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SOME PROPERTIES OF SOLUTIONS OF THE PSEUDO-PARABOLIC EQUATION

(submitted by F. Avkhadiev)

Abstract. In this paper we discuss properties of solutions for a class of pseudo-parabolic equation. Some results on the asymptotic behavior and monotonicity of support are established.

1. Introduction

In this paper, we investigate the pseudo-parabolic equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \quad x \in \Omega,\ p > 2, \tag{1.1}$$

with boundary condition

$$u\big|_{\partial\Omega} = 0, \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.3}$$

Here Ω is a bounded domain in R^N, $k > 0$ is the viscosity coefficient. The term $k\frac{\partial \Delta u}{\partial t}$ in (1.1) is interpreted as due to viscous relaxation effects, or viscosity; the well-known p-Laplacian equation is obtained by setting $k = 0$.

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Equation (1.1) arises as a regularization of the pseudo-parabolic equation

$$\frac{\partial u}{\partial t} - k\frac{\partial \Delta u}{\partial t} = \Delta u, \tag{1.4}$$

which arises in various physical phenomena. (1.4) can be assumed as a model for diffusion of fluids in fractured porous media [1, 2, 3], or as a model for heat conduction involving a thermodynamic temperature $\theta = u - k\Delta u$ and a conductive temperature u [4, 5].

To derive (1.4), B. D. Coleman, R. J. Duffin and V. J. Mizel considered a special kinematical situation, of nonsteady simple shearing flow [2]. In fact, when the influence of many factors, such as the molecular and ion effects, are considered, one has the nonlinear relation $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$ instead of Δu in the right-hand side of (1.4). Hence, we obtain (1.1).

During the past years, many authors have paid much attention to the equation (1.4), see [2, 3, 6, 7, 8, 9]. However, only a few papers are devoted to the pseudo-parabolic equation (1.1). It was Liu [10] who first studied the equation (1.1). With the use of the time discrete method, he proved the existence of weak solutions.

This paper is a further step in the study of the properties of solutions, we discuss asymptotic behavior of weak solutions and monotonicity of support of weak solutions. Our approach is based on the energy equality and comparison principle. For simplicity we set $k = 1$ in this paper.

2. Asymptotic Behavior

To investigate the asymptotic behavior of weak solutions, we need the following lemmas

Lemma 2.1. *The weak solutions u of the problem* (1.1)-(1.3)*, satisfy*

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx \quad & +\tfrac{1}{2}\textstyle\int_\Omega |u(x,t)|^2 dx - \\ & -\tfrac{1}{2}\textstyle\int_\Omega |\nabla u_0(x)|^2 dx - \tfrac{1}{2}\int_\Omega |u_0(x)|^2 dx = \\ & = -\iint_{Q_t} |\nabla u(x,t)|^p dx, \end{aligned} \tag{2.1}$$

where $Q_t = \Omega \times (0,t)$.

Proof. In the proof of Theorem 2.1 ([10]), we have

$$f(t) = \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \frac{1}{2}\int_\Omega |u(x,t)|^2 dx \in C([0,T]).$$

We consider the functionals

$$\Phi[v] = \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{1}{2}\int_\Omega |v|^2 dx,$$

it is easily seen that $\Phi[v]$ is convex functionals on $H_0^1(\Omega)$.

Hence, for any $\tau \in (0, T)$ and $h > 0$, we have

$$\Phi[u(\tau+h)] - \Phi[u(\tau)] \geq \langle u(\tau+h) - u(\tau), -\Delta u(x,\tau) + u(x,\tau)\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes inner product, and we used $\frac{\delta\Phi[v]}{\delta v} = -\Delta v + v$. For any fixed $t_1, t_2 \in [0, T], t_1 < t_2$, integrating the above inequality with respect to τ over (t_1, t_2), we have

$$\begin{aligned} &\int_{t_2}^{t_2+h} \Phi[u(\tau)]d\tau - \int_{t_1}^{t_1+h} \Phi[u(\tau)]d\tau \\ \geq \quad &\int_{t_1}^{t_2} \langle u(\tau+h) - u(\tau), -\Delta u + u\rangle d\tau. \end{aligned}$$

Multiply the both side of above equality by $\frac{1}{h}$, letting $h \to 0$, then

$$\Phi[u(t_2)] - \Phi[u(t_1)] \geq \int_{t_1}^{t_2} \langle \frac{\partial u}{\partial t}, -\Delta u + u\rangle d\tau.$$

Similarly, we have

$$\Phi[u(\tau)] - \Phi[u(\tau-h)] \leq \langle (u(\tau) - u(\tau-h)), -\Delta u + u\rangle.$$

Thus

$$\Phi[u(t_2)] - \Phi[u(t_1)] \leq \int_{t_1}^{t_2} \langle \frac{\partial u}{\partial t}, -\Delta u + u\rangle d\tau.$$

Hence

$$\Phi[u(t_2)] - \Phi[u(t_1)] = \int_{t_1}^{t_2} \langle \frac{\partial u}{\partial t}, -\Delta u + u\rangle d\tau.$$

Take $t_1 = 0, t_2 = t$. Since u satisfies the equation in the sense of distributions, we get

$$\begin{aligned} \Phi[u(t)] - \Phi[u(0)] &= \int_0^t \langle \frac{\partial u}{\partial t} - \frac{\partial \Delta u}{\partial t}, u(\tau)\rangle d\tau \\ &= \int_0^t \langle \operatorname{div}(|\nabla u|^{p-2}\nabla u), u(\tau)\rangle d\tau \\ &= -\int_0^t \langle |\nabla u|^{p-2}\nabla u, \nabla u(\tau)\rangle d\tau. \end{aligned}$$

The proof is complete.

We are now in a position to demonstrate the following theorem.

Theorem 2.1. *If u is a weak solution of the problem* (1.1)-(1.3), *and $p > 2$, then we have*

$$\int_\Omega |u(x,t)|^2 dx \leq \frac{1}{(C_1 t + C_2)^\alpha},$$

where $C_1, C_2, \alpha > 0$.

Proof. In (2.1), we have

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + & \tfrac{1}{2}\textstyle\int_\Omega |u(x,t)|^2 dx - \\ & -\tfrac{1}{2}\textstyle\int_\Omega |\nabla u_0(x)|^2 dx - \tfrac{1}{2}\int_\Omega |u_0(x)|^2 dx = \\ & = -\int_0^t \int_\Omega |\nabla u(x,t)|^p dx dt. \end{aligned} \tag{2.2}$$

Let $f(t) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u(x,t)|^2 dx + \dfrac{1}{2}\int_\Omega |u(x,t)|^2 dx$, then by (2.2) we have

$$f'(t) = -\int_\Omega |\nabla u(x,t)|^p dx \leq 0.$$

By the Poincaré inequality, we get

$$\int_\Omega |u|^2 dx \leq C \int_\Omega |\nabla u|^2 dx.$$

By the Hölder inequality and $u \in W_0^{1,p}(\Omega)$, we obtain

$$\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^2 dx \leq C \left(\int_\Omega |\nabla u|^p dx \right)^{2/p},$$

that is $f(t) \leq C|f'(t)|^{2/p}$. Again by$f'(t) \leq 0$, we have $f'(t) \leq -C f^{p/2}(t)$ hence

$$\int_\Omega |\nabla u(x,t)|^2 dx + \int_\Omega |u(x,t)|^2 dx \leq \frac{1}{(C_1 t + C_2)^\alpha},$$

where $\alpha > 0, C_i > 0, i = 1, 2$. That is

$$\int_\Omega |u(x,t)|^2 dx \leq \frac{1}{(C_1 t + C_2)^\alpha}, \quad \alpha > 0, C_i > 0, i = 1, 2.$$

The proof is complete.

3. Monotonicity of Support of Weak Solutions

In this section we study the problem (1.1)-(1.3) for one-dimensional case, i.e. we consider the following problem

$$\frac{\partial u}{\partial t} - \frac{\partial D^2 u}{\partial t} = \frac{\partial}{\partial x}\left(\left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right), \quad x \in I = (-1, 1), \tag{3.1}$$

$$u(\pm 1, t) = 0, \quad u(x, 0) = u_0(x), \tag{3.2}$$

where $p > 2$ is a given real number, and $u_0(x)$ is a nonzero nonnegative continuous function in I with $u_0(\pm 1) = 0$. We are going to prove the following theorem

Theorem 3.1. *Let u be a nonnegative weak solution of the problem* (3.1)-(3.2)*, and $p > 2$, then*

$$\operatorname{supp} u(\cdot, s) \subset \operatorname{supp} u(\cdot, t)$$

for all s, t with $0 < s < t$.

The monotonicity of support of weak solutions for p-Laplacian equation has been obtained by Yuan[11]. To prove the theorem, we need the following lemmas

Lemma 3.1. (**Comparison principle**) *Let u be a weak solutions of* (3.1)-(3.2)*. If v satisfies*

$$\frac{\partial v}{\partial t} - \frac{\partial D^2 v}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial v}{\partial x}\right|^{p-2}\frac{\partial v}{\partial x}\right),$$

in the sense of distributions, and

$$v(x,0) \le u(x,0), \qquad Dv(x,0) \le Du(x,0),$$
$$v(\pm 1, t) \le u(\pm 1, t),$$

then we have

$$v(x,t) \le u(x,t), \quad \textit{for all } \ (x,t) \in Q_T = I \times (0,T).$$

Proof. By the definition of weak solution, in [10] we have for $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega (u_h(x,\tau))_\tau \varphi(x)dx + \int_\Omega ((\nabla u)_h(x,\tau))_\tau \varphi(x)dx$$
$$+ \int_\Omega (|\nabla u|^{p-2}\nabla u)_h(x,\tau)\nabla\varphi dx = 0,$$

where

$$u_h(x,t) = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} u(\cdot,\tau)d\tau, & t \in (0, T-h), \\ 0, & t > T-h. \end{cases}$$

Hence, we have

$$\int_{-1}^{1} (v(x,\tau) - u(x,\tau))_{h\tau}\varphi(x)dx + \int_{-1}^{1} ((Dv - Du)_h(x,\tau))_\tau D\varphi(x)dx$$
$$+ \int_{-1}^{1} (|Dv|^{p-2}Dv - |Du|^{p-2}Du)_h(x,\tau)D\varphi dx = 0.$$

For a fixed τ, we take $\varphi(x) = [(v-u)_h]_+$. By the property of the Steklov mean value and noting that $v(\pm 1,t) \le u(\pm 1,t)$, we see that $\varphi(x) = [(v-u)_h]_+ \in W_0^{1,p}(\Omega)$. Substituting this function into the above integral equality, we obtain

$$\begin{aligned}
&\int_{-1}^{1} (v(x,\tau)-u(x,\tau))_{h\tau}[(v-u)_h]_+ dx \\
&+ \int_{-1}^{1} D(v(x,\tau)-u(x,\tau))_{h\tau} D[(v-u)_h]_+ dx \\
&= -\int_{-1}^{1} [(|Dv|^{p-2}Dv - |Du|^{p-2}Du)_h](x,\tau) D[(v-u)_h]_+ dx.
\end{aligned}$$

Integrating the above equality with respect to τ over $(0,t)$,

$$\begin{aligned}
&\int_{-1}^{1} [(v-u)_h]_+^2(x,t)dx + \int_{-1}^{1} |D[(v-u)_h]_+|^2(x,t)dx \\
&- \int_{-1}^{1} [(v-u)_h]_+^2(x,0)dx - \int_{-1}^{1} |D[(v-u)_h]_+|^2(x,0)dx \\
= &-\int_{-1}^{1} [(|Dv|^{p-2}Dv - |Du|^{p-2}Du)_h](x,\tau) D[(v-u)_h]_+ dx,
\end{aligned} \tag{3.3}$$

It is easily seen that

$$\lim_{h\to 0} \int_{-1}^{1} [(v-u)_h]_+(x,0)dx = 0,$$

and

$$\lim_{h\to 0} \int_{-1}^{1} [D(v-u)_h]_+(x,0)dx = 0.$$

Letting $h \to 0$ in (3.3), we have

$$\int_{-1}^{1} |(v-u)_+|^2(x,t)dx + \int_{-1}^{1} |D(v-u)_+|^2(x,t)dx \le 0,$$

that is $\int_{-1}^{1} |(v-u)_+|^2 dx = 0$; therefore, $v \le u$. The proof is complete.

Lemma 3.2. *Let u be a nonnegative weak solution of the problem* (1.1)-(1.3)*, If $p > 2$, then*

$$\frac{\partial u}{\partial t} \ge -\frac{u}{(p-2)t}$$

in the sense of distributions.

Proof. Denote

$$u_r(x,t) = r u(x, r^{p-2}t), \text{ for all } \quad (x,t) \in Q_T, \quad r \in (\frac{1}{2}, 1).$$

Clearly, u_r is a weak solution of the equation (3.1) with the following initial-boundary condition

$$u_r(x,0) = ru_0(x), \quad Du_r(x,0) = rDu_0(x), \tag{3.4}$$

$$u_r(\pm 1, t) = 0. \tag{3.5}$$

Noting $r \in (0, \frac{1}{2})$, and using (3.2) and (3.4), (3.5) we get

$$u_r(x,0) \le u_0(x), \quad Du_r(x,0) \le Du_0(x), \tag{3.6}$$

$$u_r(\pm 1, t) = u(\pm 1, t). \tag{3.7}$$

Applying the comparison principle, we have

$$u_r(x,t) \le u(x,t). \tag{3.8}$$

For $p > 2$, by (3.8), we obtain

$$\frac{[u(x,\lambda t)]^{p-2} - [u(x,t)]^{p-2}}{\lambda t - t} \ge \frac{(1/\lambda - 1)[u(x,t)]^{p-2}}{\lambda t - t}$$

where $\lambda = r^{p-2}$. Letting $\lambda \to 1^-$, we get

$$\frac{\partial}{\partial t}[u(x,t)]^{p-2} \ge -\frac{1}{t}[u(x,t)]^{p-2},$$

in the distribution, which implies that Lemma holds. Thus the proof is completed.

Proof of Theorem 3.1. The proof follows from Lemma 3.2.

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