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Approximation of values of hypergeometric functions by restricted rationals

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Résumé. Nous calculons des bornes supérieures et inférieures pour l'approximation de fonctions hyperboliques aux points $1/s$ $(s = 1, 2, \dots)$ par des rationnels x/y, tels que x, y satisfassent une équation quadratique. Par exemple, tous les entiers positifs x, y avec $y \equiv 0 \pmod 2$, solutions de l'équation de Pythagore $x^2 + y^2 = z^2$, satisfont

$$|y \sinh(1/s) - x| \gg \frac{\log\log y}{\log y} .$$

Réciproquement, pour chaque $s = 1, 2, \dots$, il existe une infinité d'entiers x, y, premiers entre eux, tels que

$$|y \sinh(1/s) - x| \ll \frac{\log\log y}{\log y}$$

et $x^2 + y^2 = z^2$ soient réalisés simultanément avec z entier. Une généralisation à l'approximation de $h(e^{1/s})$, pour $h(t)$ fonction rationnelle, est incluse.

Abstract. We compute upper and lower bounds for the approximation of hyperbolic functions at points $1/s$ $(s = 1, 2, \dots)$ by rationals x/y, such that x, y satisfy a quadratic equation. For instance, all positive integers x, y with $y \equiv 0 \pmod 2$ solving the Pythagorean equation $x^2 + y^2 = z^2$ satisfy

$$|y \sinh(1/s) - x| \gg \frac{\log\log y}{\log y} .$$

Conversely, for every $s = 1, 2, \dots$ there are infinitely many coprime integers x, y, such that

$$|y \sinh(1/s) - x| \ll \frac{\log\log y}{\log y}$$

and $x^2 + y^2 = z^2$ hold simultaneously for some integer z. A generalization to the approximation of $h(e^{1/s})$ for rational functions $h(t)$ is included.

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1. Introduction and statement of the results

In this paper, we investigate the approximation of hyperbolic functions at points $1/s$ for $s \geq 1$. For this purpose we apply the concept of leaping convergents to the irrational numbers $e^{1/s}$, and use the continued fraction expansion

$$e^{1/s} = \exp\left(\frac{1}{s}\right) = \Big[1; s-1, 1, 1, \overline{s(2k-1)-1, 1, 1}\,\Big]_{k\geq 2} . \tag{1.1}$$

For $n = 2, 3, \ldots$, set

$$\frac{P_n}{Q_n} := \Big[1; s-1, 1, 1, \overline{s(2k-1)-1, 1, 1}\,\Big]_{k=2}^{n} , \qquad (P_n, Q_n) = 1 . \tag{1.2}$$

We call the fractions P_n/Q_n leaping convergents. By Theorem 1 of [5] one has for all integers n that the leaping convergents satisfy both

$$P_{n+2} = 2s(2n+3)P_{n+1} + P_n \tag{1.3}$$

and

$$Q_{n+2} = 2s(2n+3)Q_{n+1} + Q_n \qquad (n \geq 0) ,$$

of course with $P_0 = 1$, $P_1 = 2s+1$, and $Q_0 = 1$, $Q_1 = 2s-1$ as initial values. We note that the well known expansion $e = [2; \overline{1, 2k, 1}]_{k\geq 1} = [1; 0, 1, 1, \overline{2k-2, 1, 1}]_{k\geq 2}$ and the recursion formulae, Theorem 1 of [1], for its leaping convergents, is just the special case $s = 1$ above, in that always $[1; 0, 1, 1, \ldots] = [2; 1, \ldots]$. As a consequence of these recurrence formulae, all the integers P_n and Q_n are odd.

We study rational approximation to certain values of hyperbolic functions by restricted rationals x/y. For example, if $x^2 + y^2$ is a square then always $|y \sinh(1/s) - x| \gg \log\log y/\log y$. However, conversely, we can detail infinitely many x and y with $x^2 + y^2$ square so that $|y \sinh(1/s) - x| \ll \log\log y/\log y$. The implied constants depend only on the positive integer s.

One readily sees that our methods yield analogous results for $\cosh(1/s)$ and $\tanh(1/s)$ with $x^2 - y^2$ square and $y^2 - x^2$ square, respectively.

Our results rely in significant part on the Hurwitz-periodic continued fraction expansion (1.1) of $e^{1/s}$ with the convergents P_k/Q_k at the end of each Hurwitz period being those of interest to us. These are therefore 'leaping convergents' of $e^{1/s}$; their properties are detailed in [5]. Our reliance on (1.1) explains why we restrict our functions to the value $1/s$.

Our results may seem eccentric but are in fact quite natural. Given that p/q is a good approximation to $\exp(a)$ it is not too surprising that, say for q sufficiently large, $x/y = (p^2 - q^2)/2pq$ will be a fairly good approximation to $\sinh(a)$. Our producing infinitely many x and y with $x^2 + y^2$ square so that $|y\sinh(1/s) - x| \ll \log\log y/\log y$ is therefore a matter of spelling out the details of this remark.

The converse claim, that always $|y\sinh(1/s) - x| \gg \log\log y/\log y$ for x^2+y^2 square is more delicate. We develop the tools to show, in either case $y = 2pq$ or $y = p^2 - q^2$, that p/q is a convergent of $\exp(1/s)$, and that this suffices to prove the claim.

Here, and in the sequel, s always is some fixed positive integer. Just so, here and throughout x and y denote relatively prime positive integers – of course large enough so that our related remarks make sense, for instance so that $\log\log y$ is positive.

In [2, Theorem 1.1] a general result on diophantine approximation with rationals restricted on Pythagorean numbers is proved.

Proposition 1.1 (C.E., 2003). *Let $\xi > 0$ be a real irrational number such that the quotients of the continued fraction expansion of at least one of the numbers $\eta_1 := \xi + \sqrt{1+\xi^2}$ and $\eta_2 := (1+\sqrt{1+\xi^2})/\xi$ are not bounded. Then there are infinitely many pairs of positive integers x, y satisfying*

$$|\xi y - x| \,=\, o(1) \quad \textit{with} \quad x^2+y^2 \quad \textit{square}\,.$$

Conversely, if the quotients of both of the numbers η_1 and η_2 are bounded, then there exists some $\delta > 0$ such that

$$|\xi y - x| \,\geq\, \delta$$

for all positive integers x, y with x^2+y^2 square.

It can easily be seen that the irrationality of ξ does not allow the numbers η_1 and η_2 to be rationals. The following result ([2, Corollary 1.1]) can be derived from the preceding proposition and from the metric theory of continued fractions:

Proposition 1.2. *To almost all real numbers ξ (in the sense of the Lebesgue measure) there are infinitely many pairs of integers $x \neq 0$, $y > 0$ satisfying*

$$|\xi y - x| \,=\, o(1) \quad \textit{and} \quad x^2+y^2 \quad \textit{square}\,.$$

Many exceptional numbers ξ not belonging to that set of full measure are given by certain quadratic surds ([2, Corollary 1.2]):

Proposition 1.3. *Let $r > 1$ be a rational such that $\xi := \sqrt{r^2-1}$ is an irrational number. Then the inequality*

$$|\xi y - x| \,>\, \delta$$

holds for some $\delta > 0$ (depending only on r) and for all positive integers x, y with x^2+y^2 square.

The lower bound δ can be computed explicitly. Proposition 1.3 follows from Proposition 1.1 by setting $\xi := \sqrt{r^2-1}$.

Lemma 1.1. *Let ξ be an irrational number. Let $\varphi(t)$ and $\psi(t)$ be positive decreasing functions of $t \geq t_0 (> 0)$ tending to zero as t tends to infinity such that, for any fixed $\alpha > 0$,*

$$\varphi(\alpha t) < A\varphi(t) , \quad \psi(\alpha t) > B\psi(t)$$

for all $t \geq t_0$, where $A, B > 0$ are constants depending only on α. Assume that

$$|y\xi - x| < \varphi(y) \qquad \text{for infinitely many} \quad x, y(\geq t_0) \in \mathbb{Z} , \tag{1.4}$$

$$|y\xi - x| > \psi(y) \qquad \text{for all} \quad x, y(\geq t_0) \in \mathbb{Z} . \tag{1.5}$$

Then, for any $a, b, c, d \in \mathbb{Z}$ with $ad - bc \neq 0$, the inequalities (1.4) and (1.5) hold with

$$\xi' = \frac{a\xi + b}{c\xi + d}$$

and $D_1\varphi(t)$, $D_2\psi(t)$ in place of ξ and $\varphi(t)$, $\psi(t)$, respectively, where D_1 and D_2 are constants depending possibly on $a, b, c,$ and d.

We omit the proof of this lemma, since it can be easily done in each of the cases $\xi' = a\xi$, $\xi + b$, ξ/c, and $1/\xi$.

Lemma 1.2. *For $n \geq 1$ we have*

$$n \log(ns) < \log Q_n < n \log(2ns) .$$

The preceding lemmas are used to prove the results stated below in Theorems 1.1, 1.2, and 1.3. Let $h(x)$ be a function with

$$h \in C^{(1)}[1+\delta; 3] \rightarrow \mathbb{R} , \qquad \min_{1+\delta \leq t \leq 3} |h'(t)| > 0 , \tag{1.6}$$

where δ is some arbitrary small positive real number. Particularly, $h'(x)$ takes its minimum and maximum for $1 + \delta \leq x \leq 3$. In our applications we choose $h(x)$ as rational functions.

Lemma 1.3. *Let $s \geq 1$ be an integer, and let $h(x)$ be as above. Then there are positive real constants C_1 and C_2 satisfying*

$$C_1 \cdot \frac{\log\log Q_n}{Q_n^2 \log Q_n} \leq \left| h(e^{1/s}) - h\left(\frac{P_n}{Q_n}\right)\right| \leq C_2 \cdot \frac{\log\log Q_n}{Q_n^2 \log Q_n} \qquad (n \geq 3) ,$$

where P_n and Q_n are defined by (1.3). The constants C_1 and C_2 can be computed effectively and they depend on the function h and on s.

The application of this theorem to various functions h leads to the following approximation results.

Theorem 1.1. *Let s be a positive integer and x and y relatively prime positive integers with $y \equiv 0 \pmod 2$ such that $x^2 + y^2$ is a square. Then*

$$\left| y \sinh\left(\frac{1}{s}\right) - x \right| \gg \frac{\log\log y}{\log y} \tag{1.7}$$

with an effectively computable implied constant depending only on s. Nonetheless, there are infinitely many pairs x, y as just described, so that

$$\left| y \cdot \sinh\left(\frac{1}{s}\right) - x \right| \ll \frac{\log\log y}{\log y}\ , \tag{1.8}$$

again with the implied constant depending only on s.

Indeed we explicitly compute the upper bound (see the proof below) as function of s and moreover note that in the inequality all the x may be supposed restricted to be divisible by some arbitrary nominated positive integers.

Theorem 1.2. *Let s be a positive integer and x and y relatively prime positive integers with $y \equiv 0 \pmod 2$ such that $x^2 - y^2$ is a square. Then*

$$\left| y \cosh\left(\frac{1}{s}\right) - x \right| \gg \frac{\log\log y}{\log y}$$

with an effectively computable implied constant depending only on s. Nonetheless, there are infinitely many pairs x, y as just described, so that

$$\left| y \cdot \cosh\left(\frac{1}{s}\right) - x \right| \ll \frac{\log\log y}{\log y}\ ,$$

again with the implied constant depending only on s.

Theorem 1.3. *Let s be a positive integer and x and y relatively prime positive integers with $y \equiv 1 \pmod 2$ such that $y^2 - x^2$ is a square. Then*

$$\left| y \tanh\left(\frac{1}{s}\right) - x \right| \gg \frac{\log\log y}{\log y}$$

with an effectively computable implied constant depending only on s. Nonetheless, there are infinitely many pairs x, y as just described, so that

$$\left| y \cdot \tanh\left(\frac{1}{s}\right) - x \right| \ll \frac{\log\log y}{\log y}\ ,$$

again with the implied constant depending only on s.

We shall prove Lemmas 1.2 and 1.3 in section 2, and Theorem 1.1 in section 3. The arguments in the proofs of Theorems 1.2 and 1.3 are similarly as in the proof of Theorem 1.1. Therefore we restrict ourselves to some remarks.

2. Proof of Lemmas 1.2 and 1.3.

Proof of Lemma 1.2: The proof is done by induction and by easy estimations. For $s \geq 1$ and $n \geq 1$ we have

$$Q_n > \frac{(2n-1)!}{(n-1)!} s^n \geq (ns)^n$$

and

$$Q_n < \frac{(2n)!}{n!} s^n \leq (2ns)^n \,.$$

Proof of Lemma 1.3: Let a_k and p_k/q_k be the kth partial quotient and the kth convergent of an irrational number ξ. Then one gets

$$\frac{1}{(2+a_{k+1})q_k^2} < \left|\xi - \frac{p_k}{q_k}\right| < \frac{1}{a_{k+1}q_k^2} \qquad (k \geq 1)\ . \tag{2.1}$$

For all $s \geq 1$, we choose $k = 3n$, which yields

$$a_{3n+1} = (2n+1)s - 1\ , \qquad p_{3n} = P_n\ , \qquad q_{3n} = Q_n \qquad (n \geq 0)\ .$$

Then (2.1) implies that

$$\frac{1}{((2n+1)s+1)Q_n^2} < \left|e^{1/s} - \frac{P_n}{Q_n}\right| < \frac{1}{((2n+1)s-1)Q_n^2} \qquad (n \geq 0)\ . \tag{2.2}$$

Let

$$D_1 := 1 + \frac{\log(2s)}{\log 2}\ .$$

Then from Lemma 1.2, and $\log(2ns) \leq D_1 \log n$ $(n \geq 2)$, one has

$$n \log n < \log Q_n < D_1 n \log n \qquad (n \geq 2)\ . \tag{2.3}$$

We also get from (2.2) that

$$\frac{1}{4nsQ_n^2} < \left|e^{1/s} - \frac{P_n}{Q_n}\right| < \frac{1}{2nsQ_n^2} \qquad (n \geq 2)\ . \tag{2.4}$$

Next, it follows from the right inequality in (2.3) that $\log\log Q_n < D_2 \log n$ holds for some positive constant D_2, e.g.

$$D_2 := 1 + \frac{\log\log 15}{\log 15} + \frac{\log D_1}{\log 2} \geq 1 + \frac{\log\log n}{\log n} + \frac{\log D_1}{\log n} \qquad (n \geq 2)\ .$$

Then the left inequality in (2.3) yields

$$n < \frac{\log Q_n}{\log n} < \frac{D_2 \log Q_n}{\log\log Q_n}\ , \tag{2.5}$$

or

$$\frac{1}{n} > \frac{D_3 \log\log Q_n}{\log Q_n} \qquad (n \geq 2\,,\ D_3 := D_2^{-1})\ .$$

Conversely, we get by similar arguments for $n \geq 3$ from (2.3) that $\log n < \log\log Q_n$. Hence

$$n > \frac{\log Q_n}{D_1 \log n} > \frac{\log Q_n}{D_1 \log\log Q_n}\ , \tag{2.6}$$

or

$$\frac{1}{n} < \frac{D_1 \log\log Q_n}{\log Q_n} \qquad (n \geq 3)\ .$$

From (2.4), (2.5) and (2.6) we get

$$\frac{1}{4s} \cdot \frac{D_3 \log\log Q_n}{Q_n^2 \log Q_n} < \left| e^{1/s} - \frac{P_n}{Q_n} \right| < \frac{1}{2s} \cdot \frac{D_1 \log\log Q_n}{Q_n^2 \log Q_n}\ ,$$

and therefore we have for positive constants $D_4 := D_3/4s$ and $D_5 := D_1/2s$:

$$\frac{D_4 \log\log Q_n}{Q_n^2 \log Q_n} < \left| e^{1/s} - \frac{P_n}{Q_n} \right| < \frac{D_5 \log\log Q_n}{Q_n^2 \log Q_n} \qquad (s \geq 1\ ,\ n \geq 3)\ . \tag{2.7}$$

For every integer $n \geq 1$ there exists a real number α satisfying simultaneously

$$\left| h(e^{1/s}) - h\left(\frac{P_n}{Q_n}\right) \right| = |h'(\alpha)| \cdot \left| e^{1/s} - \frac{P_n}{Q_n} \right|\ , \tag{2.8}$$

$$e^{1/s} \leq \alpha \leq \frac{P_n}{Q_n} \qquad \text{or} \qquad \frac{P_n}{Q_n} \leq \alpha \leq e^{1/s}\ . \tag{2.9}$$

By

$$\frac{P_2}{Q_2} = \frac{p_6}{q_6} \leq \frac{P_{2k}}{Q_{2k}} < e^{1/s} < \frac{P_{2k-1}}{Q_{2k-1}} \leq \frac{p_9}{q_9} = \frac{P_3}{Q_3} \qquad (k \geq 2)$$

we conclude from (2.9) with $n \geq 2$ that

$$1 < \frac{P_2}{Q_2} \leq \alpha \leq \frac{P_3}{Q_3}\ . \tag{2.10}$$

Let

$$t_1 := \frac{P_2}{Q_2} \quad \text{and} \quad t_2 := \frac{P_3}{Q_3}\ . \tag{2.11}$$

By (2.10) we know that

$$t_1 \leq \alpha \leq t_2\ , \tag{2.12}$$

and by the hypotheses on the function h the positive numbers

$$D_6 := \min_{t_1 \leq t \leq t_2} |h'(t)| \qquad \text{and} \qquad D_7 := \max_{t_1 \leq t \leq t_2} |h'(t)|$$

exist; put $\delta := t_1 - 1$ in (1.6). D_6 and D_7 depend on h only. Thus we get from (2.7), (2.8) and (2.12) the bounds in Lemma 1.3 with constants $C_1 := D_4 D_6$ and $C_2 := D_5 D_7$. This completes the proof of the lemma.

3. Proof of Theorems 1.1, 1.2, 1.3.

Proof of Theorem 1.1: For the upper bound in (1.8) let

$$h(t) := \frac{1}{2}\cdot\left(t-\frac{1}{t}\right) \qquad (1 < e^{1/2s} \le t \le 3)\ . \tag{3.1}$$

One easily computes

$$h'(t) = \frac{1}{2}+\frac{1}{2t^2} \quad \text{such that} \quad \frac{5}{9} \le h'(t) \le 1 \qquad (1 \le t \le 3)\ . \tag{3.2}$$

Now we apply Lemma 1.3. Put $x_n := P_n^2 - Q_n^2$, $y_n := 2P_nQ_n$. Particularly, $P_n > Q_n$. Furthermore,

$$h(e^{1/s}) = \sinh\left(\frac{1}{s}\right)\ , \qquad h\left(\frac{P_n}{Q_n}\right) = \frac{P_n^2-Q_n^2}{2P_nQ_n} = \frac{x_n}{y_n}\ . \tag{3.3}$$

We know for all integers $n \ge 2$ that $1 < t_1 \le P_n/Q_n \le t_2 < 3$ with t_1, t_2 defined in (2.11). This implies

$$t_1Q_n^2 \le P_nQ_n = \frac{y_n}{2} \le t_2Q_n^2 \qquad (n \ge 2)\ .$$

Using $Q_n > 3$ and $t_2 < 3$, we get $\log Q_n > \log(y_n/2)/3$. Additionally, we have $\log\log Q_n \le \log\log(y_n/2)$. Hence the application of Lemma 1.3 yields

$$\left|\sinh\left(\frac{1}{s}\right) - \frac{x_n}{y_n}\right| \le C_2\cdot\frac{\log\log Q_n}{Q_n^2\log Q_n} < 3C_2t_2\cdot\frac{\log\log(y_n/2)}{(y_n/2)\log(y_n/2)} \qquad (n \ge 3)\ .$$

Setting $C_3 := 3C_2t_2$ and $x = x_n/2$ and $y = y_n/2$, we get (1.8).

Finally, we restrict n on integers of the form $n = 2kt$ with $k \ge 1$. Then it follows from Theorem 1.5 in [1] and from a Remark in [5], respectively, that $Q_{2kt} \equiv (-1)^{2kt} \mod (2t)$ and $P_{2kt} \equiv 1 \mod (2t)$. We get by (1.2),

$$\begin{aligned} x_n &= p_{3n}^2 - q_{3n}^2 = P_n^2 - Q_n^2 = P_{2kt}^2 - Q_{2kt}^2 \\ &\equiv 1^2 - (-1)^{4kt} = 1 - 1 = 0 \qquad \mod (2kt)\ . \end{aligned}$$

Hence x_n is divisible by $2t$. Since all the integers P_n and Q_n are odd, we know that x_n and y_n are divisible by 2 both. Put

$$x := \frac{x_n}{2} \qquad \text{and} \qquad y := \frac{y_n}{2}\ . \tag{3.4}$$

Then x is divisible by t, and x, y are coprime, which follows from $(P_n, Q_n) = 1$ and $P_nQ_n \equiv 1 \mod 2$. Moreover, we have

$$x^2 + y^2 = \frac{(P_n^2-Q_n^2)^2}{4} + P_n^2Q_n^2 = \frac{P_n^4 + 2P_n^2Q_n^2 + Q_n^4}{4} = \left(\frac{P_n^2+Q_n^2}{2}\right)^2.$$

In order to prove the first statement in Theorem 1.1 we consider two positive integers x and y such that $x^2 + y^2$ is a square. It suffices to treat the case with coprime numbers x and y: When the lower bound holds with a constant C_4' for such *coprime* integers x and y with $y \geq 3$, we get for every integer $D > 1$

$$\left|(Dy)\sinh\left(\frac{1}{s}\right) - (Dx)\right| > C_4' D \cdot \frac{\log\log y}{\log y} \geq \frac{C_4'}{4} \cdot \frac{\log\log(Dy)}{\log(Dy)} \,. \tag{3.5}$$

Since $y \equiv 0 \mod 2$, there are two positive coprime integers p and q with $p > q$ exist satisfying $x = p^2 - q^2$ and $y = 2pq$ (because x and y are coprime integers). For the function h defined in (3.1) we then have, using the lower bound given in (3.2),

$$\left|\sinh\left(\frac{1}{s}\right) - \frac{x}{y}\right| > \frac{5}{9} \cdot \left|e^{1/s} - \frac{p}{q}\right| \,. \tag{3.6}$$

It suffices to consider integers x and y satisfying

(3.7)
$$\left|\sinh\left(\frac{1}{s}\right) - \frac{x}{y}\right| \leq \frac{\log\log y}{y\log y} \qquad \text{and} \qquad y \geq \max\{2\,000\,000\,;\, 8Q_3^2\} \,.$$

Particularly we have from (3.6) and (3.7) that

$$\left|e^{1/s} - \frac{p}{q}\right| < \frac{9}{5} \cdot \frac{\log\log y}{y\log y} < \frac{1}{2} \,.$$

This implies that

$$-\frac{1}{2} + e^{1/s} < \frac{p}{q} < \frac{1}{2} + e^{1/s} \,,$$

and then, by $1 < e^{1/s} < 3$,

$$\frac{q}{2} = q \cdot \left(-\frac{1}{2} + 1\right) < p < q \cdot \left(\frac{1}{2} + 3\right) < 4q \,.$$

It follows that $y = 2pq < 8q^2$, which yields $\log y < \log 8 + 2\log q \leq 5\log q$ for $q \geq 2$, or

$$\frac{\log y}{5} < \log q \qquad (q \geq 2) \,. \tag{3.8}$$

Taking logarithms again, we get by (3.7)

$$\log\log q > \log\log y \cdot \left(1 - \frac{\log 5}{\log\log y}\right) > \frac{\log\log y}{3} \,.$$

Collecting together the relationships between q and y, we have

$$\left.\begin{array}{ccccc} q^2 & < & 2pq = y & < & 8q^2 \\[2ex] \dfrac{\log y}{5} & < & \log q & < & \dfrac{\log y}{2} \\[2ex] \dfrac{\log\log y}{3} & < & \log\log q & < & \log\log y \end{array}\right\} \tag{3.9}$$

for $y \geq 2\,000\,000$. Next, we apply (3.6) and (3.7) for the second time to get a new upper bound for $|e^{1/s} - p/q|$ by (3.9):

$$\left| e^{1/s} - \frac{p}{q} \right| < \frac{9}{5} \cdot \frac{\log\log y}{y \log y} < \frac{9}{5} \cdot \frac{\log\log y}{\log y} \cdot \frac{1}{q^2} < \frac{9}{5} \cdot \frac{5}{27} \cdot \frac{1}{q^2} = \frac{1}{3q^2} \,. \tag{3.10}$$

It follows from the well-known facts of the elementary theory of continued fractions that p/q is a convergent of $e^{1/s}$ ([3], Theorem 184); namely there is an integer $k \geq 0$ with

$$\frac{p}{q} = \frac{p_k}{q_k} \,,$$

where p_k/q_k denotes the kth convergent of $e^{1/s}$. Since p and q are coprime and positive, we additionally know that $p = p_k$ and $q = q_k$. Let a_{k+1} be the corresponding quotient of the continued fraction expansion of $e^{1/s}$. We have $a_{k+1} > 1$, since otherwise it follows from $a_{k+1} = 1$ and (2.1) that

$$\left| e^{1/s} - \frac{p}{q} \right| = \left| e^{1/s} - \frac{p_k}{q_k} \right| > \frac{1}{3q_k^2} \,,$$

which contradicts (3.10). Therefore the convergent p/q equals to some fraction P_n/Q_n with positive integers $p = P_n$ and $q = Q_n$ defined by (1.3). Thus we may apply Lemma 1.3 and (3.2), (3.3), and (3.9) in order to prove (1.7). In Lemma 1.3 the hypothesis $n \geq 3$ is fulfilled by $y \geq 8Q_3^2$ and (3.9), since $Q_n = q > \sqrt{y/8} \geq Q_3$. Therefore, we have

$$\left| \sinh\left(\frac{1}{s}\right) - \frac{x}{y} \right| = \left| h(e^{1/s}) - h\left(\frac{P_n}{Q_n}\right) \right| > C_1 \cdot \frac{\log\log Q_n}{Q_n^2 \log Q_n} = C_1 \cdot \frac{\log\log q}{q^2 \log q}$$

$$> \frac{2C_1}{3} \cdot \frac{\log\log y}{y \log y} = D_8 \cdot \frac{\log\log y}{y \log y} \,. \tag{3.11}$$

The proofs of Theorem 1.2 and 1.3 differ from that one of Theorem 1.1 by use of different functions h corresponding to the solutions of the diophantine equations. All the remaining arguments are the same as in the proof of Theorem 1.1 and will be omitted.

Proof of Theorem 1.2: The basic function in the proof of Theorem 1.2 is

$$h(t) := \frac{1}{2} \cdot \left(t + \frac{1}{t} \right) \qquad (1 \leq t \leq 3) \,.$$

Then one has

$$h(e^{1/s}) = \cosh\left(\frac{1}{s}\right) \qquad \text{and} \qquad h\left(\frac{p}{q}\right) = \frac{p^2 + q^2}{2pq} \,.$$

Putting $x := p^2 + q^2$, $y := 2pq$ with $p > q$, one gets $x \geq y > 0$, and $x^2 - y^2 = (p^2 - q^2)^2$. Let

$$h'(t) = \frac{1}{2} - \frac{1}{2t^2}\ , \qquad h''(t) = \frac{1}{t^3} \qquad (1 \leq t \leq 3)\ .$$

Clearly, $h'(t)$ increases monotonously for $t \geq 1$. Thus we have

$$\frac{1}{2} - \frac{1}{2e^{1/s}} = h'(e^{1/2s}) \leq h'(t) \leq h'(3) = \frac{4}{9} \qquad (e^{1/2s} \leq t \leq 3)\ .$$

Proof of Theorem 1.3: The basic function in the proof of Theorem 1.3 is

$$h(t) := \frac{t^2 - 1}{t^2 + 1} \qquad (1 \leq t \leq 3)\ .$$

Then one has

$$h(e^{1/s}) = \tanh\left(\frac{1}{s}\right) \qquad \text{and} \qquad h\left(\frac{p}{q}\right) = \frac{p^2 - q^2}{p^2 + q^2}\ .$$

Putting $x := p^2 - q^2$, $y := p^2 + q^2$ with $p > q$, one gets $0 < x \leq y$, and $y^2 - x^2 = (2pq)^2$. Since $h'(t)$ decreases monotonously for $t \geq 1$, we have

$$\frac{3}{25} = h'(3) \leq h'(t) \leq h'(1) = 1 \qquad (1 \leq t \leq 3)\ .$$

Remark on Theorem 1.1: All the solutions of

$$x^2 + y^2 = 2z^2\ , \quad (x, y) = 1\ , \quad z > 0 \qquad (x, y \in \mathbb{Z})$$

are given by

$$x = p^2 - q^2 \pm 2pq\ , \quad y = p^2 - q^2 \mp 2pq\ , \quad z = p^2 + q^2 \qquad (p, q \in \mathbb{Z})$$

(cf [6], p.13). Putting

$$h(t) := \frac{t^2 + 2t - 1}{t^2 - 2t - 1}\ , \quad h\left(\frac{p}{q}\right) = \frac{p^2 - q^2 + 2pq}{p^2 - q^2 - 2pq} =: \frac{x}{y}\ ,$$

we can prove Theorem 1.1 with $\xi := \big(\sinh(1/s) + 1\big)/\big(\sinh(1/s) - 1\big)$ in place of $\sinh(1/s)$. The details and more applications will be discussed in our following paper.

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