A PU-INTEGRAL ON AN ABSTRACT METRIC SPACE

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Abstract. In this paper, we define a PU-integral, i.e. an integral defined by means of partitions of unity, on a suitable compact metric measure space, whose measure μ is compatible with its topology in the sense that every open set is μ-measurable. We prove that the PU-integral is equivalent to μ-integral. Moreover, we give an example of a noneuclidean compact metric space such that the above results are true.

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Introduction

In 1984, Jarník and Kurzweil gave the idea of PU-integral on $\mathbb{R}^m$ (cf. [8]), which permitted to define an integral on an abstract space. In fact they used partitions of unity by suitable functions instead of simple objects of the topology of the space, as rectangles in $\mathbb{R}^m$.

In [12], Pfeffer used a PU-integral on an abstract measure space $(X, \mathcal{M}, \mu)$ and proved some of its properties under the following assumptions:

i) each μ-integrable function is PU-integrable,

ii) each PU-integrable function is μ-measurable.

In [7], Henstock defined the Davies-McShane integral on a measure space $(X,\mathcal{M},\mu)$, using partitions of X by means of measurable sets. He proved the equivalence between this integral and the μ-integral for μ-measurable functions, however, leaving open the question whether Davies-McShane integrable (D-McSh-integrable) functions are μ-measurable.

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In this paper [Sections 1, 2, 3], we define a PU-integral on a measure space $(X, \mathcal{M}, \mu)$ and give some conditions on the measure space to prove the assumptions i) and ii) used by Pfeffer [12]. Moreover [Section 4], using the results of Henstock [7], we show the equivalence between the μ-integral and the PU-integral. This last result gives an affirmative answer to the open question of Henstock on the μ-measurability of a D-McSh-integrable function.

Finally [Section 5], we give an example of a noneuclidean measure space on which our results can be verified.

Preliminaries

In this paper X denotes a compact metric space, $\mathcal{M}$ a σ-algebra of subsets of X such that each open set is in $\mathcal{M}$, μ a non-atomic, finite, Radon measure on $\mathcal{M}$ such that:

α) each ball $U(x,r)$ centered at x with radius r has a positive measure,

β) for every x in X there is a number $h(x) \in \mathbb{R}$ such that $\mu(U[x,2r]) \leqslant h(x) \times \mu(U[x,r])$ for all $r > 0$ (where $U[x,r]$) is the closed ball),

γ) $\mu(\partial U(x,r)) = 0$ where $\partial U(x,r)$ is the boundary of $U(x,r)$.

We introduce the following basic concepts.

Definition 1. A *partition of unity* (PU-partition) in X is, by definition, a finite collection $P = \{(\theta_i, x_i)\}_{i=1}^p$ where $x_i \in X$ and θ_i are non negative, μ-measurable and μ-integrable real functions on X such that $\sum\limits_{i=1}^p \theta_i(x) = 1$ a.e. in X.

Definition 2. Let δ be a positive function on X. A PU-partition is said to be *δ-fine* if

$$S_{\theta_i} = \{x \in X \colon \theta_i(x) \neq 0\} \subset U(x_i, \delta(x_i)),\ i = 1, 2, \ldots, p.$$

Definition 3. A real function f on X is said to be PU-*integrable* on X if there exists a real number I with the property that, for every given $\varepsilon > 0$, there is a positive function $\delta \colon X \longrightarrow \mathbb{R}$ such that $\left| \sum\limits_{i=1}^p f(x_i) \cdot \int\limits_X \theta_i \, \mathrm{d}\mu - I \right| < \varepsilon$ for each δ-fine PU-partition $P = \big\{(\theta_i, x_i)\big\}_{i=1}^p$.

The number I is called the PU-integral of f and we write $I = (\mathrm{PU}) \int\limits_X f$.

It is easy to verify that if I exists, it is unique.

If $A \subset X$ is compact, we can view it as a compact space in its own right, and define the integrability of a function f on A as above.

Proposition 1.1. *If δ is a positive function on X, there exists a δ-fine* PU-*partition.*

Proof. Consider the family $\mathcal{U} = \{U(x, \frac{\delta(x)}{2}), x \in X\}$. Since $\mathcal{U}$ is an open cover of X, there is a finite subcover $\{U(x_i, \frac{\delta(x_i)}{2}), i = 1, 2, \ldots, n\}$ of X.

Set

$$V_1 = U[x_1, \tfrac{\delta(x_1)}{2}], \quad V_i = U[x_i, \tfrac{\delta(x_i)}{2}] - \bigcup_{k=1}^{i-1} U[x_k, \tfrac{\delta(x_k)}{2}] \quad i = 2, \ldots, n.$$

The functions

$$\theta_i(x) = \chi_{V_i}(x) = \begin{cases} 1 & \text{if } x \in V_i \\ 0 & \text{if } x \notin V_i \end{cases}$$

verify the relation $\sum\limits_{i=1}^{n} \theta_i(x) = 1$ and the partition $P = \{(\theta_i, x_i)\}_{i=1}^{p}$ is δ-fine. □

Proposition 1.2. *A real function f on X is* PU-*integrable if and only if for every $\varepsilon > 0$ there exists a function $\delta\colon X \longrightarrow \mathbb{R}^+$ such that, if $P = \{(\theta_i', x_i')\}_{i=1}^{p}$ and $Q = \{(\theta_j'', x_j'')\}_{j=1}^{m}$ are two δ-fine* PU-*partitions then the inequality*

$$\left| \sum_{i=1}^{p} f(x_i') \cdot \int_X \theta_i' \,\mathrm{d}\mu - \sum_{j=1}^{m} f(x_j'') \cdot \int_X \theta_j'' \,\mathrm{d}\mu \right| < \varepsilon$$

holds.

Proof. The proposition can be easily proved using the technique of Prop. 2.1.8 in [13]. □

Proposition 1.3. *If $f\colon X \longrightarrow \mathbb{R}$ is* PU-*integrable on X and if $A \subset X$ is a compact subset of X then $f/A\colon A \longrightarrow \mathbb{R}$ is* PU-*integrable on A.*

Proof. If $\varepsilon > 0$, there exists $\delta\colon X \longrightarrow \mathbb{R}^+$ and we set $\delta/A\colon A \longrightarrow \mathbb{R}^+$ to be the restriction of δ to A. Since A is compact, there exist δ-fine PU-partitions in A, and let $P_1 = \{(\theta_j^{(1)}, x_j^{(1)})\}_{j=1}^{p}$ and $P_2 = \{(\theta_k^{(2)}, x_k^{(2)})\}_{k=1}^{n}$ be two such partitions. Extend $\theta_j^{(1)}$ and $\theta_k^{(2)}$ onto X setting

$\theta_j^{(1)}(x) = \theta_k^{(2)}(x) = 0$ for $x \notin A$; we have then $S_{\theta_j^{(1)}}, S_{\theta_k^{(2)}} \subset A$, for each j and k.

By Prop. 1.1, there exists a δ-fine PU-partition in X, $P = \{(\theta_i, x_i)\}_{i=1}^{m}$, such that $\mu(S_{\theta_i} \cap S_{\theta_j}) = 0$ for $i \neq j$. If we set $A_i' = S_{\theta_i} \cap A$ and $A_i'' = S_{\theta_i} \cap (X - A)$ for each i we get :

$A_i' \cup A_i'' = S_{\theta_i}, \theta_i = \theta_i' + \theta_i''$ where $\theta_i' = \theta_i \cdot \chi_{A_i'}$ and $\theta_i'' = \theta_i \cdot \chi_{A_i''}$.

The family $\overline{P} = \{(\theta_i', x_i)\}_{i=1}^m \cup \{(\theta_i'', x_i)\}_{i=1}^m$ is δ-fine; therefore $\overline{P}$ is a PU-partition in X.

Now if we consider

$$\overline{P}_1 = \{(\theta_j^{(1)}, x_j^{(1)})\}_{j=1}^p \cup \{(\theta_i'', x_i)\}_{i=1}^m,$$

$$\overline{P}_2 = \{(\theta_k^{(2)}, x_k^{(2)})\}_{k=1}^n \cup \{(\theta_i'', x_i)\}_{i=1}^m,$$

it is clear that $\overline{P}_1$ and $\overline{P}_2$ are δ-fine PU-partitions in X.

By Prop. 1.2 we have

$$\begin{aligned}\varepsilon > \Bigg| &\sum_j f(x_j^{(1)}) \cdot \int_X \theta_j^{(1)} \,\mathrm{d}\mu + \sum_i f(x_i) \cdot \int_X \theta_i'' \,\mathrm{d}\mu \\ &- \sum_k f(x_k^{(2)}) \cdot \int_X \theta_k^{(2)} \,\mathrm{d}\mu - \sum_i f(x_i) \cdot \int_X \theta_i'' \,\mathrm{d}\mu \Bigg| \\ = \Bigg| &\sum_j f(x_j^{(1)}) \cdot \int_X \theta_j^{(1)} \,\mathrm{d}\mu - \sum_k f(x_k^{(2)}) \cdot \int_X \theta_k^{(2)} \,\mathrm{d}\mu \Bigg|,\end{aligned}$$

which proves the PU-integrability on A of f/A. □

2. μ-MEASURABILITY OF A PU-INTEGRABLE FUNCTION

In this section we are going to study the μ-measurability of PU-integrable functions. For this purpose, we recall some classical results.

Theorem 2.1. *Let X be a metric space, $\mathcal{M}$ a σ-algebra of subsets of X and $\mathcal{B}$ a Vitali system with $\mathcal{B} \subset \mathcal{M}$. If φ and μ are totally additive set functions on $\mathcal{M}$, then the derivative of φ with respect to μ and the system $\mathcal{B}$ at a point $x \in X$ is a μ-measurable function.*

Proof. See [5] p. 247. □

Theorem 2.2. *If X is a metric separable space, if the closed sets belong to a measure space $(X, \mathcal{M}, \mu)$ of subsets of X, and if to every $x \in X$ there is a finite number $p(x) > 0$, such that $\mu(U[x, 2r]) \leqslant p(x) \cdot \mu(U[x, r])$, then the system $\mathcal{C}$ of the closed balls $U[x, r]$ (for all $x \in X$ and all $r > 0$) is a Vitali system.*

Proof. See [5] p. 263. □

Now we show our results about the μ-measurability of the PU-integrable functions.

Proposition 2.1. *If f is* PU*-integrable on X and $(C_i)_{i=1}^k$ is a finite family of compact subsets of X such that*

$$\mu\Big(X - \bigcup_{i=1}^{k} C_i\Big) = 0 \text{ and } \mu(C_i \cap C_j) = 0 \text{ for } i \neq j,$$

then $(\mathrm{PU}) \int\limits_X f = \sum\limits_{i=1}^{k} (\mathrm{PU}) \int\limits_{C_i} f$.

Proof. Using Prop. 1.3, we can say that f is PU-integrable on C_i for every i; therefore, if $\varepsilon > 0$ there exists $\delta\colon X \longrightarrow \mathbb{R}^+$ such that for each δ-fine PU-partition in $C_i, P_i = \{(\theta_j^{(i)}, x_j^{(i)})\}_{j=1}^p$ and for each δ-fine PU-partition in $X, P = \{(\theta_h, x_h)\}_{h=1}^n$ we have

$$\Big|\sum_j f(x_j^{(i)}) \cdot \int\limits_{C_i} \theta_j^{(i)} \,\mathrm{d}\mu - (\mathrm{PU}) \int\limits_{C_i} f\Big| < \frac{\varepsilon}{2k}$$

and

$$\Big|\sum_h f(x_h) \cdot \int\limits_X \theta_h \,\mathrm{d}\mu - (\mathrm{PU}) \int\limits_X f\Big| < \frac{\varepsilon}{2}.$$

Consider the δ-fine PU-partition in X given by $\overline{P} = \bigcup\limits_{i=1}^{k} P_i$ where we set $\theta_j^{(i)}(x) = 0$ for $x \in X - C_i$.

Then we have:

$$\begin{aligned}\Big|(\mathrm{PU}) \int\limits_X f - \sum_{i=1}^{k} (\mathrm{PU}) \int\limits_{C_i} f\Big| &\leqslant \Big|(\mathrm{PU}) \int\limits_X f - \sum_{i=1}^{k} \big(\sum_j f(x_j^{(i)}) \cdot \int\limits_{C_i} \theta_j^{(i)} \,\mathrm{d}\mu\big)\Big| \\ &\quad + \sum_{i=1}^{k} \Big|\sum_j f(x_j^{(i)}) \cdot \int\limits_{C_i} \theta_j^{(i)} \,\mathrm{d}\mu - \sum_{i=1}^{k} (\mathrm{PU}) \int\limits_{C_i} f\Big| \\ &< \frac{\varepsilon}{2} + k \cdot \frac{\varepsilon}{2k} = \varepsilon,\end{aligned}$$

which proves the proposition. □

Proposition 2.2. *If $\varepsilon > 0$ and if f is* PU*-integrable on X and $P = \{(\theta_i, x_i)\}_{i=1}^p$ is a δ-fine* PU*-partition in X with $\theta_i = \chi_{C_i}$ where C_i is a compact subset of X and $\mu(C_i \cap C_j) = 0$ for $i \neq j$, then*

$$\sum_{i \in J} \Big|f(x_i) \cdot \int\limits_X \chi_{C_i} \,\mathrm{d}\mu - (\mathrm{PU}) \int\limits_{C_i} f\Big| < 2\varepsilon$$

for any $J \subseteq \{1, 2, \ldots, p\}$.

Proof. If $\varepsilon > 0$, there exists $\delta\colon X \longrightarrow \mathbb{R}^+$ such that, if $P = \{(\chi_{C_i}, x_i)\}_{i=1}^p$ is a δ-fine partition (P exists by Prop. 1.1), we have

$$\Big|\sum_{i=1}^p f(x_i) \cdot \int_X \chi_{C_i}\,\mathrm{d}\mu - (\mathrm{PU})\int_X f\Big| < \frac{\varepsilon}{2}.$$

By Prop. 2.1, we have

$$\Big|\sum_{i=1}^p [f(x_i) \cdot \int_X \chi_{C_i}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_i} f]\Big| < \frac{\varepsilon}{2.}$$

For each $i \notin J$, there exists $\delta_i\colon C_i \longrightarrow \mathbb{R}^+$ such that for every δ_i-fine PU-partition in $C_i, P_i = \{(\chi_{C_{i,k}}, x_{i,k})\}_k$, we obtain

$$\Big|\sum_k f(x_k) \cdot \int_{C_i} \chi_{C_{i,k}}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_i} f\Big| < \frac{\varepsilon}{2m}$$

where $m = p - \mathrm{card}(J)$.

If we suppose $\delta_i \leqslant \delta$ for each i, the family $\overline{P} = \bigcup_{j\in J}\{(\chi_{C_j}, x_j)\} \cup \bigcup_{i\notin J}\{(\chi_{C_{i,k}}, x_{i,k})\}_k$ is a δ-fine partition in X.

By the PU-integrability of f, we have :

$$\begin{aligned}
\Big|\sum_{j\in J}\Big(f(x_j) \cdot \int_X \chi_{C_j}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_j} f\Big)\Big| &= \Big|\sum_{j\in J}\Big(f(x_j) \cdot \int_X \chi_{C_j}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_j} f\Big) \\
&\quad + \sum_{i\notin J}\Big[\sum_k\Big(f(x_k) \cdot \int_X \chi_{C_{i,k}}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_{i,k}} f\Big)\Big] \\
&\quad - \sum_{i\notin J}\Big[\sum_k\Big(f(x_k) \cdot \int_X \chi_{C_{i,k}}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_{i,k}} f\Big)\Big]\Big| \\
&\leqslant \Big|\sum_{j\in J} f(x_j) \cdot \int_X \chi_{C_j}\,\mathrm{d}\mu + \sum_{i\notin J}\Big[\sum_k\Big(f(x_k) \cdot \int_X \chi_{C_{i,k}}\,\mathrm{d}\mu\Big)\Big] \\
&\quad - \Big[\sum_{j\in J}(\mathrm{PU})\int_{C_j} f + \sum_{i\notin J}\Big(\sum_k (\mathrm{PU})\int_{C_{i,k}} f\Big)\Big]\Big| \\
&\quad + \sum_{j\notin J}\Big|\sum_k f(x_k) \cdot \int_X \chi_{C_{i,k}}\,\mathrm{d}\mu - (\mathrm{PU})\int_{C_{i,k}} f\Big| < \frac{\varepsilon}{2} + m\frac{\varepsilon}{2m} = \varepsilon.
\end{aligned}$$

Dividing the lefthand side into subsums of the positive and the negative terms and applying our result separately to each of them, we obtain the assertion of Prop. 2.2.

□

Proposition 2.3. *If f is* PU*-integrable then*

$$f(x) = \lim_{C_n \to x} \frac{(\mathrm{PU}) \int_{C_n} f}{\mu(C_n)} \quad \text{a.e.} \quad \text{in } X,$$

where C_n are closed balls centered at x with radius $r_n > 0$ and C_n converges to x in the sense that r_n converges to 0.

Proof. We use the same technique as in [11], Prop. 4.4.

Set

$$A = \left\{ x \in X\colon f(x) \neq \lim_{C_n \to x} \frac{(\mathrm{PU}) \int_{C_n} f}{\mu(C_n)} \right\},$$

for each $x_0 \in A$ there exist a sequence $\{C_n\}_n$ of closed balls converging to x_0 and a number $\varepsilon(x_0) > 0$ such that for every $\varrho > 0$ there is n_0 with the property that

$$C_{n_0} \subset U(x_0, \varrho) \quad \text{and} \quad \left| \frac{(\mathrm{PU}) \int_{C_{n_0}} f}{\mu(C_{n_0})} - f(x_0) \right| \geqslant \varepsilon(x_0).$$

Set $A_n = \{x \in A | \varepsilon(x) > \frac{1}{n}\}$, then $A = \bigcup\limits_n A_n$. Given $\varepsilon > 0$, if $\delta\colon X \longrightarrow \mathbb{R}^+$ is the corresponding positive function from the definition of integral, for each $x_0 \in A_n$ there is a closed ball $C[x_0] \subset U(x_0, \delta(x_0))$ such that

$$\left| \frac{(\mathrm{PU}) \int_{C[x_0]} f}{\mu(C[x_0])} - f(x_0) \right| > \frac{1}{n}.$$

Observe that the family $\{C[x_0]\colon x_0 \in A_n\}$ is a cover of A_n, and since the system $\mathcal{C}$ of all closed balls is a Vitali system by condition β on the space $(X, \mathcal{M}, \mu)$ and by Theorem 2.2, there exist a countably many disjoint sets $C[x_i]$, $i = 1, \dots$ such that $0 \leqslant \sum\limits_i \chi_{C[x_i]}(x) \leqslant 1$ a.e. in X. For each $k \in N$ the family $\{(\chi_{C[x_i]}, x_i)\}_{i=1}^k$ can be considered as part of a δ-fine partition in X, $P = \{(\chi_{F_i}, x_i), F_i$ closed set $\}$: so by Prop. 2.2 applied for $\frac{\varepsilon}{2n}$ instead of ε, we obtain

$$\sum_{i=1}^k \mu(C[x_i]) < n \cdot \sum_{i=1}^k \Big| \Big((\mathrm{PU}) \int\limits_{C[x_i]} f - f(x_i) \cdot \mu(C[x_i]) \Big) \Big| < n \cdot \frac{\varepsilon}{n} = \varepsilon \text{ for every } k \in N$$

and

$$\mu^*(A_n) = \mu^*\Big(A_n \cap \bigcup_i C[x_i]\Big) \leqslant \mu\Big(\bigcup_i C[x_i]\Big) = \sum_i \mu\Big(C[x_i]\Big) \leqslant \varepsilon;$$

it follows that $\mu(A_n) = 0$. □

Proposition 2.4. *If f is* PU*-integrable then f is μ-measurable.*

Proof. It is a consequence of Prop. 2.3, of the condition β on the measure space $\mathcal{M}$ and of Theorems 2.1 and 2.2. In fact, by Prop. 2.3 $f(x)$ is almost everywhere in X the derivative of the set function $\varphi\colon \mathcal{C} \longrightarrow \mathbb{R}$, where $\varphi(C) = (\mathrm{PU})\int\limits_C f$, with respect to the measure μ and the Vitali system $\mathcal{C}$. □

3. PU-integrability of a μ-integrable function

Proposition 3.1. *If f is μ-measurable and μ-integrable, given $\varepsilon > 0$ there exists a function $\delta\colon X \longrightarrow \mathbb{R}^+$ such that*

$$\sum_i \left|\left(f(x_i)\cdot \int\limits_X \theta_i \,\mathrm{d}\mu - \int\limits_X f\theta_i \,\mathrm{d}\mu\right)\right| < \varepsilon$$

for each δ-fine partition $P = \{(\theta_i, x_i)\}_i \in X$.

Proof. Given $\varepsilon > 0$, set $\eta = \frac{\varepsilon}{1+\mu(X)}$.

By Vitali-Caratheodory theorem, there are two real μ-measurable functions $a(x)$ and $b(x)$ defined on X such that

$a(x)$ is upper semicontinuous and upper bounded,

$b(x)$ is lower semicontinuous and lower bounded,

$$a(x) \leqslant f(x) \leqslant b(x) \text{ for each } x \in X, \int\limits_X (b(x) - a(x))\,\mathrm{d}\mu < \eta.$$

For each $x \in X$ there exists a ball $U(x, \delta(x))$ such that

$$a(y) < a(x) + \eta \leqslant f(x) + \eta, \quad b(y) > b(x) - \eta \geqslant f(x) - \eta \quad \text{for each } y \in U(x, \delta(x)).$$

For each δ-fine partition in $X, P = \{(\theta_i, x_i)\}_i$ the following relations hold:

$$\begin{aligned}\int\limits_X a\theta_i \,\mathrm{d}\mu - \eta\int\limits_X \theta_i \,\mathrm{d}\mu &\leqslant \int\limits_X f(x_i)\cdot\theta_i \,\mathrm{d}\mu + \eta\int\limits_X \theta_i \,\mathrm{d}\mu - \eta\int\limits_X \theta_i \,\mathrm{d}\mu \\ &= \int\limits_X f(x_i)\cdot\theta_i \,\mathrm{d}\mu \leqslant \int\limits_X b\theta_i \,\mathrm{d}\mu + \eta\int\limits_X \theta_i \,\mathrm{d}\mu,\end{aligned}$$

$$\int\limits_X a\cdot\theta_i \,\mathrm{d}\mu \leqslant \int\limits_X f\theta_i \,\mathrm{d}\mu \leqslant \int\limits_X b\theta_i \,\mathrm{d}\mu$$

Since $\sum_i \theta_i(x) = 1$ a.e in X, we have

$$\Big|\sum_i \Big(\int_X f(x_i)\cdot\theta_i \,\mathrm{d}\mu - \int_X f\theta_i \,\mathrm{d}\mu\Big)\Big| \leqslant \Big|\sum_i \Big(\int_X b\theta_i \,\mathrm{d}\mu + \eta\int_X \theta_i \,\mathrm{d}\mu - \int_X a\theta_i \,\mathrm{d}\mu\Big)\Big|$$
$$= \sum_i \Big(\int_X (b-a)\cdot\theta_i \,\mathrm{d}\mu + \eta\int_X \theta_i \,\mathrm{d}\mu\Big) \leqslant \int_X (b-a)\,\mathrm{d}\mu + \eta\int_X \mathrm{d}\mu < \eta + \eta\cdot\mu(X) = \varepsilon.$$

□

Proposition 3.2. *If f is μ-measurable and μ-integrable, then f is* PU*-integrable and we have* (PU) $\int_X f = \int_X f\,\mathrm{d}\mu$.

Proof. If $\varepsilon > 0$, we take $\delta\colon X \longrightarrow \mathbb{R}^+$ as in the previous proposition and let $P = \{(\theta_i, x_i)\}_i$ be a δ-fine partition in X. By Prop. 3.1, we have

$$\Big|\sum_i f(x_i)\cdot\int_X \theta_i \,\mathrm{d}\mu - \int_X f\,\mathrm{d}\mu\Big| = \Big|\sum_i f(x_i)\cdot\int_X \theta_i \,\mathrm{d}\mu - \sum_i\int_X f\theta_i \,\mathrm{d}\mu\Big|$$
$$= \Big|\sum_i \Big(f(x_i)\cdot\int_X \theta_i \,\mathrm{d}\mu - \int_X f\theta_i \,\mathrm{d}\mu\Big)\Big| < \varepsilon,$$

from which, recalling Definition 3, the proposition follows. □

4. Equivalence between the PU-integral and the μ-integral

In [7] the Davies-McShane integral is defined in an abstract measure space and it is proved that it is equivalent to μ-integral for each μ-measurable function.

In this section, we suppose that $(X, \mathcal{M}, \mu)$ is a topological compact measure space, with a finite non atomic measure μ and with topology $T \subset \mathcal{M}$. A real function f on X is said to be Davies-Mc-Shane integrable if there exists a number I with the property that given $\varepsilon > 0$, there is a function $G\colon X \longrightarrow T$ where $G(x)$ is an open neighborhood of x, such that for every finite family $\{(I_i, x_i)\}_i$ with $I_i \in \mathcal{M}$ and $I_i \subset G(x_i)$, $\mu(X - \bigcup_i I_i) = 0$ and $\mu(I_i \cap I_j) = 0$ for $i \neq j$, we have

$$\Big|\sum_i f(x_i)\cdot\mu(I_i) - I\Big| < \varepsilon,$$

and in this case we write $I =$(D-McSh)$\int_X f$.

Proposition 4.1. *If f is* PU-*integrable then f is (D-McSh)-integrable and the two integrals coincide.*

P r o o f. Given $\varepsilon > 0$, we find the positive function δ from Definition 3 and set $G(x) = U(x, \delta(x))$. If $\{(I_i, x_i)\}_{i=1}^{h}$ is a family with $I_i \in \mathcal{M}$, $I_i \subset U(x_i, \delta(x_i))$, $\mu(I_i \cap I_j) = 0$ for $i \neq j$ and $\mu(X - \bigcup_{i=1}^{h} I_i) = 0$ then the family $\{(\chi_{I_i}, x_i)_{i=1}^{h}, x_i \in X\}$ is a δ-fine partition in X . Therefore we have

$$\varepsilon > \left| \sum_{i=1}^{h} f(x_i) \cdot \int_X \chi_{I_i} \,\mathrm{d}\mu - (\mathrm{PU}) \int_X f \right| = \left| \sum_{i=1}^{h} f(x_i) \cdot \mu(I_i) - (\mathrm{PU}) \int_X f \right|.$$

□

Proposition 4.2. *If f is* PU-*integrable then f is μ-integrable and* $(\mathrm{PU}) \int_X f = \int_X f \,\mathrm{d}\mu$.

P r o o f. This proposition is a consequence of Prop. 2.4, Prop. 4.1, [7] and Prop. 3.2. □

R e m a r k. In [7] the μ-measurability of D-McSh integrable function is left an open problem. Prop. 2.4 and Prop. 4.1 give an affirmative answer to this question if we modify the definition of PU-integral and use only functions $\theta_i = \chi_{I_i}$ with $I_i \in \mathcal{M}$.

5. Example of a noneuclidean compact metric space

In this section we give an example of a noneuclidean compact metric space, satisfying the conditions α, β, γ of this paper.

Let us consider the space $E^{(\omega)}$ of the infinite strings from the alphabet $E = \{0, 1\}$ (cf. [4]).

Fixing two symbols, say 0 and 1, we consider finite strings made up of these elements. For example 100010. The set $E = \{0, 1\}$ is called the alphabet of the strings. The number of symbols in a finite string α is called the *length* of the string and denoted $|\alpha|$. By convention we say that there is a unique string of length 0 called *empty* string and denoted by Λ. The string 100010 cited above has the length 6.

We write $E^{(n)}$ for the set of all the strings of length n and we set

$$E^{(*)} = E^{(0)} \cup E^{(1)} \cup E^{(2)} \cup \ldots \cup E^{(n)} \cup \ldots$$

If α and β are two strings, we may form the string $\alpha\beta$, called the concatenation of α and β, by listing the symbols of string α followed by the symbols of the string

β. For example if $\alpha = 010, \beta = 10101$ then $\alpha\beta = 01010101$. If $|\alpha| > n$ we write $\alpha|n$ for the initial segment of α of length n.

Now, let $E^{(\omega)}$ be the set of infinite strings of the alphabet E. We define in $E^{(\omega)}$ a metric ϱ as follows:

if $\beta, \gamma \in E^{(\omega)}$ and $\beta = \alpha\beta'$, $\gamma = \alpha\gamma'$, where $\alpha \in E^{(*)}$ is the longest common initial string and $\beta', \gamma' \in E^{(\omega)}$, we set $\varrho(\beta, \gamma) = (\frac{1}{2})^{|\alpha|}$ and $\varrho(\beta, \beta) = 0$. If $\alpha = \Lambda$ then we set $\varrho(\beta, \gamma) = 1$.

To verify the triangle inequality it is sufficient to verify the so called *ultra-triangle inequality*, that is:

if β, γ, σ in $E^{(\omega)}$ then $\varrho(\beta, \gamma) \leqslant \max[\varrho(\beta, \sigma), \varrho(\gamma, \sigma)]$.

The last inequality allows us to say that the space $E^{(\omega)}$ is not euclidean. It is called an *ultrametric space*.

We suppose that β, γ, σ are all different and $\varrho(\beta, \sigma) = (\frac{1}{2})^n < 1$ and $\varrho(\sigma, \gamma) = (\frac{1}{2})^m < 1$. If we set $k = \min\{n, m\}$, then β and γ have at least a common initial segment of length k, and $\varrho(\beta, \gamma) \leqslant (\frac{1}{2})^k = \max\{(\frac{1}{2})^n, (\frac{1}{2})^m\} = \max\{\varrho(\beta, \sigma), \varrho(\sigma, \gamma)\}$.

The geometric and topological properties of an ultrametric space are quite different from those of a euclidean space. Let's give some properties:

1) every triangle is isosceles,

2) the ball $U(x, r)$ has diameter at most r and each of its points is a center,

3) every ball is a closed and an open set (clopen set), so its boundary is empty.

It is not difficult to verify that $E^{(\omega)}$ is a complete, separable and compact space (cf. [4]). It is important to observe that the set $[\alpha] = \{\beta \in E^{(\omega)} \colon \beta = \alpha\beta'$ for some $\beta'\}$ is an open set, and, given any ball $U(x, r)$, there exists $\alpha \in E^{(*)}$ such that $[\alpha] = U(x, r)$.

We define a measure on $E^{(\omega)}$. On $\mathcal{A} = \{[\alpha], \alpha \in E^{(*)}\}$ we define the set function $m_{\frac{1}{2}}([\alpha]) = (\frac{1}{2})^{|\alpha|}$ and extend $m_{\frac{1}{2}}$ onto X in the following way:

$$m^*_{\frac{1}{2}}(A) = \inf_{\cup_i[\alpha_i]\supset A} \left\{ m_{\frac{1}{2}}\left(\sum_{i=1}^{\infty} [\alpha_i] \right) \right\} \text{ for each } A \subset X.$$

In [4], p. 142, it was proved that $m^*_{\frac{1}{2}}$ is a metric outer measure that is an outer measure such that if $A, B \subset X$ and $\varrho(A, B) > 0$ then $m^*_{\frac{1}{2}}(A \cup B) = m^*_{\frac{1}{2}}(A) + m^*_{\frac{1}{2}}(B)$.

Let $\mathcal{M}_{\frac{1}{2}}$ be the σ-algebra of all $m^*_{\frac{1}{2}}$-measurable sets in Caratheodory sense, and put $\overline{m}_{\frac{1}{2}}(M) = m^*_{\frac{1}{2}}(M)$ for each $M \in \mathcal{M}_{\frac{1}{2}}$.

In [4], p. 138, it was proved that the open sets are $\overline{m}_{\frac{1}{2}}$-measurable, so the condition α of the Preliminaries is true. Condition γ is a consequence of the metric properties, so the condition β only must be proved.

Proposition 5.1. *For each $k \in \mathbb{R}^+$ and for every $x \in E^{(\omega)}$ we have*

$$\frac{\overline{m}_{\frac{1}{2}}(U(x,2k))}{\overline{m}_{\frac{1}{2}}(U(x,k))} = 2.$$

Proof. Observe that for each $\alpha, \beta \in E^{(\omega)}$ we have $\varrho(\alpha,\beta) \leqslant 1$ so that $2k \leqslant 1$ and $k \leqslant \frac{1}{2}$. Given $k > 0$ there exist $\gamma', \gamma \in E^{(*)}$ such that $U(x,k) = [\gamma]$ and $U(x,2k) = [\gamma']$. Let $h = \min\{t \in \mathbb{N} | (\frac{1}{2})^t \leqslant k\}$ and $h_1 = \min\{t' \in \mathbb{N} | (\frac{1}{2})^{t'} \leqslant 2k\}$. So the inequalities $h \geqslant \log_{\frac{1}{2}} k$ and $h_1 \geqslant \log_{\frac{1}{2}} 2k$ hold.

Then

$$h = \begin{cases} [\log_{\frac{1}{2}} k] + 1 & \text{if } \log_{\frac{1}{2}} k \notin \mathbb{N} \\ \log_{\frac{1}{2}} k & \text{if } \log_{\frac{1}{2}} k \in \mathbb{N} \end{cases}$$

and

$$h_1 = \begin{cases} [\log_{\frac{1}{2}} 2k] + 1 & \text{if } \log_{\frac{1}{2}} 2k \notin \mathbb{N} \\ \log_{\frac{1}{2}} 2k & \text{if } \log_{\frac{1}{2}} 2k \in \mathbb{N}, \end{cases}$$

where $[a]$ denotes the integral part of a.

Note that

$$\frac{\overline{m}_{\frac{1}{2}}(U(x,2k))}{\overline{m}_{\frac{1}{2}}(U(x,k))} = \frac{(\frac{1}{2})^{h_1}}{(\frac{1}{2})^{h}} = \left(\frac{1}{2}\right)^{h_1 - h}$$

but

$$\begin{aligned} h_1 - h &= [\log_{\frac{1}{2}} 2k] + 1 - [\log_{\frac{1}{2}} k] - 1 = [\log_{\frac{1}{2}} 2k] - [\log_{\frac{1}{2}} k] \\ &= [\log_{\frac{1}{2}} 2 + \log_{\frac{1}{2}} k] - [\log_{\frac{1}{2}} k] = [-1 + \log_{\frac{1}{2}} k] - [\log_{\frac{1}{2}} k] \\ &= -1 + [\log_{\frac{1}{2}} k] - [\log_{\frac{1}{2}} k] = -1 \quad \text{if } \log_{\frac{1}{2}} k \notin \mathbb{N}, \\ h_1 - h &= \log_{\frac{1}{2}} 2k - \log_{\frac{1}{2}} k = -1 \quad \text{if } \log_{\frac{1}{2}} k \in \mathbb{N}. \end{aligned}$$

□

References

[1] *A. M. Bruckner*: Differentiation of integrals. Amer. Math. Monthly, No. 9 Supplement *78* (1971), 1–51.

[2] *D. Caponetti and V. Marraffa*: An integral in the real line defined by BV partitions of unity. Atti Semin. Mat. Fis., Univ. Modena *42* (1994), 69–82.

[3] *R. O. Davies and Z. Schuss*: A proof that Henstock's integral includes Lebesgue's. J. London Math. Soc. (2) *2* (1970), 561-562.

[4] *G. A. Edgar*: Measure, topology and fractal geometry. Springer-Verlag, 1990.

[5] *H. Hahn and A. Rosenthal*: Set functions. Albuquerque, N. Mexico, 1948.

[6] *R. Henstock*: The general theory of integration. Clarendon Press, Oxford, 1991.

[7] *R. Henstock*: Measure spaces and division spaces. Real Anal. Exchange *19(1)* (1993/94), 121–128.

[8] *J. Jarník and J. Kurzweil*: A non absolutely convergent integral which admits transformation and can be used for integration on manifolds. Czechoslovak Math. J. *35(110)* (1985), 116–139.

[9] *J. Jarník and J. Kurzweil*: A new and more powerful concept of the PU-integral. Czechoslovak Math. J. *38(113)* (1988), 8–48.

[10] *J. Kurzweil, J. Mawhin and W. Pfeffer*: An integral defined by approximating BV partitions of unity. Czechoslovak Math. J. *41(116)* (1991), 695–712.

[11] *W. F. Pfeffer*: The divergence theorem. Trans. Amer. Math. Soc. *295(2)* (1986), 665–685.

[12]] *W. F. Pfeffer and Wei-Chi Yang*: A note on conditionally convergent integrals. Real Anal. Exchange *17* (1991/92), 815–819.

[13] *W. F. Pfeffer*: The Riemann approach to integration. Cambridge University Press, 1993.

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