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**A NOTE ON THE RATE OF COMPLETE CONVERGENCE  
FOR WEIGHTED SUMS OF ARRAYS OF BANACH  
SPACE VALUED RANDOM ELEMENTS**

(submitted by D. Mushtari)

ABSTRACT. We obtain complete convergence results for arrays of row-wise independent Banach space valued random elements. In the main result no assumptions are made concerning the geometry of the underlying Banach space. As corollaries we obtain results on complete convergence in stable type  $p$  Banach spaces.

1. INTRODUCTION.

The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows. A sequence of random variables  $\{U_n, n \geq 1\}$  is said to *converge completely* to a constant  $c$  if  $\sum_{n=1}^{\infty} P\{|U_n - c| > \varepsilon\} < \infty$  for all  $\varepsilon > 0$ . By the Borel-Cantelli lemma, this implies  $U_n \rightarrow c$  almost surely (a.s.) and the converse implication is true if the  $\{U_n, n \geq 1\}$  are independent. Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite.

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This result has been generalized and extended in several directions (see Pruitt (1966), Rohatgi (1971), Hu, Moricz, and Taylor (1989), Gut (1992), Wang, Rao, and Yang (1993), Kuczmaszewska and Szynal (1994), Sung (1997), Hu, Rosalsky, Szynal, and Volodin (1999), Hu, Li, Rosalsky and Volodin (2002), and Ahmed, Giuliano Antonini, and Volodin (2002) among others). Some of these articles concern a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Hu, Rosalsky, Szynal, and Volodin (1999) presented a general result (cf. Theorem 1.1 below) establishing complete convergence for the row sums of an array of rowwise independent but not necessarily identically distributed Banach space valued random elements. Their result also specified the corresponding rate of convergence. The Hu, Rosalsky, Szynal, and Volodin (1999) result unifies and extends previously obtained results in the literature in that many of them (for example, results of Hsu and Robbins (1947), Hu, Li, Rosalsky, and Volodin (2002), Hu, Moricz, and Taylor (1989), Gut (1992), Kuczmaszewska and Szynal (1994), Pruitt (1966), Rohatgi (1971), Sung (1997), and Wang, Rao, and Yang (1993)) follow from it.

In the following we assume that  $\{V_{nk}, k \geq 1, n \geq 1\}$  is an array of rowwise independent random elements in a separable real Banach space and  $\{a_{nk}, k \geq 1, n \geq 1\}$  is an array of constants. Denote

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk}.$$

In the next theorem the weights  $a_{nk}$  are built into the array (that is,  $a_{nk} = 1$  for all  $k$  and  $n$ ).

**Theorem 1.1** (Hu, Rosalsky, Szynal, and Volodin (1999)). *Let  $\{c_n, n \geq 1\}$  be a sequence of positive constants. Suppose that*

$$(1) \quad \sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty} P\{\|V_{nk}\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0,$$

$$(2) \quad \sum_{n=1}^{\infty} c_n \left( \sum_{k=1}^{\infty} E\|V_{nk}\|^q \right)^J < \infty \text{ for some } 0 < q \leq 2 \text{ and } J \geq 2,$$

$$\sum_{k=1}^{\infty} V_{nk} \xrightarrow{P} 0, \quad \text{and}$$

(3) if  $\liminf_{n \rightarrow \infty} c_n = 0$ , then  $\sum_{k=1}^{\infty} P\{\|V_{nk}\| > \delta\} = o(1)$  for some  $\delta > 0$ .

Then

$$\sum_{n=1}^{\infty} c_n P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

It is implicitly assumed in Theorem 1.1 that the series  $S_n$  converges a.s.

The article Hu, Li, Rosalsky, and Volodin (2002) is devoted to presenting applications of Theorem 1.1 to obtain new complete convergence results. Theorem 1.2 generalizes results of Hsu and Robbins (1947), Hu, Moricz, and Taylor (1989), Gut (1992), Kuczmaszewska and Szynal (1994), Pruitt (1966), Rohatgi (1971), Sung (1997), and Wang, Rao, and Yang (1993) in three directions, namely:

- (i) Banach space valued random elements instead of random variables are considered.
- (ii) An array rather than a sequence is considered.
- (iii) The rate of convergence is obtained.

**Theorem 1.2.** (Hu, Li, Rosalsky, and Volodin (2002)). *Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  is stochastically dominated by a random variable  $X$ . Assume that*

$$\sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0, \text{ and}$$

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^{\alpha}) \text{ for some } \alpha \in [0, \gamma).$$

If

$$E|X|^{1+\frac{1+\alpha+\beta}{\gamma}} < \infty \text{ for some } \beta \in (-1, \gamma - \alpha - 1], \text{ and}$$

$$S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

The proof of Theorem 1.2 is rather complicated once it uses the Stieltjes integral techniques, summation by parts lemma and so on. The initial

objective of an investigation resulted in the paper Ahmed, Giuliano Antonini, and Volodin (2002) was only to find a simpler proof. But it appears that they were able to establish a more general result and with simpler proof. The result presented in Theorem 1.3 below is more general than the main result of Hu, Li, Rosalsky, and Volodin (2002), since rates of convergence for moving averages can be established, which cannot be proved using Theorem 1.2.

**Theorem 1.3.** *Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  is stochastically dominated by a random variable  $X$ . Assume that*

$$\sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0,$$

and

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^{\alpha}) \text{ for some } \alpha < \gamma.$$

Let  $\beta$  be such that  $\alpha + \beta \neq -1$  and fix  $\delta > 0$  such that  $\frac{\alpha}{\gamma} + 1 < \delta \leq 2$ . If

$$E|X|^{\nu} < \infty \text{ where } \nu = \max\left(1 + \frac{1 + \alpha + \beta}{\gamma}, \delta\right),$$

and

$$S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

If we assume that the Banach space has the geometric property of being of an appropriate stable type, then we can drop the condition that  $S_n$  converges in probability. For this we need the following result.

**Theorem 1.4** (Adler, Ordóñez Cabrera, Rosalsky, and Volodin (1999)) *Assume that underlying Banach space has stable type  $p, 1 < p < 2$ . Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  has mean 0 and is stochastically dominated by a random variable  $X$ . Moreover, assume that*

$$\sup_{n \geq 1} \sum_{k=1}^{\infty} |a_{nk}|^p < \infty, \text{ and } \sup_{k \geq 1} |a_{nk}| = o(1).$$

If  $\lim_{t \rightarrow \infty} t^p P\{|X| > t\} = 0$ , then  $S_n \xrightarrow{P} 0$ .

In the present paper we generalize Theorem 1.3 in two directions. First of all, instead of condition

$$\sum_{k=1}^{\infty} |a_{nk}| = O(n^\alpha) \text{ for some } \alpha < \gamma,$$

in Theorem 3.1 we consider the condition

$$\sum_{k=1}^{\infty} |a_{nk}|^\theta = O(n^\alpha) \text{ for some suitable choice of the parameters.}$$

Secondary, in Theorem 3.2 we deal with the special case  $\alpha + \beta = -1$ .

The plan of the paper is as follows. In Section 2, we recall some well known definitions pertaining to the current work. In Section 3, we apply Theorem 1.1 to obtain complete convergence for row sums with corresponding rates of convergence (Theorems 3.1 and 3.2). As in Theorems 1.1, 1.2, and 1.3, in Theorems 3.1 and 3.2 no assumptions are made concerning the geometry of the underlying Banach space. We use the geometrical assumption (type  $p$ ) on the underlying Banach space in Theorems 3.3 and 3.4.

## 2. PRELIMINARIES

Let  $(\Omega, F, P)$  be a probability space and let  $B$  be a separable real Banach space with norm  $\|\cdot\|$ . A *random element* is defined to be an  $F$ -measurable mapping of  $\Omega$  into  $B$  equipped with the Borel  $\sigma$ -algebra (that is, the  $\sigma$ -algebra generated by the open sets determined by  $\|\cdot\|$ ). A detailed account of basic properties of random elements in separable real Banach spaces can be found in Taylor (1978).

Let  $\{V_{nk}, k \geq 1, n \geq 1\}$  be an array of rowwise independent, but not necessarily identically distributed, random elements taking values in  $B$ . The array of random elements  $\{V_{nk}, k \geq 1, n \geq 1\}$  is said to be *stochastically dominated* by a random variable  $X$  if there exists a constant  $D < \infty$  such that  $P\{\|V_{nk}\| > x\} \leq DP\{|DX| > x\}$  for all  $x > 0$  and for all  $n \geq 1$  and  $k \geq 1$ . Let  $\{a_{nk}, k \geq 1, n \geq 1\}$  be an array of constants (called *weights*) and consider the sequence of *weighted sums*  $S_n \equiv \sum_{k=1}^{\infty} a_{nk}V_{nk}, n \geq 1$ .

Let  $0 < p \leq 2$  and let  $\{\theta_n, n \geq 1\}$  be independent and identically distributed stable random variables each with characteristic function  $\phi(t) = \exp\{-|t|^p\}, -\infty < t < \infty$ . The real separable Banach space  $B$  is said to be of *stable type  $p$*  if  $\sum_{n=1}^{\infty} \theta_n v_n$  converges almost surely whenever  $\{v_n, n \geq 1\} \subseteq B$  with  $\sum_{n=1}^{\infty} \|v_n\|^p < \infty$ . Equivalent characterizations of a Banach space being of stable type  $p$ , properties of stable

type  $p$  Banach spaces, as well as various relationships between the conditions “Rademacher type  $p$ ” and “stable type  $p$ ” may be found in Adler, Ordóñez Cabrera, Rosalsky, and Volodin (1999), Section 2.

Finally, the symbol  $C$  denotes throughout a generic constant ( $0 < C < \infty$ ) which is not necessarily the same one in each appearance, for  $x \geq 0$  the symbol  $[x]$  denotes the greatest integer in  $x$ , and for a finite set  $A$  the symbol  $\#A$  denotes the number of elements in the set  $A$ .

### 3. MAIN RESULTS

With the preliminaries accounted for, the main result may now be established. For the case  $\theta > 1$  it is implicitly assumed that the series  $S_n$  converges a.s. (cf. Remark (i) after Theorem 3.2)

**Theorem 3.1.** *Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  is stochastically dominated by a random variable  $X$ . Assume that*

$$(4) \quad \sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0, \text{ and}$$

$$(5) \quad \sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha})$$

for some  $0 < \theta \leq 2$  and any  $\alpha$  such that  $\theta + \frac{\alpha}{\gamma} < 2$ .

Let  $\beta$  be such that  $\alpha + \beta \neq -1$  and fix  $\delta > \theta$  such that  $\frac{\alpha}{\gamma} + \theta < \delta \leq 2$ . If

$$(6) \quad E|X|^{\nu} < \infty \text{ where } \nu = \max\left(\theta + \frac{1 + \alpha + \beta}{\gamma}, \delta\right), \text{ and}$$

$$S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

**Proof.** Note that the result is of interest only for  $\beta \geq -1$ . In addition, since  $\delta > \theta + \frac{\alpha}{\gamma}$ , we have that  $\alpha - \gamma(\delta - \theta) < 0$ .

Let  $c_n = n^{\beta}$ ,  $n \geq 1$ . Then we only need to verify that the conditions (1), (2), and (3) (if  $\beta < 0$ ) of Theorem 1.1 hold with  $a_{nk}V_{nk}$  playing the role of  $V_{nk}$  in the formulation of that theorem. Without loss of generality,  $\varepsilon$  can be taken to be 1 and in view of (4) and (5) we can assume that

$$(7) \quad \sup_{k \geq 1} |a_{nk}| = n^{-\gamma},$$

and

$$(8) \quad \sum_{k=1}^{\infty} |a_{nk}|^{\theta} = n^{\alpha}.$$

Now to verify (1), let  $b_{nj} = 1/|a_{nj}|$ , ( where  $1/0 = \infty$ ),

$$I_{nj} = \{k : (nj)^{\gamma} \leq b_{nk} < (n(j+1))^{\gamma}\} \text{ and}$$

$$J_{nj} = \cup_{k=1}^j I_{nk} = \{k : b_{nk} < (n(j+1))^{\gamma}\}, j \geq 1, n \geq 1.$$

Noting that by (7)  $\cup_{j \geq 1} I_{nj} = \mathbf{N}$  for all  $n \geq 1$ , where  $\mathbf{N}$  is the set of positive integers.

Note that for all  $m \geq 1, n \geq 1$  by (8)

$$n^{\alpha} \geq \sum_{k \in J_{nm}} |a_{nk}|^{\theta} = \sum_{k \in J_{nm}} 1/|b_{nk}| \geq \frac{\#J_{nm}}{(n(m+1))^{\gamma}},$$

and hence

$$\#J_{nm} \leq n^{\alpha+\gamma\theta}(m+1)^{\gamma\theta}.$$

Observe that since the sets  $I_{nj}, j \geq 1$  are disjoint, for any fixed  $n \geq 1$ , we have that

$$\sum_{j=1}^m \#I_{nj} = \#J_{nm} \leq n^{\alpha+\gamma\theta}(m+1)^{\gamma\theta}.$$

Therefore

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P\{\|a_{nk}V_{nk}\| \geq 1\} = \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P\{\|V_{nk}\| \geq b_{nk}\} \\
& \leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P\{|X| \geq b_{nk}\} \leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\#I_{nj}) P\{|X| \geq (nj)^{\gamma}\} \\
& \leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\#I_{nj}) P\{|X|^{1/\gamma} \geq nj\} \\
& \leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{j=1}^{\infty} (\#I_{nj}) \sum_{k=nj}^{\infty} P\{k \leq |X|^{1/\gamma} < k+1\} \\
& = C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=n}^{\infty} P\{k \leq |X|^{1/\gamma} < k+1\} \sum_{j=1}^{\lfloor k/n \rfloor} (\#I_{nj}) \\
& \leq C \sum_{n=1}^{\infty} n^{\beta} \sum_{k=n}^{\infty} P\{k \leq |X|^{1/\gamma} < k+1\} n^{\alpha+\gamma} (\lfloor k/n \rfloor + 1)^{\gamma\theta} \\
& \leq C \sum_{n=1}^{\infty} n^{\beta} n^{\alpha+\gamma\theta} n^{-\gamma\theta} \sum_{k=n}^{\infty} k^{\gamma\theta} P\{k \leq |X|^{1/\gamma} < k+1\} \\
& \leq C \sum_{k=1}^{\infty} k^{\gamma\theta} P\{k \leq |X|^{1/\gamma} < k+1\} \sum_{n=1}^k n^{\alpha+\beta} \tag{*} \\
& \leq C \sum_{k=1}^{\infty} k^{\alpha+\beta+\gamma\theta+1} P\{k \leq |X|^{1/\gamma} < k+1\} \text{ because } \alpha + \beta \neq -1 \\
& \leq CE|X|^{\theta+\frac{\theta+\alpha+\beta}{\gamma}} < \infty \text{ by (6)}.
\end{aligned}$$

To verify (2), note that for any  $J > \frac{\beta+1}{\gamma(\delta-\theta)-\alpha}$

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta} \left( \sum_{k=1}^{\infty} E\|a_{nk}V_{nk}\|^{\delta} \right)^J = \sum_{n=1}^{\infty} n^{\beta} \left( \sum_{k=1}^{\infty} |a_{nk}|^{\delta} E\|V_{nk}\|^{\delta} \right)^J \\
& \leq \sum_{n=1}^{\infty} n^{\beta} \left( \sup_{k \geq 1} |a_{nk}|^{\delta-\theta} \sum_{k=1}^{\infty} |a_{nk}|^{\delta-\theta} E\|V_{nk}\|^{\delta} \right)^J \\
& \leq C \sum_{n=1}^{\infty} n^{\beta} \left( n^{-\gamma(\delta-\theta)} n^{\alpha} E|X|^{\delta} \right)^J \text{ (by (7), (8), and stochastic domination)} \\
& = C \sum_{n=1}^{\infty} n^{\beta+J(\alpha-\gamma(\delta-\theta))} < \infty.
\end{aligned}$$

Finally, to verify (3) if  $\beta < 0$ , note that for any  $\lambda > 0$

$$\begin{aligned} \sum_{k=1}^{\infty} P\{|a_{nk}V_{nk}| > \lambda\} &\leq \sum_{k=1}^{\infty} \lambda^{-\delta} E\|a_{nk}V_{nk}\|^{\delta} \text{ (by the Markov inequality)} \\ &\leq C \sup_{k \geq 1} |a_{nk}|^{\delta-\theta} \sum_{k=1}^{\infty} |a_{nk}|^{\theta} E\|V_{nk}\|^{\delta} \\ &\leq C n^{-\gamma(\delta-\theta)+\alpha} E|X|^{\delta} = o(1) \text{ (by (7), (8), and stochastic domination)}. \end{aligned}$$

For the special case  $\alpha + \beta = -1$  we can establish the following result.

**Theorem 3.2.** *Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  is stochastically dominated by a random variable  $X$ . Assume that*

$$\sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0,$$

and

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha}) \text{ for some } 0 < \theta \leq 2 \text{ and any } \alpha, \text{ such that } \theta + \frac{\alpha}{\gamma} < 2.$$

Let  $\beta = -1 - \alpha$  and fix  $\delta > \theta$  such that  $\frac{\alpha}{\gamma} + \theta < \delta \leq 2$ . If  $E|X|^{\delta} < \infty$  and

$$S_n \xrightarrow{P} 0,$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

**Proof.** Let  $c_n = n^{\beta}, n \geq 1$ . Then we only need to verify that the conditions (1), (2), and (3) (if  $\beta < 0$ ) of Theorem 1.1 hold with  $a_{nk}V_{nk}$  playing the role of  $V_{nk}$  in the formulation of that theorem. We mention that conditions (2) and (3) can be checked exactly in the same way as in the theorem above. While for condition (1) we need to make the following

changes right after the inequality (\*):

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\beta} \sum_{k=1}^{\infty} P\{\|a_{nk}V_{nk}\| \geq 1\} \\
& \leq C \sum_{k=1}^{\infty} k^{\gamma\theta} P\{k \leq |X|^{1/\gamma} < k+1\} \sum_{n=1}^k n^{\alpha+\beta} \\
& \leq C \sum_{k=1}^{\infty} k^{\gamma\theta} \log k P\{k \leq |X|^{1/\gamma} < k+1\} \text{ because } \alpha + \beta = -1 \\
& \leq CE|X|^{\theta} \log |X| \leq CE|X|^{\delta} < \infty \text{ and since } \delta > \theta.
\end{aligned}$$

**Remarks.** (i) We can verify that if  $\theta \leq 1$ , then it follows from the assumptions of Theorems 3.1 and 3.2 that the series  $S_n$  converges a.s. Note at the outset that the stochastic domination hypothesis ensures that  $E\|V_{nk}\| \leq CE|X|, k \geq 1, n \geq 1$  and hence for all  $n \geq 1$ , by the Beppo Levi theorem and the assumptions of Theorem 3.1

$$\begin{aligned}
E \left( \sum_{k=1}^{\infty} \|a_{nk}V_{nk}\| \right) &= \sum_{k=1}^{\infty} E\|a_{nk}V_{nk}\| \leq CE|X| \sum_{k=1}^{\infty} |a_{nk}| \\
&\leq C \sup_{k \geq 1} |a_{nk}|^{1-\theta} \sum_{k=1}^{\infty} |a_{nk}|^{\theta} \leq Cn^{\alpha-\gamma(1-\theta)} < \infty.
\end{aligned}$$

Thus for all  $n \geq 1, \sum_{k=1}^{\infty} \|a_{nk}V_{nk}\| < \infty$  a.s. and so for all  $n \geq 1$  and all  $K \geq 1$ ,

$$\begin{aligned}
& \sup_{L>K} \left\| \sum_{k=1}^L a_{nk}V_{nk} - \sum_{k=1}^K a_{nk}V_{nk} \right\| = \sup_{L>K} \left\| \sum_{k=K+1}^L a_{nk}V_{nk} \right\| \\
& \leq \sup_{L>K} \sum_{k=K+1}^L \|a_{nk}V_{nk}\| = \sum_{k=K+1}^{\infty} \|a_{nk}V_{nk}\| \xrightarrow{K \rightarrow \infty} 0 \text{ a.s.}
\end{aligned}$$

Thus for all  $n \geq 1$ , with probability 1,  $\{\sum_{k=1}^K a_{nk}V_{nk}, K \geq 1\}$  is a Cauchy sequence in  $X$  whence  $\sum_{k=1}^{\infty} a_{nk}V_{nk}$  converges a.s.

(ii) Take  $\theta = 1$  in order to obtain Theorem 1.3.

(iii) Since Theorem 3.1 is stronger than Theorem 1.3, which is, in its turn is stronger than Theorem 1.2, the results of the papers Hsu and Robbins (1947), Hu, Moricz, and Taylor (1989), Gut (1992), Wang, Rao, and Yang (1993), Kuczmaszewska and Szynal (1994), Pruitt (1966), Rohatgi (1971), and Sung (1997) follow from it in the same way as was proved in Hu, Li, Rosalsky, and Volodin (2002).

(iv) Of course, the conditions of Theorems 3.1 and 3.2 can be slightly generalized on the case  $\sup_{k \geq 1} |a_{nk}| = f(n)$  and  $\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = g(n)$  where  $f(n)$  and  $g(n)$  are regularly varying functions with indexes  $-\gamma$  and  $\alpha$ , correspondenly. The same is true for Theorems 3.3 and 3.4 below.

We can drop the convergence in probability condition if we assume that underling Banach space is of stable type.

**Theorem 3.3.** *Assume that underlying Banach space has stable type  $p, 1 < p < 2$ . Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  has mean zero and is stochastically dominated by a random variable  $X$ . Moreover, assume that*

$$\sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0,$$

and

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha}) \text{ for some } 0 < \theta \leq p \text{ and any } \alpha \text{ such that } \theta + \frac{\alpha}{\gamma} \leq p.$$

Let  $\beta$  be such that  $\alpha + \beta \neq -1$  and fix  $\delta > \theta$  such that  $\theta + \frac{\alpha}{\gamma} < \delta \leq 2$ . If

$$E|X|^{\nu} < \infty, \text{ where } \nu = \max\left(\theta + \frac{1 + \alpha + \beta}{\gamma}, \delta, p\right)$$

then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

**Proof.** We need to check that

$$S_n \equiv \sum_{k=1}^{\infty} a_{nk} V_{nk} \xrightarrow{P} 0.$$

To do this we apply Theorem 1.4. Since  $p \geq \theta + \frac{\alpha}{\gamma}$ , we have

$$\begin{aligned} \sup_{n \geq 1} \sum_{k=1}^{\infty} |a_{nk}|^p &= \sup_{n \geq 1} \sum_{k=1}^{\infty} |a_{nk}|^{p-\theta} |a_{nk}|^{\theta} \\ &= \sup_{n \geq 1} (\sup_{k \geq 1} |a_{nk}|)^{p-\theta} \sum_{k=1}^{\infty} |a_{nk}|^{\theta} \\ &\leq C \sup_{n \geq 1} n^{\alpha - \gamma(p-\theta)} < \infty. \end{aligned}$$

All other assumptions of Theorem 3.1 are obviously satisfied.

For the special case  $\alpha + \beta = -1$  we can establish the following result. The proof repeats the proof of Theorem 3.3 with the only difference being, that we need to refer to Theorem 3.2 instead of Theorem 3.1. Because of this we omit the proof.

**Theorem 3.4.** *Assume that underlying Banach space has stable type  $p$ ,  $1 < p < 2$ . Suppose that the array  $\{V_{nk}, k \geq 1, n \geq 1\}$  has mean zero and is stochastically dominated by a random variable  $X$ . Moreover, assume that*

$$\sup_{k \geq 1} |a_{nk}| = O(n^{-\gamma}) \text{ for some } \gamma > 0,$$

and

$$\sum_{k=1}^{\infty} |a_{nk}|^{\theta} = O(n^{\alpha}) \text{ for some } 0 < \theta \leq p \text{ and any } \alpha \text{ such that } \theta + \frac{\alpha}{\gamma} \leq p.$$

Let  $\beta = -1 - \alpha$  and fix  $\delta > \theta$  such that  $\theta + \frac{\alpha}{\gamma} < \delta \leq 2$ . If  $E|X|^{\max(p, \delta)} < \infty$  then

$$\sum_{n=1}^{\infty} n^{\beta} P\{\|S_n\| > \varepsilon\} < \infty \text{ for all } \varepsilon > 0.$$

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