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The cubics which are differences of two conjugates of an algebraic integer

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Résumé. On montre qu'un entier algébrique cubique sur un corps de nombres K, de trace nulle est la différence de deux conjugués sur K d'un entier algébrique. On prouve aussi que si N est une extension cubique normale du corps des rationnels, alors tout entier de N de trace zéro est la différence de deux conjugués d'un entier de N si et seulement si la valuation $3-$adique du discriminant de N est différente de 4.

Abstract. We show that a cubic algebraic integer over a number field K, with zero trace is a difference of two conjugates over K of an algebraic integer. We also prove that if N is a normal cubic extension of the field of rational numbers, then every integer of N with zero trace is a difference of two conjugates of an integer of N if and only if the $3-$adic valuation of the discriminant of N is not 4.

1. Introduction

Let K be a number field, β an algebraic number with conjugates $\beta_1 = \beta, \beta_2, ..., \beta_d$ over K and $L = K(\beta_1, \beta_2, ..., \beta_d)$ the normal closure of the extension $K(\beta)/K$. In [2], Dubickas and Smyth have shown that β can be written $\beta = \alpha - \alpha'$, where α and α' are conjugates over K of an algebraic number, if and only if there is an element σ of the Galois group $G(L/K)$ of the extension L/K, of order n such that $\sum_{0\leq i\leq n-1} \sigma^i(\beta) = 0$. In this case $\beta = \alpha - \sigma(\alpha)$,where $\alpha = \sum_{0\leq i\leq n-1}(n-i-1)\sigma^i(\beta)/n$ is an element of L and the trace of β for the extension $K(\beta)/K$, namely $Tr_{K(\beta)/K}(\beta) = \beta_1 + \beta_2 + ... + \beta_d$, is 0. Furthermore, the condition on the trace of β to be 0 is also sufficient to express $\beta = \alpha - \alpha'$ with some α and α' conjugate over K of an algebraic number, when the extension $K(\beta)/K$ is normal (i. e. when $L = K(\beta)$) and its Galois group is cyclic (in this case we say that the extension $K(\beta)/K$ is cyclic) or when $d \leq 3$.

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Let D be a positive rational integer and $\mathcal{P}(D)$ the proposition : *For any number field K and for any algebraic integer β of degree $\leq D$ over K, if β is a difference of two conjugates over K of an algebraic number, then β is a difference of two conjugates over K of an algebraic integer.* In [1], Smyth asked whether $\mathcal{P}(D)$ is true for all values of D. It is clear that if $Tr_{K(\beta)/K}(\beta) = 0$ and $\beta \in \mathbb{Z}_K$,where $\mathbb{Z}_K$ is the ring of integers of K, then $\beta = 0 = 0 - 0$ and $\mathcal{P}(1)$ is true. For a quadratic extension $K(\beta)/K$, Dubickas showed that if $Tr_{K(\beta)/K}(\beta) = 0$, then β is a difference of two conjugates over K of an algebraic integer of degree ≤ 2 over $K(\beta)$ [1]. Hence, $\mathcal{P}(2)$ is true. In fact, Dubickas proved that if the minimal polynomial of the algebraic integer β over K, say $Irr(\beta, K)$, is of the form $P(x^m)$, where $P \in \mathbb{Z}_K[x]$ and m is a rational integer greater than 1, then β is a difference of two conjugates over K of an algebraic integer.

Consider now the assertion $\mathcal{P}_c(D)$: *For any number field K and for any algebraic integer β of degree $\leq D$ over K such that the extension $K(\beta)/K$ is cyclic, if $Tr_{K(\beta)/K}(\beta) = 0$, then β is a difference of two conjugates over K of an algebraic integer.*

The first aim of this note is to prove :

Theorem 1. *The assertions $\mathcal{P}(D)$ and $\mathcal{P}_c(D)$ are equivalent, and $\mathcal{P}(3)$ is true.*

Let $\mathbb{Q}$ be the field of rational numbers. In [5], the author showed that if the extension $N/\mathbb{Q}$ is normal with prime degree, then every integer of N with zero trace is a difference of two conjugates of an integer of N if and only if $Tr_{N/\mathbb{Q}}(\mathbb{Z}_N) = \mathbb{Z}_\mathbb{Q}$. It easy to check that if $N = \mathbb{Q}(\sqrt{m})$ is a quadratic field (m is a squarefree rational integer), then $Tr_{N/\mathbb{Q}}(\mathbb{Z}_N) = \mathbb{Z}_\mathbb{Q}$ if and only if $m \equiv 1[4]$. For the cubic fields we have :

Theorem 2. *Let N be a normal cubic extension of $\mathbb{Q}$. Then, every integer of N with zero trace is a difference of two conjugates of an integer of N if and only if the $3-$adic valuation of the discriminant of N is not 4.*

2. Proof of Theorem 1

First we prove that the propositions $\mathcal{P}(D)$ and $\mathcal{P}_c(D)$ are equivalent. It is clear that $\mathcal{P}(D)$ implies $\mathcal{P}_c(D)$, since by Hilbert's Theorem 90 [3] the condition $Tr_{K(\beta)/K}(\beta) = 0$ is sufficient to express $\beta = \alpha - \alpha'$ with some α and α' conjugate over K of an algebraic number. Conversely, let β be an algebraic integer of degree $\leq D$ over K and which is a difference of two conjugates over K of an algebraic number. By the above result of Dubickas and Smyth, and with the same notation, there is an element $\sigma \in G(L/K)$, of order n such that $\sum_{0 \leq i \leq n-1} \sigma^i(\beta) = 0$. Let $< \sigma >$ be the cyclic subgroup of $G(L/K)$ generated by σ and $L^{<\sigma>} = \{x \in L, \sigma(x) = x\}$ the fixed field of $< \sigma >$. Then, $K \subset L^{<\sigma>} \subset L^{<\sigma>}(\beta) \subset L$, the degree of β over $L^{<\sigma>}$ is

$\leq D$ and by Artin's theorem [3], the Galois group of the normal extension $L/L^{<\sigma>}$ is $<\sigma>$. Hence, the extensions $L/L^{<\sigma>}$ and $L^{<\sigma>}(\beta)/L^{<\sigma>}$ are cyclic since their Galois groups are respectively $<\sigma>$ and a factor group of $<\sigma>$. Furthermore, the restrictions to the field $L^{<\sigma>}(\beta)$ of the elements of the group $<\sigma>$ belong to the Galois group of $L^{<\sigma>}(\beta)/L^{<\sigma>}$ and each element of $G(L^{<\sigma>}(\beta)/L^{<\sigma>})$ is a restriction of exactly d elements of the group $<\sigma>$, where d is the degree of $L/L^{<\sigma>}(\beta)$. It follows that

$$dTr_{L^{<\sigma>}(\beta)/L^{<\sigma>}}(\beta) = Tr_{L/L^{<\sigma>}}(\beta) = \sum_{0\leq i\leq n-1} \sigma^i(\beta) = 0,$$

and β is a difference of two conjugates over $L^{<\sigma>}$ of an algebraic number. Assume now that $\mathcal{P}_c(D)$ is true. Then, β is difference of two conjugates over $L^{<\sigma>}$, and a fortiori over K, of an algebraic integer and so $\mathcal{P}(D)$ is true.

To prove that $\mathcal{P}(3)$ is true, it is sufficient to show that if β a cubic algebraic integer over a number field K with $Tr_{K(\beta)/K}(\beta) = 0$ and such that the extension $K(\beta)/K$ is cyclic, then β is a difference of two conjugates of an algebraic integer, since $\mathcal{P}(2)$ is true and the assertions $\mathcal{P}(3)$ and $\mathcal{P}_c(3)$ are equivalent. Let

$$Irr(\beta, K) = x^3 + px + q,$$

and let σ be a generator of $G(K(\beta)/K)$. Then, $p = Tr_{K(\beta)/K}(\beta\sigma(\beta))$ and the discriminant $disc(\beta)$ of the polynomial $Irr(\beta, K)$ satisfies

$$disc(\beta) = -4p^3 - 3^3q^2 = \delta^2,$$

where $\delta = (\beta - \sigma^2\beta)(\sigma\beta - \beta)(\sigma^2\beta - \sigma\beta) \in \mathbb{Z}_K$. Set $\gamma = \beta - \sigma^2(\beta)$. Then, γ is of degree 3 over K and

$$Irr(\gamma, K) = x^3 + 3px - \delta.$$

As the polynomial $-27t + x^3 + 3px - 26\delta$ is irreducible in the ring $K(\beta)[t, x]$, by Hilbert's irreducibility theorem [4], there is a rational integer s such that the polynomial $x^3 + 3px - (26\delta + 27s)$ is irreducible in $K(\beta)[x]$. Hence, if $\theta^3 + 3p\theta - (26\delta + 27s) = 0$, then

$$Irr(\theta, K(\beta)) = x^3 + 3px - (26\delta + 27s) = Irr(\theta, K),$$

since $Irr(\theta, K(\beta)) \in K[x]$. Set $\alpha = \frac{\gamma}{3} + \frac{\theta}{3}$. Then, $\frac{\sigma(\gamma)}{3} + \frac{\theta}{3}$ is a conjugate of α over $K(\beta)$ (and a fortiori over K) and

$$\beta = \frac{\gamma}{3} + \frac{\theta}{3} - (\frac{\sigma(\gamma)}{3} + \frac{\theta}{3}).$$

From the relations $(\frac{\theta}{3})^3 + \frac{p}{3}(\frac{\theta}{3}) - \frac{26\delta+27s}{27} = 0$ and $(\frac{\gamma}{3})^3 + \frac{p}{3}(\frac{\gamma}{3}) = \frac{\delta}{27}$, we obtain that α is a root of the polynomial

$$x^3 - \gamma x^2 + (\frac{\gamma^2 + p}{3})x - (\delta + s) \in K(\beta)[X]$$

and α is an algebraic integer (of degree ≤ 3 over $K(\beta)$) provided $\frac{\gamma^2+p}{3} \in \mathbb{Z}_{K(\beta)}$. A short computation shows that from the relation $\gamma(\gamma^2+3p)=\delta$, we have $Irr(\frac{\gamma^2}{3}, K) = x^3+2px^2+p^2x-\frac{disc(\beta)}{27}$ and $\frac{\gamma^2+p}{3}$ is a root of the polynomial $x^3+px^2+q^2$ whose coefficients are integers of K. $\square$

Remark 1. It follows from the proof of Theorem 1, that if β is a cubic algebraic integer over a number field K with zero trace, then β is a difference of two conjugates over K of an algebraic integer of degree ≤ 3 over $K(\beta)$. The following example shows that the constant 3 in the last sentence is the best possible. Set $K=\mathbb{Q}$ and $Irr(\beta,\mathbb{Q}) = x^3-3x-1$. Then, $disc(\beta)=3^4$ and the extension $\mathbb{Q}(\beta)/\mathbb{Q}$ is normal, since β^2-2 is also a root of $Irr(\beta,\mathbb{Q})$. By Theorem 3 of [5], β is not a difference of two conjugates of an integer of $\mathbb{Q}(\beta)$ (the $3-$adic valuation of $disc(\beta)$ is 4) and if $\beta=\alpha-\alpha'$, where α is an algebraic integer of degree 2 over $\mathbb{Q}(\beta)$ and α' is a conjugate of α over $\mathbb{Q}(\beta)$, then there exists an element τ of the group $G(\mathbb{Q}(\beta,\alpha)/\mathbb{Q}(\beta))$ such that $\tau(\beta)=\beta$, $\tau(\alpha)=\alpha'$, $\tau(\alpha')=\alpha$ and $\beta=\tau(\alpha-\alpha')=\alpha'-\alpha=-\beta$.

Remark 2. With the notation of the proof of Theorem 1 (the second part) we have: *Let β be a cubic algebraic integer over K with zero trace and such that the extension $K(\beta)/K$ is cyclic. Then, β is a difference of two conjugates of an integer of $K(\beta)$, if and only if there exists $a \in \mathbb{Z}_K$ such that the two numbers $\frac{a^2+p}{3}$ and $\frac{a^3+3pa+\delta}{27}$ are integers of K.* Indeed, suppose that $\beta=\alpha-\sigma(\alpha)$, where $\alpha \in \mathbb{Z}_{K(\beta)}$ (if $\beta=\alpha-\sigma^2(\alpha)$, then $\beta=\alpha+\sigma(\alpha)-\sigma(\alpha+\sigma(\alpha))$). Then, $\alpha-\sigma(\alpha)=\frac{\gamma}{3}-\sigma(\frac{\gamma}{3})$, $\alpha-\frac{\gamma}{3}=\sigma(\alpha-\frac{\gamma}{3})$, $\alpha-\frac{\gamma}{3} \in K$ and there exists an integer a of K such that $3\alpha-\gamma=a$. Hence, $\frac{\gamma+a}{3}=\alpha \in \mathbb{Z}_{K(\beta)}$, $Irr(\frac{\gamma+a}{3},K) = x^3-ax^2+\frac{a^2+p}{3}x-\frac{a^3+3pa+\delta}{27} \in \mathbb{Z}_K[X]$ and so the numbers $\frac{a^2+p}{3}$ and $\frac{a^3+3pa+\delta}{27}$ are integers of K. The converse is trivial, since $\beta=\frac{\gamma}{3}-\sigma(\frac{\gamma}{3})=\frac{\gamma+a}{3}-\sigma(\frac{\gamma+a}{3})$ for all integers a of K. It follows in particular when $\frac{disc(\beta)}{3^6} \in \mathbb{Z}_K$, that β is a difference of two conjugates of an integer of $K(\beta)$ ($a=0$). Note finally that for the case where $K=\mathbb{Q}$ a more explicit condition was obtained in [5].

3. Proof of Theorem 2

With the notation of the proof of Theorem 1 (the second part) and $K=\mathbb{Q}$, let N be a cubic normal extension of $\mathbb{Q}$ with discriminant Δ and let v be the $3-$adic valuation. Suppose that every non-zero integer β of N with zero trace is a difference of two conjugates of an integer of N. Then, $N=\mathbb{Q}(\beta)$ and by Theorem 3 of [5], $v(disc(\beta)) \neq 4$. Assume also $v(\Delta)=4$. Then, $v(disc(\beta))>4$ and hence $v(disc(\beta)) \geq 6$, since $\frac{disc(\beta)}{\Delta} \in \mathbb{Z}_{\mathbb{Q}}$ and $disc(\beta)$ is a square of a rational integer. It follows that $\frac{\gamma}{3}$ is an algebraic integer, since its minimal polynomial over $\mathbb{Q}$ is $x^3 + \frac{p}{3}x - \frac{\delta}{27} \in \mathbb{Z}_{\mathbb{Q}}[X]$ and β can be written $\beta = \alpha - \sigma(\alpha)$, where $\alpha = \frac{\gamma}{3}$ is an integer of N with zero trace. Thus, $v(disc(\alpha)) \geq 6$ and there is an integer η of N with zero trace, such that $\alpha = \eta - \sigma(\eta)$. It follows that $\beta = \eta - \sigma(\eta) - \sigma(\eta - \sigma(\eta)) = \eta - 2\sigma(\eta) + \sigma^2(\eta) = -3\sigma(\eta)$ and $\frac{\beta}{3}$ is also an integer of N with zero trace. The last relation leads to a contradiction since in this case $\frac{\beta}{3^n} \in \mathbb{Z}_N$ for all positive rational integers n. Conversely, suppose $v(\Delta) \neq 4$. Assume also that there exists an integer β of N with zero trace which is not a difference of two conjugates of an integer of N. Then, $N = \mathbb{Q}(\beta)$ and by Theorem 1 of [5], we have $Tr_{N/\mathbb{Q}}(\mathbb{Z}_N) = 3\mathbb{Z}$, since $Tr_{N/\mathbb{Q}}(1) = 3$ and $Tr_{N/\mathbb{Q}}(\mathbb{Z}_N)$ is an ideal of $\mathbb{Z}$. If $\{e_1, e_2, e_3\}$ is an integral basis of N, then from the relation $\Delta = \det(Tr(e_i e_j))$, we obtain $v(\Delta) \geq 3$ and hence $v(\Delta) \geq 6$, since Δ is a square of a rational integer. The last inequality leads to a contradiction as in this case we have $v(disc(\beta)) \geq 6$ and $\beta = \frac{\gamma}{3} - \sigma(\frac{\gamma}{3})$ where $\frac{\gamma}{3} \in \mathbb{Z}_N$. □

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