $y \leqslant x$ and $x' \leqslant y'$. By (ii), there exists $u \in P$ with $y \leqslant u$, $y' \leqslant u$ and $u \in [b]_\Theta$.

Then $u, y \in [b]_\Theta$ and $x' \in [a]_\Theta$ such that $x' \leqslant u$ and $y \leqslant u$.

By (iii), there exists $s \in P$ with $s \leqslant x'$, $s \leqslant y$ and $s \in [a]_\Theta$. By (i), $[a]_\Theta$ is convex, i.e. $s \leqslant y \leqslant x$ implies $y \in [a]_\Theta$. Since equivalence classes are pairwise disjoint, this gives $[a]_\Theta = [b]_\Theta$ proving antisymmetry of $\leqslant_{/\Theta}$.

Let us prove transitivity of $\leqslant_{/\Theta}$. Let $[a]_\Theta \leqslant_{/\Theta} [b]_\Theta$ and $[b]_\Theta \leqslant_{/\Theta} [c]_\Theta$. Then there exist $x \in [a]_\Theta, y, y' \in [b]_\Theta$ and $z \in [c]_\Theta$ such that $x \leqslant y$ and $y' \leqslant z$. By (ii), there is $u \in [b]_\Theta$ with $y \leqslant u$, $y' \leqslant u$. Hence $x \leqslant u$. By (iii), there exists $v \in P$ with $u \leqslant v$, $z \leqslant v$ and $v \in [c]_\Theta$. Hence $x \leqslant v$ proving $[a]_\Theta \leqslant_{/\Theta} [c]_\Theta$. $\square$

Theorem 3. *Let P, Q be ordered sets.*

(a) *If $f\colon P \to Q$ is an LU-morphism then Θ_f is a congruence on P and $P/\Theta_f \simeq Q$,*

(b) *If Θ is a congruence on P then the canonical map $h\colon P \to P/\Theta$ given by $a \to [a]_\Theta$ is an LU-morphism.*

Proof. (a) We are going to check the conditions of Definition 2. The condition (i) is evident since f is an order preserving map. For (ii) suppose $x, y \in [a]_\Theta$.

Then $f(x)=f(y)$ and, by Theorem 1, there exists $u\in P$ with $x\leqslant u$, $y\leqslant u$ and $f(u)=f(x)$. Hence $u\in[a]_\Theta$. Dually we can show the second part of (ii). Let us prove (iii): Let $u\leqslant a$, $u\leqslant b$ and $u\Theta_f a$. Then $f(u)=f(a)$, i.e. $f(U(a,b))=U(f(a),f(b))=U(f(u),f(b))=U(f(b))=f(U(b))$. Hence, there exist $t\in U(a,b)$ with $f(t)=f(b)$, thus $b\Theta_f t$ and $a\leqslant t$, $b\leqslant t$.

Dually the second part of (iii) can be proved. Hence, Θ_f is a congruence on P and clearly $Q\simeq P/\Theta_f$.

(b) Of course, the canonical map is order preserving. Let us prove the second part of (2) of Theorem 1. Let $x,y\in P$ and $h(x)\leqslant h(y)$. Then there exist $c,d\in P$ with $c\leqslant d$ and $h(c)=h(x)$, $h(d)=h(y)$. By (ii), there is $v\in P$ with $v\leqslant x$, $v\leqslant c$ and $v\in[x]\Theta$, and further, there is $t\in P$ with $d\leqslant t$, $y\leqslant t$ and $t\in[y]\Theta$. By (iii), there exists $u\in P$ such that $t\leqslant u$, $x\leqslant u$ and $u\Theta t$. Hence $x\leqslant u$, $y\leqslant u$ and $h(u)=h(y)$. Analogously, there is $s\in P$ with $s\leqslant x$, $s\leqslant y$ and $h(s)=h(x)$. By Theorem 1, h is an LU-morphism. □

A nice characterization of a lattice congruence was settled by G. Dorfer [Dor]. It is of some interest that this characterization does not involve lattice operations. We show that the same characterization is valid also for ordered sets and congruences introduced by our Definition 2. Beside other things it witnesses that if $(P,\leqslant)$ is a lattice, our definition of congruence in P coincides with the lattice congruence.

Recall that an ordered set A is *directed* if $U(a,b)\neq\emptyset\neq L(a,b)$ for every $a,b\in A$.

Theorem 4. *An equivalence Θ on an ordered set P is a congruence if and only if $\Theta=P\times P$ or it satisfies the following three conditions:*

(a) *if $a\leqslant b$ and $a\Theta a_1$ then there exists $b_1\in P$ such that $a_1\leqslant b_1$ and $b\Theta b_1$;*
(b) *if $a\leqslant b$ and $b\Theta b_1$ then there exists $a_1\in P$ such that $a_1\leqslant b_1$ and $a\Theta a_1$;*
(c) *for each $a\in P$, $[a]_\Theta$ is a convex and directed subset of P.*

P r o o f. We prove that (i), (ii), (iii) of Definition 2 imply (a), (b), (c) and vice versa. (1) Let $a\leqslant b$ and $a\Theta a_1$ for some $a_1\in P$. By (ii), there exists $d\in[a]_\Theta$ with $d\leqslant a_1$, $d\leqslant a$ and hence $d\leqslant b$. By (iii) there is $b_1\in[b]_\Theta$ with $a_1\leqslant b_1$, proving (a). Dually we can show (b). By (i) and (ii) it is almost evident that every class $[a]_\Theta$ is a convex and directed subset of P.

(2) Suppose that an equivalence Θ on P satisfies (a), (b), (c). This immediately yields (i) and (ii) of Definition 2. Proving (iii) by using (a) and (b) is an easy computation. □

Corollary. *Let P be an ordered set and Θ an equivalence on P. Then Θ is a congruence on P if and only if*

(1) *P/Θ is an ordered set (with the order $\leqslant_{/\Theta}$);*

(2) $[L_P(x,y)]_\Theta = L_{P/\Theta}([x]\Theta,[y]\Theta)$ *and* $[U_P(x,y)]_\Theta = U_{P/\Theta}([x]_\Theta,[y]_\Theta)$ *for every* x, y *of* P.

Proof. If Θ is a congruence on P then, by Theorem 2, the relation $\leqslant_{/\Theta}$ is an order on the factor set P/Θ. By Theorem 3, Θ induces a canonical mapping which is an LU-morphism, thus also (2) is satisfied.

Conversely, if Θ satisfies (1) and (2) then the canonical mapping $h\colon P \to P/\Theta$ is an LU-morphism. Since $\Theta = \Theta_h$, Theorem 3 (a) completes the proof. □

Remark 1. The Corollary witnesses that our definition of congruence is the only possible to satisfy the following assumptions:

- the factor set is again an ordered set (with the factor order);
- it preserves upper and lower bounds and hence coincides with a lattice congruence provided $(P, \leqslant)$ is a lattice.

Remark 2. There exist other definitions of congruences or homomorphisms in ordered sets. E. g. M. Kolibiar [Kol] has a useful definition of congruence. However, our definition is different, see the following example: let $P = (\{0, a, b, c, d, 1\}, \leqslant)$ be an ordered set visualized in Fig. 1.

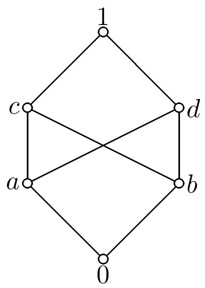

Fig. 1

By Kolibiar's definition, the only congruences on this set are $P \times P$ and the identity relation. By our definition, this set has the aforementioned congruences together with the following ones (given by their partitions):

$$\Theta_1 \ldots \{0, a, b, d\}, \{c, 1\}$$
$$\Theta_2 \ldots \{0, a, b, c\}, \{d, 1\}$$
$$\Theta_3 \ldots \{0, a\}, \{b, c, d, 1\}$$
$$\Theta_4 \ldots \{0, b\}, \{a, c, d, 1\}$$
$$\Theta_5 \ldots \{0, a\}, \{b, c\}, \{d, 1\}$$
$$\Theta_6 \ldots \{0, b\}, \{a, d\}, \{c, 1\}$$

Also there exist various definitions of morphisms. If f is a mapping preserving sup and inf (provided they exist), see e.g. G. Grätzer [GR], then f need not be an LU-morphism. If e.g. in our set (in Fig. 1) we have a mapping f of P into the two-element chain $\{0, 1\}$ defined by setting

$$f(0) = f(a) = f(b) = 0$$
$$f(1) = f(c) = f(d) = 1$$

then f preserves sup and inf but it is not an LU-morphism.

References

[Dor] *G. Dorfer*: Sublattice-extensions by means of convex sublattices. Contributions to General Algebra 9. Verlag Hölder-Pichler-Temski, Wien, 1995, pp. 127–132.

[GR] *G. Grätzer*: General Lattice Theory. Birkhäuser, Basel-Stuttgart, 1987.

[Kol] *M. Kolibiar*: Congruence relations and direct decomposition of ordered sets. Acta Sci. Math. (Szeged) *51* (1987), 129–135.

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